A MIXED BOUNDARY VALUE PROBLEM FOR AN ANNULUS

By Yûsaku KOMATU and Imsik HONG

Among ring domains, i.e. doubly-connected domains bounded by two continua, an annulus plays often a particular role of canonical domain. Several functions, analytic or harmonic in an annulus, satisfying some types of preassigned boundary conditions can be expressed within a range of elementary or elliptic functions. Especially, the formulas due to Villat and Dini [1] for Dirichlet problem belong to this category, and a formula for Neumann problem does also [2].

Once Tomotika[3] has derived an expression of such a type for a related problem which has been shown to be useful for solving several problems on hydro- and aerodynamics. The problem may be stated as follows:

To determine an analytic function $f(z)$ regular single-valued and bounded in an annulus $q<|z|<1$ and satisfying the boundary conditions

$$\Re f(e^{i\varphi})=\Phi(\varphi)$$

and

$$\Im f(qe^{i\varphi})=0$$

for

$$0\leqq\varphi<2\pi.$$

In order to derive an expression for the solution, he followed faithfully first the Villat's method and then the Demotchenko's in his respective papers cited in [3]. However, this problem can be immediately reduced to a Dirichlet problem after an analytic prolongation by inversion and then solved quite briefly. In fact, the problem is evidently equivalent to determine an analytic function $f(z)$ regular single-valued and bounded in the duplicated annulus $q^2<|z|<1$ and satisfying the boundary conditions

$$\Re f(e^{i\varphi})=\Re f(q^2e^{i\varphi})=\Phi(\varphi)$$
$$\text{for}\quad 0\leqq\varphi<2\pi,$$

an arbitrary purely imaginary constant which is additively involved in the solution is to be determined by $\Im f(q)=0$; since $\Im f(z)$ remains constant along $|z|=q$, it then vanishes out along $|z|=q$.

Now, in the last problem, the condition for single-valuedness (monodromy condition) is surely satisfied:

$$\int_0^{2\pi}\Re f(e^{i\varphi})d\varphi=\int_0^{2\pi}\Re f(qe^{i\varphi})d\varphi.$$

Consequently, by means of Villat's formula, the solution is obtained in the form

$$f(z)=\frac{\hat{\omega}_1}{i\pi^2}\int_0^{2\pi}\Phi(\varphi)\cdot\left\{\hat{\zeta}\left(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)\right)-\hat{\zeta}_3\left(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)\right)\right\}d\varphi,$$

in conformity with the formula of Tomotika, where the notations for Weierstrassian theory of elliptic functions, marked by ^ , refer to those with primitive periods $2\hat{\omega}_1$ and $2\hat{\omega}_3$ satisfying a relation

$$\hat{\omega}_3/\hat{\omega}_1=-2i\lg q/\pi;$$

an additive constant has been adjusted so as $\Im f(q)=0$.

In the present Note, we shall deal with a slightly general type of mixed boundary value problem by means of a similar method as explained above. It is formulated as follows:

To determine a function $u(z)$, $z=re^{i\theta}$, harmonic and bounded in an annulus $q<|z|<1$ and satisfying the boundary conditions

$$u(qe^{i\varphi})=N(\varphi)$$

and

$$\frac{\partial u}{\partial \nu}(e^{i\varphi}) = P(\varphi)$$

for

$$0 \leqq \varphi < 2\pi,$$

$\partial/\partial\nu$ designating the differentiation along inward normal at $e^{i\varphi}$, and $N(\varphi)$ and $P(\varphi)$ being the assigned functions bounded and continuous for $0 \leqq \varphi < 2\pi$ and $N(2\pi - 0) = N(0)$, $P(2\pi - 0) = P(0)$.

The problem of Tomotika corresponds to a special case with $P(\varphi) \equiv 0$ after interchanging both circumferences by inversion with respect to $|z| = \sqrt{q}$.

The solution of our present problem may be regarded as the superposition of the functions $u^{(1)}(z)$ and $u^{(2)}(z)$, both harmonic and bounded in the annulus and satisfying special boundary conditions

$$u^{(1)}(qe^{i\varphi}) = N(\varphi) \quad \text{and} \quad \frac{\partial u^{(1)}}{\partial \nu}(e^{i\varphi}) = 0,$$

$$u^{(2)}(qe^{i\varphi}) = 0 \quad \text{and} \quad \frac{\partial u^{(2)}}{\partial \nu}(e^{i\varphi}) = P(\varphi).$$

The function $u^{(1)}(z)$ can be determined, after a harmonic prolongation, by solving a Dirichlet problem for the duplicated annulus $q < |z| < q^{-1}$ with boundary conditions

$$u^{(1)}(qe^{i\varphi}) = u^{(1)}(q^{-1}e^{i\varphi}) = N(\varphi).$$

By means of Villat's formula, it is readily expressed in the form

$$u^{(1)}(z) = \mathfrak{J}\frac{\hat{\omega}_1}{\pi^2}\int_0^{2\pi} N(\varphi) \cdot \left\{ \hat{\zeta}\left(\frac{\hat{\omega}_1}{\pi}(i \lg z + \varphi) - \frac{\hat{\omega}_3}{2}\right) - \hat{\zeta}_3\left(\frac{\hat{\omega}_1}{\pi}(i \lg z + \varphi) - \frac{\hat{\omega}_3}{2}\right)\right\} d\varphi,$$

where the notations from Weierstrassian theory of elliptic functions, marked by $\wedge$, refer here again to those with primitive periods $2\hat{\omega}_1$ and $2\hat{\omega}_3$ satisfying a relation

$$\hat{\omega}_3 / \hat{\omega}_1 = -2i \lg q / \pi.$$

On the other hand, the function $u^{(2)}(z)$ can be determined, after a harmonic prolongation, by solving a Neumann problem for the duplicated annulus $q^2 < |z| < 1$ with boundary conditions

$$\frac{\partial u^{(2)}}{\partial \nu}(e^{i\varphi}) = -q^2 \frac{\partial u^{(2)}}{\partial \nu}(q^2 e^{i\varphi}) = P(\varphi).$$

