ON BOUNDED ANALYTIC FUNCTIONS AND CONFORMAL MAPPING, I.

By Mitsuru OZAWA.

(Communicated by Y. Komatu)

§ 1. An extension of Schwarz's lemma. One of the most important theorems in the theory of functions is the so-called Schwarz's lemma, various extensions of which have been established in various directions by many authors. Garabedian has recently established an important and elegant extension of the lemma for a finitely-connected domain, but his result concerns with a corollary of this lemma, namely, the first coefficient of the expansion of function at a given point. On the other hand, Robinson has attempted to extend the lemma itself, that is, the absolute value of $f(z)$, especially, in the doubly-connected domain. In the present note we shall first establish some extensions which correspond to Robinson's one in an n-ply connected domain.

We shall simply explain some domain-functions.

Let D be a given n-ply connected domain bounded by n simple closed Jordan curves Γ_ν $(\nu=1,\cdots,n)$, and $G(z,\zeta)$ and $\Omega_\nu(z)$ be the analytic functions whose real parts $g(z,\zeta)$ and $\omega_\nu(z)$ are the Green's function of D with a simple logarithmic pole ζ and the harmonic measure of Γ_ν respectively. Moreover, we put

$$G(z,\zeta) = g(z,\zeta) + i\,\tilde{g}(z,\zeta)$$

and

$$\Omega_\nu(z) = \omega_\nu(z) + i\,\tilde{\omega}_\nu(z)$$

respectively. The following two relations are well-known:

$$\omega_\nu(z) = -\frac{1}{2\pi}\int_{\Gamma_\nu}\frac{\partial}{\partial n}g(\zeta;z)\,ds,\quad \sum_{\nu=1}^{n}\omega_\nu(z)=1,$$

$\frac{\partial}{\partial n}$ being the outer normal derivative. Regarding to the periodicity moduli

$$p_{\nu\mu} = \frac{-1}{2\pi}\int_{\Gamma_\nu}\frac{\partial}{\partial n}\omega_\mu(z)\,ds,$$

we know

$$p_{\nu\mu} = p_{\mu\nu},\quad \sum_{\mu=1}^{n} p_{\nu\mu} = 0 .$$

We are now in a position to attack the explained problem.

Theorem 1. Suppose that $f(z)$ is a single-valued analytic function, regular and non-vanishing in D except eventual poles a_μ^∞ $(\mu=1,\cdots,l)$ and zeros a_μ^0 $(\mu=1,\cdots,m)$, and that it satisfies the conditions

$$|f(z)| \leqq e^{C_\nu},\quad \text{for } z\in\Gamma_\nu .$$

Then we have the inequality:

$$|f(z)| \leqq \exp\Big(\sum_{\nu=1}^{n}C_\nu\omega_\nu(z) - \sum_{\mu=1}^{m}g(z,a_\mu^0) + \sum_{\mu=1}^{l}g(z,a_\mu^\infty)\Big).$$

Proof. The function $\lg f(z)$ is not single-valued in D, in general, on account of its poles and zeros. In order to avoid the many valuedness we settle a set of cuts Δ which consists of simple Jordan curves, connecting the points a_μ^0, a_μ^∞ to the boundary points b_μ^0, b_μ^∞, respectively. Then $\lg f(z)$ becomes single-valued in the domain D_Δ obtained from D by cutting along Δ. Now we consider the integral

$$I = \frac{1}{2\pi i}\int_{\Gamma+\Delta}\lg f(z)\,G'(z,\zeta)\,dz$$

By the residue theorem we can immediately calculate the value of I, and get

$$I = -\lg f(\zeta)$$

On the other hand, we have

$$I = \frac{1}{2\pi i}\Big(\int_{\Gamma}+\int_{\Delta}\Big)\lg f(z)\,G'(z,\zeta)\,dz.$$

