ON LACUNARY NON-HARMONIC TRIGONOMETRIC SERIES

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I. The following theorem of Zygmund on lacunary trigonometric series is well known.[1)]

A. If $n_{k+1}/n_k > \lambda > 1$, n_k's being integers, and the series $\sum(a_k^2+b_k^2)$ converges, then

$$(1) \qquad \sum_{k=1}^{\infty}(a_k \cos n_k x + b_k \sin n_k x)$$

is the Fourier series of a function $f(x)$ belonging to the class L^r, r being any positive number and

$$\left\{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^r dx\right\}^{1/r} \leq A_{r\lambda}\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2}$$

where $A_{r\lambda}$ depends only on r and λ.

B. Under the conditions of A, we have

$$B_{r\lambda}\left\{\sum_{k=1}^{\infty}(a_k^2+b_k^2)\right\}^{1/2} \leq \left\{\frac{1}{\pi}\int_0^{2\pi}|f(x)|^r dx\right\}^{1/r},$$

where $B_{r\lambda}$ is positive and depends only on r, and λ.

We shall prove that a theorem of same type is valid, even if the integral character of the numbers is not assumed.

Theorem 1. If $\lambda_{k+1}/\lambda_k \geq \lambda > 1$, $\lambda_1 \geq 1$ (this is not on essential restriction, so for our convention we suppose this in the following all theorems), and if the series $\sum_{k=1}^{\infty}|C_k|^2$ converges, then

$$(2) \qquad \sum_{k=1}^{\infty} C_k e^{i\lambda_k x}$$

is almost everywhere convergent to a function $f(x)$ belonging to every $L^{r,\sigma}(-\infty,\infty)$, $L^{r,\sigma}$ means the class of functions whose r th power in absolute values are integrable with respect to a monotone function $\sigma(x)$,

$$\sigma(x) = \frac{1}{\pi}\int_{-\infty}^{x}\frac{1-\cos t}{t^2}dt \text{ , and}$$

$$(3) \qquad \left\{\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x)\right\}^{1/r} \leq A'_{r\lambda}\left\{\sum_{k=1}^{\infty}|C_k|^2\right\}^{1/2}$$

holds, where $A'_{r\lambda}$ depends only on r and λ[4)].

Theorem 2. Under the conditions of Theorem 1,

$$(4) \qquad B'_{r\lambda}\left\{\sum_{k=1}^{\infty}|C_k|^2\right\}^{1/2} \leq \left\{\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x)\right\}^{1/r}, \quad r \geq 1$$

where $B'_{r\lambda}$ is a positive constant which depends only on r and λ.

Theorem 3. Under the conditions of Theorem 1, the series (2) converges in the mean with exponent r over every finite interval to $f(x)$.

We shall prove more strong results than Theorem 1, that is;

Theorem 4. If the conditions of Theorem 1 are satisfied, and

$$S^*(x) = \sup_n |S_n(x)|,$$

where

$$S_n(x) = \sum_{k=1}^{n} C_k e^{i\lambda_k x}$$

then

$$(5) \qquad \int_{-\infty}^{\infty}|S^*(x)|^r d\sigma(x) \leq D_{r,\lambda}\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x) \quad r > 1$$

where $D_{r,\lambda}$ depends only on r and λ.

Theorem 5. Under the conditions of Theorem 4, we have

$$\int_a^b |S^*(x)|^r dx \leq D\int_a^b |f(x)|^r dx, \quad r > 1$$

where D depends on r, λ, a and b.

Theorem 6. There exists a constant $\mu (>0)$ which depends only $\sum|C_k|^2$ and λ such that the function $e^{\mu|f(x)|}$ belonging to the class $L^{1,\sigma}(-\infty,\infty)$.

Remark. All the above theorems still hold even if we take $\sigma_h(x)$ $(0<h\leq 1)$ for $\sigma(x)$, where

$$\sigma_h(x) = \frac{1}{\pi}\int_{-\infty}^{x}\frac{\sin^2 ht}{ht^2}dt$$

2. Proof of Theorem 1. The almost everywhere convergence are already proved by M.Kac,[2)] and so we shall prove the inequality (3).

