ON CLASSIFICATION OF THE FUNCTION-THEORETIC NULL-SETS ON RIEMANN SURFACES OF INFINITE GENUS

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In a recent paper, P.J.Myrberg (1) stated a wonderful fact:

On a certain type of Riemann surface of infinite genus, even if it has the null-boundary, there may exist a compact set E containing a continuum, onto which every single-valued bounded analytic function outside E can be prolonged in the sense of Painlevé.

This is a surprising result. In fact, such a phenomenon can never occur in case of finite genus, that is, every prolongable compact set in Painlevé's sense can never contain a continuum in case of finite genus.

In the Painlevé-problem we must recognize an essential difference between the Riemann surfaces of finite genus and those of infinite genus, and on account of this different character, we are apt to consider that any quantitative treatments do not meet with success to the Painlevé-problem of infinite genus. Though there is an essential difference, the present note will show that the methods, applied by many authors to researches in the Painlevé-problem of finite genus, especially, of zero genus, and the results are also powerful in cases of infinite genus. We shall show that there is a fine relation of parallelism between the Riemann surfaces of finite genus and of infinite genus.

We are now in the position to explain the definitions.

We start from an arbitrary transcendental integral function $g(z)$, having infinitely many simple real zero-points e_ν. An equation

$$y^m = g(z) \quad , \quad m \text{ being integer } \geq 2 \quad ,$$

defines then an m-sheeted Riemann surface $\mathfrak{F}$, having an infinite number of branch-points e_ν of $(m-1)$-th orders. For the sake of simplicity, we assume that the point at infinity is the only limiting point of the branch-points e_ν. Thus, the infinity is an only one ideal boundary point of $\mathfrak{F}$. (This restriction is not essential in our discussion and we are able to extend our basic Riemann surface to that, similar to Lauri Myrberg's one, which may have infinitely many limiting points provided that the linear measure of the set of those limiting points is zero for Painlevé problem.)

Let F be the complex plane, E_1 be any given closed compact set on $\mathfrak{F}$, and further E_0 be the projection of E_1 on F (symbolically denoted by $\mathbf{P}E_1 = E_0$). Let E_ν $(\nu = 2, \dots, m)$ be $m-1$ replicas of E_1, that is, $\mathbf{P}E_\nu = E_0$. We then put

$$\prod_{\nu=1}^{m} E_\nu = E_{1,s}, \sum_{\nu=1}^{m} E_\nu = E_{1,s} + E_{1,c},$$
$$E_{1,c} \cap E_{1,s} = \phi,$$

ϕ denoting the null-set in the sense of point-set theory. We further put

$$S_{E_1} = \mathbf{P}E_{1,s}, \ C_{E_1} = \mathbf{P}E_{1,c}, \ S_{E_1} = \mathbf{P}SE_1$$

and $$C_{E_1} = \mathbf{P}CE_1,$$

then we get

$$E_0 = S_{E_1} + C_{E_1}, \ S_{E_1} \cap C_{E_1} = \phi$$
$$E_1 = SE_1 + CE_1$$

and $$SE_1 \cap CE_1 = \phi$$

Two classes of one-valued analytic functions, having the bounded absolute value or bounded Dirichlet integral on $\mathfrak{F} - E_1$, are denoted by the symbols $\mathfrak{B}(\mathfrak{F} - E_1)$ or $\mathfrak{D}(\mathfrak{F} - E_1)$, respectively.

If any $f \in \mathfrak{B}(\mathfrak{F} - E_1)$ $(\mathfrak{D}(\mathfrak{F} - E_1))$ can be analytically prolonged onto E_1, then we call that E_1 belongs to $N_{\mathfrak{B}}(\mathfrak{F})$ $(N_{\mathfrak{D}}(\mathfrak{F}))$.

