ON PRIMARY ELEMENTS OF A MODULAR LATTICE

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It is well known that, in a Noetherean ring, every ideal can be written as an intersection of primary ideals. The theorem was extended by M.Ward and R.P.Dilworth to integral modular lattice ordered semigroups with maximum condition.([2], [3]) The purpose of this paper is to discuss it for modular lattices with maximum condition.

§1. Definitions.

Let L be a modular lattice with maximum condition and Θ be a set of congruences on L such that every meet of congruences in Θ is also in Θ. For a congruence θ the class containing an element a is denoted by $\theta(a)$ and the greatest element of $\theta(a)$ by a_θ.

Definition 1. An element q of L is primary (with respect to Θ) if and only if $q = q_\theta$ or $q \equiv I\ (\theta)$ for every θ in Θ.

Definition 2. A congruence θ in Θ is a radical (with respect to Θ) of an element a in L if and only if θ is the smallest one among the congruences by which a is congruent to I.

The radical of a is denoted by $\rho(a)$. Evidently, $\theta \geq \rho(a)$ if and only if $a \equiv I\ (\theta)$. For a primary element q we have $q \equiv I\ (\theta)$ if $\theta \geq \rho(q)$, and $q_\theta = q$ if $\theta \ngeq \rho(q)$.

Definition 3. By a short representation (with respect to Θ) of an element a in L, it is meant a representation of a as an irredundant meet of a finite number of primary elements all of which radicals are different.

Definition 4. A congruence on L is said to be neutral if (a) the class containing I is a neutral dual ideal ([I]) and (b) a is congruent to b if and only if $a \wedge x = b \wedge y$ for some x and y congruent to I.

§2. UNIQUENESS

Lemma 1. $(a \wedge b)_\theta = a_\theta \wedge b_\theta$

Theorem 1. Let $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ where every q_i be primary and θ be in Θ. If $q_j \not\equiv I\ (\theta)$ for $j = 1, 2, \cdots, r$ and $q_k \equiv I\ (\theta)$ for $k = r+1, \cdots, n$ then $a_\theta = q_1 \wedge q_2 \wedge \cdots \wedge q_r$

Proof. By the assumption, $q_{r+1} \wedge \cdots \wedge q_n \equiv I\ (\theta)$ and hence $a \equiv q_1 \wedge q_2 \wedge \cdots q_r (\theta)$. But $q_{j\theta} = q_j$ for $j = 1, 2, \cdots, r$. Thus $a_\theta = q_{1\theta} \wedge q_{2\theta} \wedge \cdots \wedge q_{r\theta} = q_1 \wedge q_2 \wedge \cdots \wedge q_r$

Theorem 2. In two short representations of an element a in L the radicals of the components coincide.

Proof. Assume, two short representations $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n = q'_1 \wedge q'_2 \wedge \cdots \wedge q'_m$ are given. Let the minimal one among the $\rho(q_i)$ and $\rho(q'_j)$ be, say, $\rho(q_1) = \theta$. Now $a_\theta = q_{1\theta} \wedge q_{2\theta} \wedge \cdots \wedge q_{n\theta} = q'_{1\theta} \wedge q'_{2\theta} \wedge \cdots q'_{m\theta}$. If every $\rho(q'_j)$ were different from θ then $q_2 \wedge \cdots \wedge q_n = q'_1 \wedge q'_2 \wedge \cdots \wedge q'_m = a$ which contradicts our assumption. Hence among the $\rho(q'_j)$ some one, say, $\rho(q'_1)$ is equal to θ. Whence $q_2 \wedge \cdots \wedge q_n = q'_2 \wedge \cdots \wedge q'_m$ which completes the proof by the finite induction.

§3. DECOMPOSABILITY

Theorem 3. A meet of a finite number of primary elements which have the same radicals is also primary and has the same radical.