By Holder's inequality, if $r < r'$, we have

$$\left\{\int_{-\infty}^{\infty}|f(x)|^r d\sigma(x)\right\}^{1/r} \leq \left\{\int_{-\infty}^{\infty}|f(x)|^{r'} d\sigma(x)\right\}^{1/r'},$$

and hence it is sufficient to consider $r = 2m$, $m = 1, 2, \cdots$.

Moreover it is well known that if we prove the inequality

$$(3') \quad \left\{ \int_{-\infty}^{\infty} \left| \sum_{k=1}^{n} c_k e^{i\lambda_k x} \right|^{2m} d\sigma(x) \right\}^{1/2m} \leq A'_{2m,\lambda} \left\{ \sum_{k=1}^{n} |c_k|^2 \right\}^{1/2},$$

then the inequality (3) will be valid. Hence we shall prove the inequality (3'). Suppose first $\lambda_{k+1}/\lambda_k \geq \lambda > 4m$.

Then

$$\left(\sum_k c_k e^{i\lambda_k x} \right)^m = \sum d_\nu e^{i\mu_\nu x}$$

where μ_ν is the number of the form

$$\alpha_1 \lambda_{k_1} + \alpha_2 \lambda_{k_2} + \cdots, \text{ with}$$
$$\lambda_{k_1} > \lambda_{k_2} > \cdots, \quad \alpha_i \geq 0$$
$$\alpha_1 + \alpha_2 + \cdots = m.$$

And under our assumptions every can be represented uniquely in the form (6). For otherwise we should have an equation

$$\beta_1 k_1 + \beta_2 k_2 + \quad \cdots = 0$$

where

$$\beta_1 > 0, \quad 0 \leq |\beta_i| \leq m \ (i = 2, 3 \quad)$$
$$\lambda_{k_1} > \lambda_{k_2} > \qquad , \text{ and so}$$
$$\lambda_{k_1} \leq m(\lambda_{k_2} + \lambda_{k_3} + \quad)$$

Thus

$$1 \leq m(\lambda^{-1} + \lambda^{-2} + \cdots) \leq \frac{m}{\lambda - 1}.$$

This is impossible if $\lambda > m + 1$. Moreover

$$|\mu_\nu - \mu_{\nu'}| \geq 1 \quad \text{for } \nu \neq \nu'.$$

Because if

$$|\mu_\nu - \mu_{\nu'}| < 1,$$

then we should have an inequality

$$(\beta_1 \lambda_{k_1} + \beta_2 \lambda_{k_2} + \cdots \quad) < 1$$

where

$$\beta_1 > 0, \ 0 \leq |\beta_i| \leq m, \ \lambda_{k_1} > \lambda_{k_2} > \cdots,$$

and so

$$\lambda_{k_1} < 1 + m(\lambda_{k_2} + \lambda_{k_3} + \cdot \quad).$$

Thus

$$1 < \frac{1}{\lambda_{k_1}} + \frac{m}{\lambda - 1}$$

This is impossible if $\lambda \geq 3m$. Thus under our assumption $\lambda_{k+1}/\lambda_k \geq \lambda \geq 3m$, $\{e^{i\mu_\nu x}\}$ are an orthonormal system in $(-\infty, \infty)$ with respect to $\sigma(x)$. Thus by Parseval's theorem,

$$\int_{-\infty}^{\infty} \left| \sum c_k e^{i\lambda_k x} \right|^{2m} d\sigma(x) = \sum |d_\nu|^2$$

where d_ν is of the form

$$d_\nu = \frac{m!}{\alpha_1! \alpha_2! \cdots} c_{k_1}^{\alpha_1} c_{k_2}^{\alpha_2} \cdots$$
$$|d_\nu|^2 = \left(\frac{m!}{\alpha_1! \alpha_2! \cdots} \right)^2 |c_{k_1}|^{2\alpha_1} |c_{k_2}|^{2\alpha_2} \cdots$$
$$\leq m! \frac{m!}{\alpha_1! \alpha_2! \cdots} |c_{k_1}|^{2\alpha_1} |c_{k_2}|^{2\alpha_2} \cdots$$

