# ON AN ASYMPTOTIC PROPERTY OF A GAP SEQUENCE

By Gisiro MARUYAMA

(Communicated by T.Kawata)

Let $f(t)$ be a real measurable function of a real variable $t$, and $\{\lambda_k\}$ be an increasing positive sequence with certain gap conditions. Asymptotic properties of the sequence of functions $f(\lambda_k t)$, such as the asymptotic distribution of their partial sums and almost everywhere convergence or divergence of the series $\sum c_k f(\lambda_k t)$, $c_k$ constant coefficients, have been discussed by M.Kac,[1] R.Fortet,[2] R.Salem and A.Zygmund,[3] R.Fortet and J.Ferrand,[4] and T.Kawata.[5] The object of this note is to prove the following theorem which corresponds to the law of the iterated logarithm in the theory of probability. The proof given here depends on Kac's method of approximating a gap sequence by a set of independent functions.

Theorem. Let $f(t)$, $0 \le t \le 1$, be a periodic continuous function which has the vanishing mean $\int_0^1 f(t)\,dt = 0$ and satisfies a Lipschitz condition of order $\alpha$, $\alpha > 0$. Then we have

$$\overline{\lim_{n\to\infty}} \frac{f(2t)+f(2^2 t)+\cdots+f(2^n t)}{\sqrt{2n\log\log n}} = \sigma$$

for almost all $t$, where

$$\sigma^2 = \lim_{n\to\infty} n^{-1}\int_0^1 \{f(2t)+\cdots+f(2^n t)\}^2 dt. \quad {}^{6)}$$

Proof. Let us put $\lambda(x) = [c \log x]$, where $c = 2/\alpha\log 2$ and $[a]$ denotes the integral part of $a$, and define non-negative sequences of integers $\{d_\nu\}, \{p_\nu\}, \{m_\nu\}, \{n_\nu\}$ in the following way. Choose an integer $N$ so large, that we have

$$c\log\{2\nu(\log\nu)^2\} < [(\log\nu)^2] \qquad \text{for } \nu \ge N$$

and let $1 \le d_1 \le \cdots \le d_{N-1} \le [(\log N)^2]$ but $d_\nu$ $(1 \le \nu \le N-1)$ otherwise arbitrary; $d_\nu = [(\log\nu)^2]$ for $\nu \ge N$; $p_\nu = d_\nu$ for $1 \le \nu \le N-1$;

$$p_\nu = \lambda(d_1+\cdots+d_\nu+p_1+\cdots+p_{\nu-1})$$

for $\nu \ge N$, and put

$$(1) \qquad \begin{aligned} m_\nu &= d_1+\cdots+d_\nu+p_1+\cdots+p_{\nu-1} \\ n_\nu &= m_\nu + \lambda(m_\nu) \end{aligned}$$

Then by the above choice of $N$ and definition of $p_\nu$

$$(2) \qquad p_N \le \lambda(2Nd_N) \le c\log\{2N(\log N)^2\} \le d_N$$

$$(3) \qquad p_\nu > \lambda(d_1+\cdots+d_\nu) \qquad \text{for } \nu \ge 1,$$

and hence, starting from (2), we can inductively show that we have

$$(4) \qquad p_\nu \le d_\nu \qquad \text{for all } \nu \ge 1.$$

Now, for any positive sequences $\{a_\nu\}$ and $\{b_\nu\}$ let us agree to write $a_\nu \sim b_\nu$, when $a_\nu/b_\nu \to 1$, as $\nu \to \infty$. Then, taking into account of (3) and (4) we obtain

$$(5) \qquad p_\nu \sim c\log\nu, \quad d_\nu \sim (\log\nu)^2, \quad m_\nu \sim n_\nu \sim \nu(\log\nu)^2.$$

Consider the dyadic expansions of all real numbers $t$, $0 \le t \le 1$,

$$t = \frac{\varphi_1(t)}{2} + \frac{\varphi_2(t)}{2^2} + \cdots, \quad \varphi_i(t) = \frac{1+r_i(t)}{2},$$

where by $r_i(t)$ we denote Rademacher's system of independent functions, and put

$$\theta_m(t) = \frac{\varphi_m(t)}{2} + \cdots + \frac{\varphi_{m+\lambda(m)}(t)}{2^{1+\lambda(m)}},$$

$$\mu_\nu = \int_0^1 f(\theta_\nu(t))\,dt, \quad x_\nu = x_\nu(t) = f(\theta_\nu(t)) - \mu_\nu,$$

$$(6) \qquad S_\nu = \sum_{p=n_{\nu-1}+1}^{m_\nu} x_p, \quad T_\nu = \sum_{p=m_\nu+1}^{n_\nu} x_p,$$

$$\sigma_\nu^2 = \int_0^1 S_\nu^2\,dt, \quad \tau_\nu^2 = \int_0^1 T_\nu^2\,dt$$

Then by the periodicity and Lipschitz condition imposed on $f(t)$ we obtain

$$(7) \qquad f(2^\nu t) - f(\theta_{\nu+1}(t)) = O(2^{-\alpha\lambda(\nu)}) = O(\nu^{-2}),$$

and therefore

$$(8) \qquad \mu_\nu = \int_0^1 f(2^{\nu-1}t)\,dt + \int_0^1 \{f(\theta_\nu(t)) - f(2^{\nu-1}t)\}\,dt = O(\nu^{-2}).$$