If $f_1(z)$ is a single-valued analytic function on $\mathfrak{F} - E_1$, then there is a relation

$$\text{(a)} \quad f_1(z) = \sum_{n=1}^{m} A_n(z)\, y^{n-1},$$

where $A_n(z)$ $(n = 1, \dots, m)$ is the single-valued analytic function of z. Let $f_\nu(z)$ $(\nu = 2, \dots, m)$ be $m-1$ branches of $f_1(z)$, then we can write

$$\text{(b)} \quad f_\nu(z) = \sum_{n=1}^{m} A_n(z) \left(\omega^{\nu-1} y\right)^{n-1},$$
$$\omega = \exp\left(\frac{2\pi i}{m}\right)$$

Without loss of generality we may assume that $f_\nu(z)$ are defined in $\mathfrak{F} - E_\nu$.

If we solve the simultaneous equations (a) and (b), then we have

$$\text{(c)} \quad A_\mu = A_\mu(z) = \frac{1}{m\, y^{\mu-1}} \sum_{n=1}^{m} f_n(z)\, \omega^{m-(\mu-1)n},$$
$$\mu = 1, \dots, m.$$

After these preparations we can state our results as follows:

Theorem 1.

$$E_1 \in N_{\mathfrak{D}}(\mathfrak{F}) \rightleftarrows S_{E_1} \in N_{\mathfrak{D}}(F).$$

Theorem 2.

$$E_1 \in N_{\mathfrak{B}}(\mathfrak{F}) \rightleftarrows S_{E_1} \in N_{\mathfrak{B}}(F).$$

Proof of Theorem 1.

The proof will be performed by slight modifications of Myrberg's. In view of (c) and our assumption that $f_1(z) \in \mathfrak{D}(\mathfrak{F}-E_1)$, we have

$$y^{\mu-1}A_\mu = \frac{1}{m}\sum_{n=1}^{m} f_n(z)\omega^{m-(\mu-1)n} \in \mathfrak{D}(\mathfrak{F}-\sum_{\nu=1}^{m}E_\nu),$$

$$\mu = 1, \cdots, m.$$

Therefore,

$$y^{(\mu-1)m}A_\mu^m = (y^m)^{\mu-1}A_\mu^m, \ \mu \neq 1,$$

belongs to $\mathfrak{D}(F-E_0)$. By means of this fact we shall prove that A_μ is regular and bounded. If A_μ had a pole, which must lie on e_ν, then it would be of at least first order, therefore A_μ^m had a pole of at least m-th order on e_ν. On the other hand $(y^m)^{\mu-1}$ has the zeros e_ν of at most $(\mu-1)$-th order $(\mu-1 \leq m-1 < m)$. Therefore, $A_\mu^m(y^m)^{\mu-1}$ had a pole e_ν of at least first order, which is a contradiction. Thus A_μ must be regular in $F-E_0$, that is, $A_\mu \in \mathfrak{D}(F-E_0)$, $\mu \neq 1$.

E_ν $(\nu = 1, \cdots, m)$ being m closed compact sets on $\mathfrak{F}$, thus $\sum_{\nu=1}^{m} E_\nu$ is also a closed compact set, and hence there is an infinite number of e_ν, which does not belong to the set $\sum_{\nu=1}^{m} E_\nu$.

At the branch-points $e_\nu (\notin \sum_{\nu=1}^{m} E_\nu)$, $f_\mu(e_\nu) \equiv$ const, $\mu = 1, \cdots, m$, we thus get

$$(y^{\mu-1}A_\mu)^m_{z=e_\nu} = \frac{\text{const.}}{m}\sum_{\nu=1}^{m}\omega^{m-(\mu-1)\nu} = 0, \ \mu \neq 1.$$

Therefore, we have

$$(y^{\mu-1}A_\mu)^m \equiv 0, \quad \mu \neq 1,$$

and $A_\mu \equiv 0$, $\mu = 2, \cdots, m, \neq 1.$

Thus, we get

$$f_1 = f_2 = \cdots = f_m = A_1,$$

and hence A_1 can be continued onto the domain $\sum(\mathfrak{F}-E_\nu) = \mathfrak{F} - \prod_{\nu=1} E_\nu = \mathfrak{F} - E_{1,s}$. Thus A_1 belongs to $\mathfrak{D}(F-S_{E_1})$.