Hence

$$\sum |d_\nu|^2 \leq m! \left(\sum |c_k|^2 \right)^m$$

Thus we have

$$\int_{-\infty}^{\infty} \left| \sum c_k e^{i\lambda_k x} \right|^{2m} d\sigma(x) \leq m! \left(\sum |c_k|^2 \right)^m$$

that is

$$\left\{ \int_{-\infty}^{\infty} \left| \sum c_k e^{i\lambda_k x} \right|^{2m} d\sigma(x) \right\}^{1/2m} \leq (m!)^{1/2m} \left(\sum |c_k|^2 \right)^{1/2}$$

To prove (3) for general $\lambda > 1$, we break up (2) into a finite number s of series, which the gap λ greater than $3m$.

Then for each series, we have

$$(7) \quad \left\{ \int_{-\infty}^{\infty} \left| \sum_j c_{sj+l} e^{i\lambda_{sj+l} x} \right|^{2m} d\sigma(x) \right\}^{1/2m} \leq (m!)^{1/2m} \left(\sum_j |c_{sj+l}|^2 \right)^{1/2}$$
$$(l = 1, 2, \cdots, s)$$

Thus by Minkowski's inequality

$$(8) \quad \left\{ \int_{-\infty}^{\infty} \left| \sum_k c_k e^{i\lambda_k x} \right|^{2m} d\sigma(x) \right\}^{1/2m}$$
$$\leq \sum_{l=1}^{s} \left\{ \int_{-\infty}^{\infty} \left| \sum_j c_{sj+l} e^{i\lambda_{sj+l} x} \right|^{2m} d\sigma(x) \right\}^{1/2m}$$
$$\leq s(m!)^{1/2m} \left(\sum_k |c_k|^2 \right)^{1/2}$$

Thus this completes the proof.

<u>Proof of Theorem 2.</u> Let

$$f = \sum_{k=1}^{\infty} c_k e^{i\lambda_k x}$$

Since

$$|f|^2 = |f|^{2/3} |f|^{4/3}$$

we have, by Holder's inequality

$$\left\{ \int_{-\infty}^{\infty} |f|^2 d\sigma(x) \right\}^{1/2} \leq \left\{ \int_{-\infty}^{\infty} |f| \, d\sigma(x) \right\}^{1/3} \left\{ \int_{-\infty}^{\infty} |f|^4 d\sigma(x) \right\}^{\frac{1}{4} \cdot \frac{2}{3}}$$

Thus

$$\int_{-\infty}^{\infty} |f| \, d\sigma(x) \geq \left\{ \int_{-\infty}^{\infty} |f|^2 d\sigma(x) \right\}^{3/2} \Big/ \left\{ \int_{-\infty}^{\infty} |f|^4 d\sigma(x) \right\}^{1/2}$$

$$\geq \left\{ \sum_{k=1}^{\infty} |c_k|^2 \right\}^{3/2} \Big/ A'_\lambda \left\{ \sum |c_k|^2 \right\}$$
$$\geq B_\lambda \left\{ \sum_{k=1}^{\infty} |c_k|^2 \right\}^{1/2}$$

That is

$$\int_{-\infty}^{\infty} |f| \, d\sigma(x) \geqq B_\lambda \left\{ \sum_{k=1}^{\infty} |C_k|^2 \right\}^{1/2}$$

Thus by Holder's inequality

$$\left\{ \int_{-\infty}^{\infty} |f|^r d\sigma(x) \right\}^{1/r} \geqq B_\lambda \left\{ \sum_{k=1}^{\infty} |C_k|^2 \right\}^{1/2}$$

Proof of Theorem 3. By Theorem 1

$$\int_{-\infty}^{\infty} \left| \sum_{k=m} C_k e^{i\lambda_k x} \right|^r \frac{\sin x}{\pi x^2} dx \leqq C_{r\lambda} \left\{ \sum_m^n |C_k|^2 \right\}^{r/2}$$