Proof. Assume, $q_1, q_2, \cdots, q_n$ are primary and $\rho(q_1) = \rho(q_2) = \cdots = \rho(q_n)$ Let θ be in Θ. If $\theta \geq \rho(q_1)$ then $q_i \equiv I\ (\theta)$ for $i = 1, 2, \cdots, n$. Hence $q_1 \wedge q_2 \wedge \cdots \wedge q_n \equiv I(\theta)$. But, if $\theta \ngeq \rho(a_1)$ then $q_i = q_{i\theta}$ for $i = 1, 2, \cdots, n$. Thus $(q_1 \wedge q_2 \wedge \cdots \wedge q_n)_\theta = q_1 \wedge q_2 \wedge \cdots \wedge q_n$. Whence $q_1 \wedge q_2 \wedge q_3 \cdots \wedge q_n$ is primary. Next, $q_1 \wedge q_2 \wedge \cdots \wedge q_n \equiv I(\rho(q_1))$ since $q_i \equiv I(\rho(q_1))$. But, if $q_1 \wedge q_2 \wedge \cdots \wedge q_n \equiv I(\varphi)$ for some φ in Θ then a fortiori $q_1 \equiv I\ (\varphi)$ thus $\rho(q_1) \leq \varphi$. Hence $q_1 \wedge q_2 \wedge \cdots \wedge q_n$ has the radical $\rho(q_1)$.

Theorem 4. An irredundant meet of a finite number of primary elements of which not all have the same radicals is not primary.

Proof. We may assume by th. 3 that all the radicals of primary components are different. Let $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ be irredundant where q_i be primary. If the minimal one among $\rho(q_i)$ is, say, $\rho(q_1) = \theta$ then

$q_{j\theta} = q_j$ for $j = 2, 3, \cdots, n$, while $q_{1\theta} = I$; this shows $(q_1 \wedge q_2 \wedge \cdots \wedge q_n)_\theta = q_2 \wedge \cdots \wedge q_n \neq$

$\neq q_1 \wedge q_2 \wedge \cdots \wedge q_n$

. But, it follows also from $q_j \not\equiv I(\theta)$ that $q_1 \wedge q_2 \wedge \cdots \wedge q_n \not\equiv I(\theta)$. Thus $q_1 \wedge q_2 \wedge \cdots \wedge q_n$ is not primary.

Lemma 2. For a congruence θ on L, $a \equiv b(\theta)$ implies $a \geq b \wedge c$ for some $c \equiv I(\theta)$ if and only if θ is neutral.

Proof. Assume that, θ satisfies the above condition. If $x \equiv y \equiv I(\theta)$ then $(x \wedge a) \vee (y \wedge b) \equiv a \vee b\ (\theta)$ and hence $(x \wedge a) \vee (y \wedge b) \geq (a \vee b) \wedge z$ and $z \equiv I(\theta)$ for some z which prove that $\theta(I)$ is neutral. The remaining part of the lemma is also easily proved.

Theorem 5. Every element of L has a short representation if and only if every congruence in Θ is neutral.

Proof. Assume that, every congruence in Θ is neutral. Let q be not primary. Then $q_\theta \neq q$ and $q \not\equiv I(\theta)$ for some θ in Θ. From $q \equiv q_\theta(\theta)$ we get $q \geq q_\theta \wedge c$ for some $c \equiv I(\theta)$. Hence $q = q_\theta \wedge (q \vee c)$, where $q_\theta \neq q$ and $q \vee c \neq q$ since $q \not\equiv I(\theta)$. Thus q is reducible. Whence every irreducible element is primary which proves a half of the theorem by the generalized induction principle and th. 3. Conversely, if a has a short representation $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ then $a = a_\theta \wedge t$ for some $t \equiv I(\theta)$ by th. I. Thus, if $a \equiv b\ (\theta)$ then $a_\theta = b_\theta \geq b$ and hence $a \geq b \wedge t$ which completes the proof.

§ 4. EXAMPLES

Let R be a noncommutative ring satisfying the ascending chain condition for two-sided ideals. The totality of ideals in R forms an integral modular lattice ordered semigroup L. A lattice congruence of L is said to be (right) regular if it satisfies the condition that $A \equiv B$ implies $AC \equiv BC$ and $A:C \equiv B:C$ for every C in L, where : shows the ideal quotient. A subset k of L is called a k-system if and only if it satisfies the conditions that (a) $k \ni R$, (b) k is J-closed, (c) k is multiplicatively closed.