Sufficiency. From the definition of $S_{E_1} \in N_{\mathfrak{D}}(F)$, $A_1(z)$ being analytically prolongable onto S_{E_1}, $A_1(z)$ belongs to $\mathfrak{D}(F)$. Thus $A_1(z) \equiv$ const. on F, and so $f_1(z) \equiv$ const. on $\mathfrak{F}$. Thus, $E \in N_{\mathfrak{D}}(\mathfrak{F})$.

Necessity. From the definition of $E_1 \in N_{\mathfrak{D}}(\mathfrak{F})$, $f_1(z)$ can be analytically prolonged onto E_1, and so $A_1(z)$ onto S_{E_1}. Therefore, $S_{E_1} \in N_{\mathfrak{D}}(F)$.

Proof of Theorem 2.

The analogous considerations as in Theorem 1 lead us to the desired results.

As a consequence of Theorem 1 and 2, if $E_1 \in N_{\mathfrak{D}}(\mathfrak{F})$ or $N_{\mathfrak{B}}(\mathfrak{F})$, then we can classify the set E_1 as follows:

$$E_1 = CE_1 + SE_1, \ CE_1 \cap SE_1 = \phi, \ \mathbf{P}SE_1 = C_{E_1}$$

and $\mathbf{P}SE_1 = S_{E_1}$, and moreover

$$S_{E_1} \in N_{\mathfrak{D}}(F) \text{ or } N_{\mathfrak{B}}(F).$$

Evidently S_{E_1} must be a totally disconnected set on F, but C_{E_1} may contain a continuum. Therefore, we can say that CE_1 is a non-essential part of E_1, while SE_1 is an essential part of E_1. Moreover, we shall be able to treat the problem quantitatively, as in the paper of Ahlfors-Beurling (3), in which they investigated thoroughly six types of conformal invariants in the case of complex planar sets. Some of their results remain also valid, if we make use of our terminology S_{E_1}.

Remark. Suppose that E_1 and E_2 be two arbitrary closed compact sets on $\mathfrak{F}$, then the relation $S_{E_1+E_2} = S_{E_1} + S_{E_2}$ does not hold in general. Therefore, it is not true that $E_1 + E_2 \in N_{\mathfrak{D}}(\mathfrak{F})$ or $N_{\mathfrak{B}}(\mathfrak{F})$, even if both E_1 and $E_2 \in N_{\mathfrak{D}}(\mathfrak{F})$ or $N_{\mathfrak{B}}(\mathfrak{F})$. Thus, the union of two closed compact sets, belonging to the family of null-set simultaneously, does not always belong to the same family. It is seen that $\mathbf{P}E_1 \cap \mathbf{P}E_2 \in N_{\mathfrak{D}}(\mathfrak{F})$ (or $N_{\mathfrak{B}}(\mathfrak{F})$) is a necessary and sufficient condition for $E_1 + E_2 \in N_{\mathfrak{D}}(\mathfrak{F})$ (or $N_{\mathfrak{B}}(\mathfrak{F})$), if $E_1, E_2 \in N_{\mathfrak{D}}(\mathfrak{F})$ (or $N_{\mathfrak{B}}(\mathfrak{F})$).

(*) Received May 28, 1951.

(1) P.J.Myrberg: Über die analytische Fortsetzung von beschränkten Funktionen. Ann. Acad. Sci. Fenn. Series A. I. 58 (1949) 7 pp.

(2) Lauri Myrberg: Normalintegrale auf zweiblättrigen Riemannschen Fläche mit reellen Verzweigungspunkten. Ann. Acad. Sci. Fenn. Series A. I. 71 (1950) 51 pp.

(3) L.Ahlfors-A.Beurling: Conformal Invariants and Function-Theoretic Null-Sets. Acta Math. 83 (1950) pp.101-129.

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