Hence

$$\int_{-\pi/2}^{\pi/2} \left| \sum_m^n C_k e^{i\lambda_k x} \right|^r dx \leqq \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left| \sum_m^n C_k e^{i\lambda_k x} \right|^r \frac{\sin^2 x}{\pi x^2} dx$$

$$(9) \qquad \leqq C_{r,\lambda} \left\{ \sum_{k=m}^n |C_k|^2 \right\}^{r/2}$$

we have, more generally

$$(10) \quad \int_{-\pi/2+y}^{\pi/2+y} \left| \sum_m^n C_k e^{i\lambda_k x} \right|^r dx \leqq C_{r,\lambda} \left\{ \sum_{k=m}^n |C_k|^2 \right\}^{r/2}$$

For

$$\int_{-\pi/2+y}^{\pi/2+y} \left| \sum_{k=m}^n C_k e^{i\lambda_k x} \right|^r dx = \int_{-\pi/2}^{\pi/2} \left| \sum_{k=m}^n C_k e^{i\lambda_k y} e^{i\lambda_k x} \right|^r dx$$

$$\leqq C_{r,\lambda} \left\{ \sum_{k=m}^n |C_k|^2 \right\}^{r/2}$$

Thus by (9), we get

$$\int_a^b \left| \sum_m^n C_k e^{i\lambda_k x} \right|^r dx \leqq \frac{b-a}{\pi} C_{r,\lambda} \left\{ \sum_m^n |C_k|^2 \right\}^{r/2}$$

Thus, if $\sum_{k=1}^{\infty} |C_k|^2 < \infty$, then the series (2) converges to $f(x)$ in the mean with exponent r over every finite interval.

We shall state a lemma for the proof of Theorem 4.

Lemma. There exist constant C_1 which is independent of k and h such that

$$\left| \int_{x_0}^{x_0+h} e^{i\lambda_k x} d\sigma(x) \right| \leqq \frac{1}{\lambda_k} C_1 \left\{ \frac{\sin^2 x_0}{x_0^2} + \frac{\sin^2(x_0+h)}{(x_0+h)^2} \right\}$$

This is proved easily by the second mean value theorem.

We shall prove Theorem 4. From

$$\int_{-\infty}^{\infty} \left| f(x) - \sum_{k=1}^{m} C_k e^{i\lambda_k x} \right|^2 d\sigma(x) \longrightarrow 0 \ (\text{as } n \to \infty)$$

we have

$$\int_{x_0}^{x_0+h} f(x) \, d\sigma(x) = \sum_{k=1}^{\infty} C_k \int_{x_0}^{x_0+h} e^{i\lambda_k x} d\sigma(x)$$

Using this fact and lemma, we have

$$\left| \int_{x_0}^{x_0+h} f(x) d\sigma(x) - \sum_1^r C_k \int_{x_0}^{x_0+h} e^{i\lambda_k x} d\sigma(x) \right|$$

$$\leqq \left[\sum_{k=r+1}^{\infty} |C_k|^2 \right]^{1/2} \left[\sum_{r+1}^{\infty} \frac{1}{\lambda_k^2} \right]^{1/2} C_1 \left\{ \frac{\sin^2 x_0}{x_0^2} + \frac{\sin^2(x_0+h)}{(x_0+h)^2} \right\}$$

$$(11) \quad \left| \sum_{k=1}^r C_k \int_{x_0}^{x_0+h} \left[e^{i\lambda_k x} - e^{i\lambda_k x_0} \right] d\sigma(x) \right|$$

$$\leqq M |h| \left(\sum_1^r |C_k| \lambda_k \right) |\sigma(x_0+h) - \sigma(x_0)|$$

Thus by (10) and (11)

$$\left| \frac{1}{[\sigma(x_0+h) - \sigma(x_0)]} \int_{x_0}^{x_0+h} f(x) \, d\sigma(x) - \sum_{k=1}^r C_k e^{i\lambda_k x_0} \right|$$

$$\leqq \left| \frac{1}{\sigma(x_0+h) - \sigma(x_0)} \int_{x_0}^{x_0+h} f(x) d\sigma(x) - \frac{1}{\sigma(x_0+h) - \sigma(x_0)} \sum_1^r C_k \int_{x_0}^{x_0+h} e^{i\lambda_k x} d\sigma(x) \right|$$