First, we make use of the fact that $\lg f(z)$ has the saltus $2\pi i$ along both sides of Δ, and hence obtain

$$\frac{1}{2\pi i}\int_{\Delta}\lg f(z)\,G'(z,\zeta)\,dz = \sum_{\mu=1}^{m}G(\zeta;a_\mu^0) - i\sum_{\mu=1}^{m}\tilde{g}(\zeta,b_\nu^0) - \sum_{\mu=1}^{l}G(\zeta,a_\mu^\infty) + i\sum_{\mu=1}^{l}\tilde{g}(\zeta,b_\mu^\infty),$$

remembering that

$$g(\zeta,b_\nu^0) = g(\zeta,b_\mu^\infty) = 0 .$$

Comparing the real parts of both expressions for I, we obtain

$$\lg|f(\zeta)| = \frac{-1}{2\pi}\int_{\Gamma}\lg|f(z)|\frac{\partial}{\partial n}g(z,\zeta)\,ds - \sum_{\mu=1}^{m}g(\zeta,a_\mu^0) + \sum_{\mu=1}^{l}g(\zeta,a_\mu^\infty).$$

But, by the assumption, $\lg|f(z)| \leqq c_\nu$ for $z\in\Gamma_\nu$ and always $-\frac{\partial}{\partial n}g(z,\zeta)\geqq 0$, and hence we obtain the desired result:

$$\begin{aligned}\lg|f(\zeta)| &\leqq \sum_{\nu=1}^{n}c_\nu\frac{1}{2\pi}\int_{\Gamma_\nu}-\frac{\partial}{\partial n}g(z,\zeta)\,ds - \sum_{\mu=1}^{m}g(\zeta;a_\mu^0) + \sum_{\mu=1}^{l}g(\zeta;a_\mu^\infty),\\ &= \sum_{\nu=1}^{n}c_\nu\omega_\nu(\zeta) - \sum_{\mu=1}^{m}g(\zeta;a_\mu^0) + \sum_{\mu=1}^{l}g(\zeta;a_\mu^\infty).\end{aligned}$$

Equality sign holds here if and only if

$$f(z) = e^{i\theta} e^{\sum_{\nu=1}^{n} c_\nu \Omega_\nu(z) - \sum_{\mu=1}^{m} G(z;a_\mu^\circ) + \sum_{\mu=1}^{\ell} G(z,a_\mu^\infty)}.$$

But, in general, the extremal function does not exist, because, if the set of parameters a_μ°, a_μ^∞ and c_ν are arbitrarily chosen, the monodromy conditions, which are necessary and sufficient for $f(z)$ being single-valued, will not be satisfied. Monodromy conditions can be stated as follows:

$$\sum_{\mu=1}^{n} c_\mu p_{\nu\mu} - \sum_{\mu=1}^{m} \omega_\nu(a_\mu^\circ) + \sum_{\mu=1}^{\ell} \omega_\nu(a_\mu^\infty) = q_\nu ,$$

$$(\nu = 1, \cdots, n)$$

where the q_ν denote some integers. Such a Diophantine character makes the problem difficult to solve but plays an important role in conformal mapping.

Theorem 1 may be considered as an extension of Schwarz's lemma and of Hadamard's three circle theorem. Some special cases will be mentioned as illustrating examples.

Example 1. If we take, as the domain D, the unit circle $|z|<1$ and suppose $|f(z)| \leqq 1$ for $|z| = 1$, then the theorem 1 yields the well-known result:

$$|f(z)| \leqq \prod_{\mu=1}^{m} \left|\frac{z-a_\mu^\circ}{1-\bar{a}_\mu^\circ z}\right| \prod_{\mu=1}^{\ell} \left|\frac{1-\bar{a}_\mu^\infty z}{z-a_\mu^\infty}\right|$$

