In the present note, first we shall supply the proof of the last part of the previous note [4] (the proof of Theorem 2 and its Corollary resp.), here we shall describe more general form than Theorem 2 of [4], and next we shall prove some relations between semi-traces and traces in $D^*$-algebra. The definitions and the notations in the note [4] will be used in this note.

1. Let $\mathfrak{A}$ be a $D^*$-algebra and $\tau$ be a finite semi-trace of $\mathfrak{A}$. And let $\{x^\tau, x^\tau_j, j\}$ be the representation of $\mathfrak{A}$ generated by $\tau$ and let $\mathfrak{R}^\tau$ be $C^*$-algebra generated by $\{x^\tau, x^\tau_j, j\}$. All other algebras $L^\tau$, $L^{\tau_j}$, $R^{\tau_j}$, and $W^\tau$ are defined by the same way in §1 of [4]. Then $\Omega$ be the character space of $\mathfrak{R}^\tau$ and $N = \{\omega \in \Omega : \omega(x^\tau) = 0\}$ for all $x \in \mathfrak{A}$ then $\Omega_\omega = (\Omega - N)$ can be embeded into the trace space of $\mathfrak{R}^\tau$ with weak* topology on $\mathfrak{R}^\tau$ by the canonical mapping $\mathfrak{R}^{\tau_j} \rightarrow \omega(A)$ for all $A \in \mathfrak{R}^\tau$. Putting $\Omega' = \Omega$ weak closure of $\omega(\Omega)$, $\Omega'$ is locally compact with respect to the weak* topology on $\mathfrak{R}^\tau$. If $K^\tau$ is a compact set in $\Omega'$, then it is covered by finite number of nbds (with compact closures in $\Omega'$) $U((\omega_1, A_1), \epsilon \in \Omega; (\omega(A_1) - \omega(A_2)) \leq \epsilon \leq x \in \Omega$).

We have called that $\mathfrak{A}$ in $\mathfrak{R}^\tau$ is bounded if and only if $\forall x \in \mathfrak{A}$ and a const. $M > 0$ (cf. §1 or [4]) in which $x^\tau$ must be replaced by $x^\tau$ at P.125, right side, lines 24 and 28). Now we describe supplementary remarks with respect to the bounded elements in $\mathfrak{R}^\tau$.

2. Let $\mathfrak{A}$ be a $D^*$-algebra with the approximate identity $\{e_x\}$, let $\tau$ be a semi-trace of $\mathfrak{A}$ and let $\{x^\tau, x^\tau_j, j\}$ be the corresponding representation of $\mathfrak{A}$. Moreover let
\( \mathcal{A} \) and \( \mathcal{W} \) be uniform and weak closure of \( \mathcal{L}(\mathcal{A}) \) respectively.

**PROPOSITION 1.** The following conditions are equivalent each other:

1. \( \mathcal{A} \) is trace.
2. There exists a constant \( M > 0 \) such that \( \|e \| \leq M \).
3. \( \mathcal{L}(\mathcal{A}) \) is dense in \( \mathcal{W} \).

**Proof.** (1\( \Rightarrow \)) is clear. First we prove (2\( \Rightarrow \)) (3\( \Rightarrow \)). Since

\[
\lim_{n \to \infty} \|e_n\| = 0,
\]

and hence \( \epsilon_n \to 0 \) (strongly). Hence \( \epsilon_n \to 0 \) (weakly). For all \( \eta \), there exists a directed set \( \mathcal{D} \) such that \( \epsilon_n \to \eta \) (strongly). Hence \( \epsilon_n \to \eta \) (weakly). For all \( \eta \), there exists a directed set \( \mathcal{D} \) such that \( \epsilon_n \to \eta \) (strongly). Hence \( \epsilon_n \to \eta \) (weakly).

Moreover, \( \mathcal{L}(\mathcal{A}) \) is dense in \( \mathcal{W} \). Therefore \( \mathcal{L}(\mathcal{A}) \) is dense in \( \mathcal{W} \). By Lemma 1 and the proof of Prop. 1 of [3], we can find a vector \( \eta \neq 0 \) in \( \mathcal{W} \) such that \( \epsilon_n \to \eta \) (weakly). For all \( \xi \), there exists a vector \( \xi_0 \) such that \( \epsilon_n \to \xi_0 \) (weakly). For all \( \eta \), there exists a vector \( \eta_0 \) such that \( \epsilon_n \to \eta_0 \) (weakly). For all \( \xi \), there exists a vector \( \xi_0 \) such that \( \epsilon_n \to \xi_0 \) (weakly). For all \( \eta \), there exists a vector \( \eta_0 \) such that \( \epsilon_n \to \eta_0 \) (weakly). For all \( \xi \), there exists a vector \( \xi_0 \) such that \( \epsilon_n \to \xi_0 \) (weakly). For all \( \eta \), there exists a vector \( \eta_0 \) such that \( \epsilon_n \to \eta_0 \) (weakly). 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and the right side $= \left( z^\tau, \tau(z) I \right)$. Since the both sides are equal for all $x$, $\xi$ and $\gamma \in \mathcal{O}$, $\eta^\tau = \tau(\xi) I$.

Let $P$ be projection onto the central manifold $\mathcal{O}$. For any $\xi \in \mathcal{O}$, there exist $x, \xi \in \mathcal{O}$ such that $\left( x^\tau - \eta \right) I$. Hence $Px^\tau \rightarrow P\eta = \eta$. Since for all $x \in \mathcal{O}$, $\eta^\tau$ and $\left( Px^\tau \right)^\tau = x^\tau$, cf. the proof of Prop. 1 of [3], $(Px^\tau)^\tau = \tau(x) I$, $Px^\tau = \tau(x) I$ and the center of $\mathcal{O}$ is scalar, i.e., $\{ x I \}$. Thus the center of $\mathcal{O}^\tau = \mathcal{W}^\tau \mathcal{W}^*(= \mathcal{W}^\tau)$ is $\{ x I \}$, and $\tau$ is pure. The proposition obtained in this remark contains the first part of Prop. 2 of [4] as a special case.

**FOOTNOTES**

(1) In a separable $D^\tau$-algebra, the decomposition of arbitrary semi-trace into a system of pure semi-traces in the form of direct integral over the real line has been shown in the previous note [5] using the reduction theory of von Neumann. Recently I.E. Segal has been published his decomposition theory "Decomposition of Operator Algebras. I and II, Mem. Amer. Math. Soc., 1951". If we apply his theory, Th. 1 of [4] may be shown in a most general form (in separable case). The precise discussion may be stated in the following in which we may prove that, in Th.1 of [4] all $\omega \in \mathcal{O}$ are characters of $A$ which is not always separable.

(2) For any $A \in \mathcal{L}^\tau$, let the corresponding bounded element $\tilde{A} \in B$ denote $A$.

(3) It is known that for semi-trace or trace of a $D^\tau$-algebra being pure, it is NASC that the corresponding two-sided representation is irreducible respectively (cf. [3]).

**REFERENCES**


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