Lemma 3. In every regular congruence θ of L, $\theta(R)$ forms a k-system and $A \equiv B\ (\theta)$ if and only if $A:B$ and $B:A$ are in $\theta(R)$. Conversely, for every k-system k of L, the relation that $A:B$ and $B:A$ are in k defines a regular congruence θ in L and $k = \theta(R)$.

Proof is omitted but it is easy.

The radical of an ideal A is defined as the intersection of all prime ideals containing A and hence it is the greatest ideal of which n-th power is contained in A for some positive integer n. An ideal Q is (right) primary if and only if $AB \leq Q$ implies $A \leq Q$ or $B \leq$ radical of A. ([4])

Now, let Θ be the set of all regular congruences of L. The k-system generated by one element A of L, that is, the set of all those X such that $A^n \leq X$ for some positive integer n, is denoted by k_A and the corresponding congruence in Θ by θ_A.

Theorem 6. An element Q of L is primary with respect to Θ if and only if the ideal Q is primary.

Proof. Let an element Q be primary with respect to Θ and $AB \leq Q$. Since $AB:A \geq B \in k_B$ and $A:AB = R \in k_B$ we get $AB \equiv A(\theta_B)$. If $Q = Q_{\theta_B}$ then $Q = Q_{\theta_B} \geq (AB)_{\theta_B} = A_{\theta_B} \geq A$ hence $Q \geq A$. But, if $Q \equiv I(\theta_B)$ then $B^n \leq Q$ for some n. Thus Q is primary as ideal. Conversely, let Q be primary as ideal and θ be in Θ. Since $Q \equiv Q_\theta(\theta)$, we have $Q:Q_\theta \in k_\theta$. From $Q \geq Q_\theta(Q:Q_\theta)$ it follows that $Q \geq Q_\theta$ or $Q \geq (Q:Q_\theta)^n$ for some n. If the former is valid then $Q = Q_\theta$ and if the latter holds then $Q \in k_\theta$. Thus Q is primary with respect to Θ.

Theorem 7. Let Q in L be primary. Then there exists a prime ideal P such that $\rho(Q) = \theta_P$. P is the radical ideal of Q.

Proof. The radical ideal P of Q is prime. ([4]) $k_P \ni Q$ since $P^n \leq Q$. But, if $\theta(R) \ni Q$ then a fortiori $\theta(R) \ni P$ and we get $\theta \geq \theta_P$. Thus $\rho(Q) = \theta_P$. If $\theta_P = \theta_{P'}$ where P and P' are prime then evidently $P = P'$.

M.Ward and R.P.Dilworth presented the following condition for the decomposability of ideals into primary ideal components. ([2], [3])

Condition (D). For every pair of ideals A and B there exists n such that $AB \geq A \wedge B^n$.

Theorem 8. Every θ in Θ is neutral if and only if the condition (D) holds.

Proof. We proved at the start in the proof of th. 6 that $AB \equiv A(\theta_B)$. If θ_B is neutral then $AB \geq A_\cap X$, for some $X \in \mathfrak{k}_B$, by the Lemma 2. But, from the definition of $\mathfrak{k}_B$, $X \geq B^n$ for some n. Hence the condition (D) holds. Conversely, assume the condition (D). Let $A \equiv B(\theta)$ for some θ in Θ. Then $A \geq B(A:B) \geq B_\cap(A:B)^n$ where $(A:B)^n \in \mathfrak{k}_\theta$.

As another example, we can apply our results to the representation of elements in a distributive lattice as intersection of irreducible elements.inIa distribn distributive lattice every element is neutral and we can adopt as Θ the set of all the neutral congruences corresponding to principal dual ideals. Then an element is primary if and only if it is irreducible. The radical of an element is the congruence corresponding to the principal dual ideal generated by this element. But this example has fewer meaning to us.

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