Example 2. In the case D is $q<|z|<1$ and $|f(z)| \leqq 1$ for $|z|=1$ and $|z|=q$, we have

$$|f(z)| \leqq \exp\left(-\sum_{\mu=1}^{m} g(z;a_\mu^\circ) + \sum_{\mu=1}^{\ell} g(z;a_\mu^\infty)\right);$$

explicit expression for $g(z,\zeta)$ is known in this case, namely

$$G(z;a) = -\lg\left(i(q^{-\frac{1}{2}}z)^{\frac{1}{2}-\frac{\lg|\zeta|}{\lg q}} \frac{\vartheta_1'\left(\frac{1}{2\pi i}\lg z/\zeta\right)}{\vartheta_0\left(\frac{1}{2\pi i}\lg(\bar{\zeta}z/q)\right)}\right)$$
$$+ iC.$$

Robinson's theorem can also be proved by our method.

Example 3. If we adopt as D the annular ring $1 \leqq |z| \leqq R$ and if $f(z)$ satisfies the conditions $|f(z)| \leqq 1$ for $|z| = 1$, $|f(z)| \leqq M$ for $|z|=R$ and $\ell = 0$, then we obtain the inequality

$$\lg|f(z)| \leqq \sum_{\nu=1}^{2} c_\nu \omega_\nu(z) - \sum_{\mu=1}^{m} g(z;a_\mu^\circ),$$

$$c_1 = \lg M, \quad c_0 = 0$$

Making use of the Teichmüller's lemma 2, we have

$$\lg|f(z)| \leqq \frac{\lg M}{\lg R} \lg|z| - g(z,-a),$$

$$a = \frac{R^m}{M}, \quad m = \lg M / \lg R.$$

This theorem is an extension of Hadamard's three circle theorem due to Teichmüller.

Example 4. If $f(z)$ has, about the point a_1°, an expansion of the form $\alpha_f(z-a_1^\circ)+\alpha_2(z-a_1^\circ)^2+\cdots$ and is bounded in D, that is, $|f(z)| \leqq 1$ for $z \in D$ and if $\ell = 0$, then there exists the relation

$$|\alpha_f| \leqq \exp\left(-\gamma(a_1^\circ) - \sum_{\nu=2}^{m} g(a_1^\circ, a_\nu^\circ)\right),$$

where $\gamma(a_1^\circ)$ is the Robin's constant. This is an extension of corollary of Schwarz's lemma.

On the subarcs $\Gamma_{\nu i}$ of Γ_ν $\left(\Gamma_\nu = \sum_{i=1}^{\ell_\nu} \Gamma_{\nu i},\ \Gamma_{\nu i} \cap \Gamma_{\nu j} = 0,\ i \neq j\right)$, let $|f(z)| \leqq e^{c_{\nu i}}$ be valid. Then, we can extend our theorem 1 to a wider theorem.

Theorem 2. Suppose that $f(z)$ is single-valued, analytic, regular and non-vanishing in D except eventual poles a_μ^∞ $(\mu=1, \cdots, \ell)$ and zeros a_μ° $(\mu=1,\cdots,m)$ and that it satisfies the conditions

$$|f(z)| \leqq e^{c_{\nu i}} \quad \text{for} \quad z \in \Gamma_{\nu i}.$$

Then we have the inequality:

$$\lg|f(z)| \leqq \sum_{\nu=1}^{n}\sum_{i=1}^{\ell_\nu} c_{\nu i}\, \omega(z,\Gamma_{\nu i},D) - \sum_{\mu=1}^{m} g(z,a_\mu^\circ)$$
$$+ \sum_{\mu=1}^{\ell} g(z;a_\mu^\infty),$$

where $\omega(z,\Gamma_{\nu i}, D)$ denotes the harmonic measure of subarc $\Gamma_{\nu i}$.

Proof. Considering the same integral I as in the proof of theorem 1, and remembering the fact

$$\omega(z;\Gamma_{\nu i}\ D) = \frac{-1}{2\pi}\int_{\Gamma_{\nu i}} \frac{\partial g(\zeta,z)}{\partial n}\, ds,$$

we obtain our theorem 2.