$$+ \left| \frac{1}{\sigma(x_0+h) - \sigma(x_0)} \sum_{k=1}^r C_k \int_{x_0}^{x_0+h} (e^{i\lambda_k x} - e^{i\lambda_k x_0}) d\sigma(x) \right|$$

$$\leqq \frac{1}{|\sigma(x_0+h) - \sigma(x_0)|} \left[\sum_{r+1}^{\infty} |C_k|^2 \right]^{\frac{1}{2}} \frac{1}{\lambda_r} C_1 \left\{ \frac{\sin^2 x_0}{x_0^2} + \frac{\sin^2(x_0+h)}{(x_0+h)^2} \right\}$$

$$+ M |h| \left(\sum_{k=1}^r |C_k| \lambda_k \right)$$

$$\leqq M_1 \left(\sum_{k=r+1}^{\infty} |C_k|^2 \right)^{1/2} \frac{1}{h \lambda_r} + M |h| \left(\sum_{k=1}^r |C_k| \lambda_k \right)$$

where M_1 and M_2 are independent of x_0, h and r. Hence we have

$$\left| \sum_{k=1}^r C_k e^{i\lambda_k x_0} \right|$$

$$\leqq \left| \frac{1}{\sigma(x_0+h) - \sigma(x_0)} \int_{x_0}^{x_0+h} f(x) d\sigma(x) \right|$$

$$+ M_1 \left(\sum_{r+1}^{\infty} |C_k|^2 \right)^{1/2} \frac{1}{h \lambda_r} + M |h| \left(\sum_{k=1}^r |C_k| \lambda_k \right)$$

From this inequality, if we take $r = r(x_0)$ such that

$$S^*(x_0) = \sup_n |S_n(x_0)| = \left| \sum_1^{r(x_0)} C_k e^{i\lambda_k x} \right|$$

then

$$S^*(x_0) \leqq \left| \frac{1}{\sigma(x_0+h) - \sigma(x_0)} \int_{x_0}^{x_0+h} f(x) d\sigma(x) \right|$$

$$+ M_1 \left(\sum_{k=r(x_0)+1}^{\infty} |C_k|^2 \right)^{1/2} \frac{1}{h \lambda_{r(x_0)}} + M |h| \left(\sum_{k=1}^{r(x_0)} |C_k| \lambda_k \right)$$

Now, let $h = \lambda_{r(x_0)}^{-1}$, then we have

$$S^*(x_0) = \left| \frac{1}{\sigma(x_0 + \lambda_{r(x_0)}^{-1}) - \sigma(x_0)} \int_{x_0}^{x_0 + \lambda_{r(x_0)}^{-1}} f(x) d\sigma(x) \right|$$

$$+ M_1 \left(\sum_1^{\infty} |C_k|^2 \right)^{1/2} + M \frac{1}{\lambda_r} \left(\sum_{k=1}^r |C_k| \lambda_k \right)$$

Now since

$$\frac{1}{\lambda_r} \left(\sum_{k=1}^r |C_k| \lambda_k \right) \leqq \frac{1}{\lambda^r} \sum_{k=1}^r |C_k| \lambda^k$$

$$\leqq \frac{1}{\lambda^r} \left(\sum_{k=1}^r |C_k|^2 \right)^{1/2} \left(\sum_{k=1}^r \lambda^{2k} \right)^{1/2}$$

$$\leqq M_\lambda \left(\sum_{k=1}^{\infty} |C_k|^2 \right)^{1/2}$$

where M_λ is constant which depends only on λ. We have

$$(12)\quad S^*(x_0) \leqq \left| \frac{1}{\sigma(x_0+\lambda^{-1}_{\gamma(x_0)}) - \sigma(x_0)} \int_{x_0}^{x_0+\lambda^{-1}_{\gamma(x_0)}} f(x)\,d\sigma(x) \right| + M_\lambda \left(\sum_{k=1}^{\infty} |C_k|^2 \right)^{1/2}$$

Using the well known maximal theorem of Hardy and Littlewood

$$(13)\quad \int_{-\infty}^{\infty} \left[\sup \left| \frac{1}{\sigma(x_0+h)-\sigma(x)} \int_{x}^{x_0+h} f(x)\,d\sigma(x) \right| \right]^r d\sigma(x) \leqq C_r \int_{-\infty}^{\infty} |f(x)|^r d\sigma(x), \quad r>1$$

where C_r is an absolute constant which depends only on r.