(6), (7) and (8) give us

$$(9) \qquad \sigma_\nu^2 = \int_0^1 \{f(t)+\cdots+f(2^{d_\nu}t)\}^2 dt + o(1) = d_\nu\sigma^2(1+o(1)),$$

and similarly

$$(10) \qquad \tau_\nu^2 = p_\nu\sigma^2(1+o(1)).$$

Since, as is seen from (6), $S_\nu$ depends only on $r_i(t)$, $n_{\nu-1}+1 \le i \le n_\nu$, $\{S_\nu\}$ is a set of independent functions, each with common mean value $0$, and the same is true of $\{T_\nu\}$. Hence, if we write

$$(11) \qquad \sum_{p=1}^{n_k} x_p = \sum_{\nu=1}^{k} S_\nu + \sum_{\nu=1}^{k} T_\nu,$$

then by (5), (9) and (10) the "variances" $B_k$ and $C_k$ of the two terms in the right-hand member of (11) are given by

$$B_k=\sum_1^k \sigma_\nu^2 \sim \sigma^2 k(\log k)^2,$$

$$C_k=\sum_1^k \tau_\nu^2 \sim \sigma^2 ck \log k,$$

and they satisfy

$$(12) \quad \frac{B_k}{n_k}\to\sigma^2, \quad \frac{C_k}{n_k}\to 0, \quad k\to\infty,$$

$$(13) \quad \frac{B_k}{\log\log B_k}\sim\frac{\sigma^2 k(\log k)^2}{\log\log k}, \quad \frac{C_k}{\log\log C_k}\sim\frac{\sigma^2 ck\log k}{\log\log k}$$

By (13)

$$\max_{0\le t\le 1}|S_k(t)|=O(d_k)=o\left(B_k/\log\log B_k\right)^{\frac{1}{2}},$$

and also

$$\max_{0\le t\le 1}|T_k(t)|=O(p_k)=o\left(C_k/\log\log C_k\right)^{\frac{1}{2}}.$$

Therefore we can apply Kolmogoroff's theorem on the law of the iterated logarithm, getting almost everywhere in $t$

$$(14) \quad \overline{\lim_{k\to\infty}}\frac{\sum_1^k S_\nu}{\sqrt{2n_k\log\log n_k}}=\overline{\lim_{k\to\infty}}\frac{\sum_1^k S_\nu}{\sqrt{2B_k\log\log B_k}}\times\left(\frac{B_k\log\log B_k}{n_k\log\log n_k}\right)^{\frac{1}{2}}=\sigma,$$

and similarly

$$(15) \quad \overline{\lim_{k\to\infty}}\frac{\sum_1^k T_\nu}{\sqrt{2n_k\log\log n_k}}=0.$$

Given $n$, let us choose $n_k$ such that $n_{k-1}<n\le n_k$, then

$$(16) \quad \sum_1^n f(2^\nu t)=\sum_1^{n_{k-1}} f(2^\nu t)+O(p_k+d_k),$$

and

$$n\log\log n\sim n_k\log\log n_k, \quad n\to\infty.$$

Hence, by (11), (14), (15) and (16) we get

$$\overline{\lim_{n\to\infty}}\frac{\sum_1^n f(2^\nu t)}{\sqrt{2n\log\log n}}=\overline{\lim_{k\to\infty}}\frac{\sum_1^{n_k} x_\nu}{\sqrt{2n_k\log\log n_k}}=\sigma$$

for almost every $t$. This proves the theroem.

(*) Received May 11, 1950.

(1) M.Kac: On the distribution of values of sums of the type $\sum f(2^k t)$, Ann.Math., 47 (1946).
(2) R.Fortet: Sur une suite egalement repartie, Studia Math., 9(1940).

(3) R.Salem and Zygmund: On lacunary trigonometrical series, I and II, Proc.Nat.Acad.Science, 33(1947), 34(1948).
(4) R.Fortet and J.Ferrand: Sur des suites arithmetiques equireparties, C.R.Acad.Sc.,Paris, 224 (1947).
(5) As yet not published.
(6) See (1).

Kyusyu University.