Theorem 2 can be considered as an extension of Doetsch's three line theorem. For, taking the strip $|\Re z| \leqq u$, $-\infty < \Im z < \infty$ as the domain D and $\ell = 0$, $n = 1$, $\ell_1 = 2$, we have

$$\lg|f(z)| \leqq c_{11}\,\omega(z, \Re z = u, D) + c_{12}\,\omega(z, \Re z = -u, D)$$
$$- \sum_{\mu=1}^{\infty} g(z;a_\mu^\circ)$$
$$\leqq c_{11}\,\omega(z;\Re z = u, D) + c_{12}\,\omega(z;\Re z = -u, D)$$

Remembering the relations

$$\omega(z,\Re z = u, D) = \frac{\Re z + u}{2u},$$

$$\omega(z,\Re z = -u, D) = \frac{u - \Re z}{2u}$$

we obtain

$$\lg|f(z)| \leqq c_{11}\frac{\Re z + u}{2u} + c_{12}\frac{u - \Re z}{2u};$$

the desired result.

Moreover, we shall explain another application of theorem 2.

Let D be a domain bounded by two radial half straight lines $\arg z = \pm \frac{\varphi\pi}{2}$, $0 < \varphi < 2$, and a Jordan curve Γ passing between them which connects a point on the line $\arg z = -\frac{1}{2}\varphi\pi$ with a point on the line $\arg z = +\frac{1}{2}\varphi\pi$. Let Γ have the maximum distance R from the origin. Let $f(z)$ be regular and single-valued in D, $|f(z)| \leqq M$ for $\arg z = \pm\frac{1}{2}\varphi\pi$, $|f(z)| \leqq m$ for $z \in \Gamma$, and $M, m \geq 1$. Then we have the inequality

$$|f(z)| \leqq M^{\left(1-\frac{4}{\pi}\arctan\left(\frac{r}{R}\right)^{\frac{1}{\varphi}}\right)} m^{\frac{4}{\pi}\arctan\left(\frac{r}{R}\right)^{\frac{1}{\varphi}}}$$

for $\arg z = 0$, $|z| = r$.

For simplicity, we may take $R = 1$. From the theorem 2, we can easily obtain the inequality:

$$\lg|f(z)| \leq \omega(z, \arg z = \pm\tfrac{\varphi}{2}\pi\ D)\lg M + \omega(z, \Gamma\ D)\lg m.$$

On the other hand, by the domain-extension principle (Prinzip der Gebietserweiterung) for the harmonic measure ω, we can replace D by the domain D_1 bounded by the lines: $\arg z = \pm\frac{1}{2}\varphi\pi$ and $|z| = 1$. For D_1, we can easily calculate the harmonic measure explicitly and we obtain

$$\omega(z, \arg z = \pm\tfrac{\varphi\pi}{2}, D_1) = 1 - \frac{4}{\pi}\arctan(r)^{\frac{1}{\varphi}}$$

and

$$\omega(z; |z| = 1\ D_1) = \frac{4}{\pi}\arctan(r)^{\frac{1}{\varphi}},$$

for $\arg z = 0, |z| = r$. Thus we obtain the desired evaluation.

Analogous result can be established for other straight lines. Carleman had established an analogous evaluation. Moreover, we can discuss an extension of the well-known Phragmen-Lindelöf's theorem in a sector domain.

§ 2. Painlevé problem.

The problem can be stated as follows:

Let E be a compact set in the complex plane. Under what condition does there exist a non-constant function which is single-valued, analytic and bounded outside of E? The corresponding problem for the existence of bounded harmonic functions has already been solved by making use of the notion of capacity and may be stated in the following manner.

A necessary and sufficient condition for the existence of a function, non-constant, bounded and harmonic outside of E is that E be of positive logarithmic capacity. Of course, this condition is also necessary for Painlevé problem. It seems, however, very important to separate a condition for Painlevé problem from that of harmonic function, and an effort may be attempted in the following manner.