From Theorem 1, (12) and (13), we completes the proof of Theorem 4.

The proof of Theorem 5 can be done in quitely similar way as in the proof of Theorem 4 by using Theorem 3.

<u>Proof of Theorem 6.</u>

<u>Lemma.</u> Under the same conditions of Theorem 1,

$$\left\{ \int_{-\infty}^{\infty} |f|^{2n} d\sigma(x) \right\}^{1/2n} \leqq C_1 [n!]^{1/2n} \left[\frac{3n}{\lambda} \right]^{1/2} \left[\sum_{k=1}^{\infty} |C_k|^2 \right]^{1/2}$$

By (6)

$$\left\{ \int_{-\infty}^{\infty} \left| \sum_j C_{s_j+\ell} e^{i\lambda_{s_j+\ell} x} \right|^{2n} d\sigma(x) \right\}^{1/2n} \leqq (n!)^{1/2n} \left(\sum_j |C_{s_j+\ell}|^2 \right)^{1/2}$$

$$(\ell = 1, 2, \cdots, s), \quad s = \left[\frac{3n}{\lambda} \right] + 1.$$

So by Minkowski's inequality we have

$$\left\{ \int_{-\infty}^{\infty} |f|^{2n} d\sigma(x) \right\}^{1/2n} \leqq \sum_{\ell=1}^{s} \left\{ \int_{-\infty}^{\infty} \left| \sum_j c_{s_j+\ell} e^{i\lambda_{s_j+\ell} x} \right|^{2n} d\sigma(x) \right\}^{1/2n}$$

$$\leqq (n!)^{1/2n} \sum_{\ell=1}^{s} \left(\sum_j |C_{s_j+\ell}|^2 \right)^{1/2}$$

$$\leqq (n!)^{1/2n} s \left[\frac{1}{s} \sum_{k=1}^{\infty} |C_k|^2 \right]^{1/2}$$

$$\leqq (n!)^{1/2n} s^{1/2} \left[\sum_{k=1}^{\infty} |C_k|^2 \right]^{1/2}$$

$$\leqq C_1 [n!]^{1/2n} \left[\frac{3n}{\lambda} \right]^{1/2} \left[\sum_{k=1}^{\infty} |C_k|^2 \right]^{1/2}$$

which proves the lemma.

We shall now prove the theorem. From the above lemma, using Stirling formula, we have

$$\int_{-\infty}^{\infty} \frac{\mu^{2n}}{(2n)!} |f(x)|^{2n} d\sigma(x) \leqq C_1 \frac{\mu^{2n}}{(2n)!} n! \left[\frac{3n}{\lambda} \right]^n \left[\sum_{k=1}^{\infty} |C_k|^2 \right]^n$$

$$\leqq C_1 \left(\frac{e^2}{4} \right)^n \left(\frac{3}{\lambda} \right)^n \mu^{2n} \left(\sum_{k=1}^{\infty} |C_k|^2 \right)^n$$

Thus, if $\quad \frac{e^2}{4} \frac{3}{\lambda} \mu^2 \left(\sum_{k=1}^{\infty} |C_k|^2 \right)^n < 1,$

that is

$$(14)\quad \mu^2 < \frac{4\lambda}{e^2 3 (\sum |C_k|^2)}$$

then the series

$$(15)\quad \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mu^{2n}}{(2n)!} |f(x)|^{2n} d\sigma(x)$$

converges. Similarly, under the hypothesis (14),

$$(16)\quad \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{\mu^{2n-1}}{(2n-1)!} |f(x)|^{2n-1} d\sigma(x)$$

converges. Hence the series

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\mu^n}{n!} |f(x)|^n d\sigma(x)$$

converges. So if $\mu^2 < \frac{4\lambda}{3e^2} \Big/ \left(\sum_{n=1}^{\infty} |C_n|^2 \right)^{1/2}$ the function $e^{\mu|f(x)|}$ belongs to the class $L^{1,\sigma}(-\infty,\infty)$. This completes the proof.

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