The condition for existence of a solution is evidently satisfied. An arbitrary additive constant involved in the solution is to be determined by $u^{(2)}(q) = 0$; $u^{(2)}(z)$ then vanishes out along $|z| = q$, since it remains constant along $|z| = q$. By means of a formula on Neumann problem for an annulus, the desired solution is expressed in the form

$$u^{(2)}(z) = \mathfrak{R}\frac{1}{\pi}\int_0^{2\pi} P(\varphi) \cdot \lg \frac{\hat{\sigma}\left(\frac{\hat{\omega}_1}{\pi}(i \lg z + \varphi)\right)}{\hat{\sigma}_3\left(\frac{\hat{\omega}_1}{\pi}(i \lg z + \varphi)\right)} d\varphi + C,$$

where the notations from Weierstrassian theory of elliptic functions refer again to those availed above and C designates a constant which is to be determined so as $u^{(2)}(q) = 0$.

Thus, our original problem has been explicitly solved by

$$u(z) = u^{(1)}(z) + u^{(2)}(z).$$

We have prefered as a basic domain for our problem the annulus $q < |z| < 1$. Functions solving several related problems for this annulus are often expressed by means of the quantities associated to the primitive periods $2\omega_1$ and $2\omega_3$ satisfying a relation

$$\omega_3 / \omega_1 = -i \lg q / \pi.$$

Accordingly, it will be convenient to transform the expression derived above into a form concerning these new primitive periods.

Since, in every case, only the ratio of the periods is essential, we may put

$$\hat{\omega}_1 = \omega_1 / 2 \quad \text{and} \quad \hat{\omega}_3 = \omega_3.$$

Then, in view of the identities

$$\hat{\zeta}(Z)=e_1 Z+\zeta(Z)+\zeta_1(Z),$$

$$\hat{\zeta}_3(Z)=e_1 Z+\zeta_2(Z)+\zeta_3(Z),$$

$$\hat{\sigma}(Z)=\exp(e_1 Z^2/2)\cdot\sigma(Z)\sigma_1(Z),$$

$$\hat{\sigma}_3(Z)=\exp(e_1 Z^2/2)\cdot\sigma_2(Z)\sigma_3(Z),$$

the quantities marked by $\wedge$ in our problem become as follows:

$$\begin{aligned}
&\hat{\zeta}\Big(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)-\frac{\hat{\omega}_3}{2}\Big) \\
&\qquad-\hat{\zeta}_3\Big(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)-\frac{\hat{\omega}_3}{2}\Big) \\
&=\zeta\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)-\frac{\omega_3}{2}\Big) \\
&\qquad+\zeta_1\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)-\frac{\omega_3}{2}\Big) \\
&-\zeta_2\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)-\frac{\omega_3}{2}\Big) \\
&\qquad-\zeta_3\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)-\frac{\omega_3}{2}\Big),
\end{aligned}$$

$$\frac{\hat{\sigma}\Big(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)\Big)}{\hat{\sigma}_3\Big(\frac{\hat{\omega}_1}{\pi}(i\lg z+\varphi)\Big)}
=\frac{\sigma\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)\Big)\,\sigma_1\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)\Big)}{\sigma_2\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)\Big)\,\sigma_3\Big(\frac{\omega_1}{2\pi}(i\lg z+\varphi)\Big)}.$$

We finally state a supplementary remark concerning an inverse problem.

In our original problem it is supposed that the boundary functions are bounded and continuous for their interval of definition. However, the integral representation for solution of the problem defines surely a function harmonic in $q<|z|<1$, provided the preassigned boundary functions are merely supposed to be integrable. It is, moreover, readily shown that the function thus defined satisfies the boundary conditions almost everywhere, though it is, of course, not necessarily bounded.

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(2) Cf. Y. Komatu, Integralformel betreffend Neumannsche Randwertaufgabe für einem Kreisring. Kōdai Math. Sem. Rep. (1953), 37-40.

(3) S. Tomotika, On certain problems of Dirichlet for an annular region, with special reference to hydrodynamical applications. Proc. Phys. Math. Soc. Japan **14** (1932), 197-213; A new derivation of the formula solving a kind of Dirichlet's problem for a ring region. Ibid. **18** (1936), 427-435.

(4) Cf., for instance, S. Tomotika and I. Imai, the interference effect of the surface of the sea on the lift of a seaplane. Rep. Aeron. Res. Inst., Tokyo Imp. Univ. No.146, **12** (1937), 71-128.

Department of Mathematics,
Tokyo Institute of Technology and
Mathematical Institute,
Tokyo University.

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