Theorem 3. A necessary and sufficient condition for the existence of a function non-constant, bounded and analytic in the exterior D of a compact set E is that

$$(a) \qquad \lim_{m\to\infty} \exp(-\gamma_m(a_1^\circ)) > 0,$$

and

$$(b) \qquad \lim_{m\to\infty} \exp\left(-\sum_{\nu=2}^{n} g(a_1^\circ, a_{\nu m}^\circ)\right) > 0,$$

where Γ_m is the level curve $g(z; a_1^\circ) = \frac{1}{m}$, and the system $a_{\nu m}^\circ$ $(\nu = 1, \cdots, n)$ satisfies the monodromy conditions

$$\sum_{\mu=1}^{n} \omega_\nu^{(m)}(a_{\mu m}^\circ) = 1, \quad a_{1m}^\circ = a_1^\circ,$$

here $\sum_{\nu=1}^{m}\Gamma_{m\nu} = \Gamma_m$ bounds a subdomain D_m of D, and $\omega_\nu^{(m)}$ are the harmonic measure of $\Gamma_{m\nu}$ with respect to D_m.

Proof. Ahlfors has established a condition necessary and sufficient for the existence of non-constant, bounded and analytic functions, which states

$$\lim_{m\to\infty}\max_f|\alpha_{fm}| > 0, \quad \lim_{z\to a_1^\circ}\frac{f(z)}{z-a_1^\circ} = \alpha_{fm}$$

$\max_f |\alpha_{fm}|$ is attained by the function

$$f(z) = \exp\left(-\sum_{\mu=1}^{m} G_m(z; a_\mu^\circ)\right),$$

satisfying the monodromy conditions

$$\sum_{\mu=1}^{m}\omega_\nu^{(m)}(a_{\mu m}^\circ) = 1.$$

Denoting by $g_m(z; z_0)$ the Green's function of D_m, and making use of the relations

$$g_m(z, z_0) = g(z, z_0) - \frac{1}{m},$$

$$\lim_{z\to z_0}\left(g(z, z_0) - \lg\frac{1}{|z-z_0|}\right) = \gamma(z_0),$$

we obtain

$$|\alpha_m| \leqq \exp(-\gamma_m(a_1^\circ))\exp\left(-\sum_{\nu=2}^{n} g(a_1^\circ, a_{\nu m}^\circ) + \frac{n-1}{m}\right);$$

equality sign holds here if and only if $f(z)$ is the function stated above. So we obtain the desired result by letting $m \to \infty$.

This theorem is only a restatement of Ahlfors' one, with a slight precision, for the separation of the condition to (a) and (b) is made. Of course, (a) is a necessary and sufficient condition for the corresponding problem for harmonic functions.

§ 3. Analogous problem and applications.

A theorem of Blaschke and Ostrowski which refers to $|z| < 1$, can be extended to the n-ply connected case.

Theorem 4. Let Γ_m be a level curve $g(z, \zeta) = \frac{1}{m}$, and let $f(z)$

have an infinite number of zero points a_μ^0, and be regular and single-valued in D, and let Γ satisfy the conditions

$$\lim_{m\to\infty} M_{2m} \neq 0 \text{ and } \lim_{m\to\infty} M_{1m} \neq \infty,$$

M_{2m} and M_{1m} being defined by the inequalities:

$$(0 \neq)\, M_{2m} < -\frac{\partial}{\partial n} g(\zeta, z_0) \leqq M_{1m} (\neq \infty)$$

for $\zeta \in \Gamma_m$.

Then a necessary and sufficient condition for the convergence of the positive term series

$$\sum_{\mu=1}^{\infty} g(a_\mu^0, \zeta)$$

is the boundedness of integral

$$\int_{\Gamma_m} \lg|f(z)|\, ds \quad \text{for all } m\ (0 \leqq m < \infty).$$

Proof. In the first place, we assume that $f(z_0) \neq 0$. In the proof of theorem 1, we have obtained

$$\lg|f(z_0)| + \sum_{\mu=1}^{T} g_m(z_0, a_\mu^0) = \frac{1}{2\pi}\int_{\Gamma_m} \lg|f(\zeta)| \times \left(-\frac{\partial}{\partial n} g_m(\zeta; z_0)\right) ds$$
$$= \frac{1}{2\pi}\int_{\Gamma_m} \lg|f(\zeta)| \left(-\frac{\partial}{\partial n} g(\zeta; z_0)\right) ds,$$

which yields

$$\frac{M_{2m}}{2\pi}\int_{\Gamma_m} \lg|f|\, ds \leqq \lg|f(z_0)| + \sum_{\mu=1}^{T} g_m(z_0, a_\mu^0) \leqq \frac{M_{1m}}{2\pi}\int_{\Gamma_m} \lg|f|\, ds.$$

On the other hand, there exists $\lim_{m\to\infty} g_m(z_0; a_\mu^0) = g(z_0; a_\mu^0)$ which gives the desired result. Next, we consider the case $f(z_0) = 0$. Then, we have only to consider the sum

$$\sum_{\mu=q+1}^{\infty} g(a_\mu^0, z_0)$$

where q denotes the multiplicity of $z_0 = a_1^0$.

Theorem 5. Let $f(z)$ be a single-valued, analytic function whose real part satisfies the inequalities $\mathcal{R}e\, f(z) \leqq c_\nu$ for $z \in \Gamma_\nu$, and has a finite number of poles a_μ^∞ $(\mu = 1, \cdots, l)$. Then we have

$$\mathcal{R}e\, f(z) \leqq \sum_{\mu=1}^{n} c_\mu \omega_\mu(z) + \sum_{\mu=1}^{l} \mathcal{R}e\left(b_\mu G'(a_\mu^\infty; z_0)\right),$$

b_μ being defined by the equalities

$$b_\mu = \lim_{z\to a_\mu^\infty} (z - a_\mu^\infty)^{\lambda_\mu} f(z),$$

where λ_μ denotes the multiplicity of the zero a_μ^∞.

Proof. We consider the integral

$$I = \frac{1}{2\pi i}\int_\Gamma f(z) G'(z, \zeta)\, dz.$$

Then, we can make use of similar discussion as in Theorem 1.

Theorem 6. A necessary and sufficient condition for $\sum_{\mu=1}^{\infty} \mathcal{R}e\,(b_\mu G'(a_\mu^\infty; z)) < \infty$ is $\int_{\Gamma_m} \mathcal{R}e\, f(z)\, ds < \infty$, where we adopt the following assumptions:

(i) $\lim_{m\to\infty} M_{2m} \neq 0$, $\lim_{m\to\infty} M_{1m} \neq 0$,

(ii) $f(z)$ has an infinite number of poles a_μ^∞ and $b_\mu = \lim_{z\to a_\mu^\infty} (z - a_\mu^\infty)^{\lambda_\mu} f(z)$.

(iii) $\mathcal{R}e\, f(z) \leqq c_\nu$ for $z \in \Gamma_\nu$.

(*) Received May 15, 1950.

(1) L.Ahlfors: Bounded Analytic Functions. Duke Math. Journ. 14(1947), pp. 1-11.
(2) P.R.Garabedian: Schwarz's lemma and the Szegö kernel function. Trans.Amer.Math.Soc. 67(1949), pp.1-35.
(3) H.Grunsky: Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten. I, II. J.D.M.-V. 50(1940), 52(1942).
(4) A.Ostrowski: Über Nullstellen gewisser im Einheitskreis regulärer Funktionen und einige Sätze über Konvergenz unendlicher Reihen. J.D.M.-V. 36(1926), pp. 161-171.
(5) R.M.Robinson: Analytic functions in circular rings. Duke Math. Journ. 10(1943), pp. 209-216.
(6) O.Teichmüller: Einige Verschärfung des Dreikreisesatzes. Deutsch. Math. 4(1939), pp. 16-22.

Tokyo Institute of Technology.