FABER'S POLYNOMIALS, II.

By Shohei NAGURA.

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## § 3. Coefficient Problem

**3.1. Preliminary remarks.** In the previous part[7] we have discussed fundamental identities on Faber's polynomials and their applications to distortion theorems. In the present section we shall deal with the coefficient problem for the family of schlicht functions (2), especially the estimations of its second and third coefficients.

In the formula (14), if we put, instead of $z$, $\varepsilon z$ with an arbitrary $\varepsilon$ such as $|\varepsilon| = 1$, it becomes

$$(20) \qquad \lg \frac{\bar{\varepsilon} f(\varepsilon z)}{z} = \sum_{\nu=1}^{\infty} \frac{P_\nu(0)\varepsilon^\nu}{\nu} z^\nu, \left(\bar{\varepsilon} = \frac{1}{\varepsilon}\right).$$

Therefore, so far as we are concerned with the estimation of absolute value of each coefficient, it may be supposed that the real part of the respective coefficient is real and non-negative.

On the other hand, differentiating (14) with respect to $z$, we easily obtain

$$(21) \qquad (n-1)\,a_n = \sum_{\nu=1}^{n-1} a_\nu P_{n-\nu}(0) \qquad (n = 2, 3, \cdots).$$

**3.2.** Parameter representation by means of Löwner's differential equation. Let

$$(22) \qquad h(z,t) = e^{-(t_0-t)}\left(z + \sum_{\nu=2}^{\infty} c_{\nu-1}(t) z^\nu\right),$$

$$(23) \qquad f(z,t) = e^{-t} \left(z + \sum_{\nu=2}^{\infty} b_{\nu-1}(t) z^\nu\right)$$

be functions defined by Prof. Y.Komatu[8] for slit mapping functions of Löwner's theory.

Löwner[9] derived the differential equation for $h(z,t)$ such that

$$\frac{\partial h(z,t)}{\partial t} = z\,\frac{\partial h(z,t)}{\partial z}\,\frac{1+\kappa(t) z}{1-\kappa(t) z},$$

where $\bar{\kappa}(t) = e^{-i\theta(t)}$ $(|\kappa(t)| = 1)$ is the starting point of the slit, the boundary condition being $h(z,t_0) = z$. From this equation we obtain

$$(24) \qquad \frac{\partial \lg h(z,t)}{\partial t} = z\,\frac{\partial \lg h(z,t)}{\partial z}\,\frac{1+\kappa(t) z}{1-\kappa(t) z}.$$

Both functions $e^{t_0-t} h(z,t)$ and $e^t f(z,t)$ are schlicht and normalized at the origin. Hence, we can make use of the formula (14) for them. The constant terms of Faber's polynomials belonging to $e^{t_0-t} h(z,t)$ and $e^t f(z,t)$ may be, without ambiguity, denoted by $Q_\nu(t)$ and $P_\nu(t)$ instead of $Q_\nu(0)$, $P_\nu(0)$ respectively, since $Q_\nu(0)$ and $P_\nu(0)$ in (14) depend now on the parameter $t$.

Substituting (22) into (14), we have the following differential equations

$$(25) \qquad Q_1'(t) = Q_1(t) + 2\kappa(t),$$

$$(26) \qquad Q_n'(t) = n Q_n(t) + 2n\left(\sum_{\nu=1}^{n-1} \kappa(t)^{n-\nu} Q_\nu(t) + \kappa(t)^n\right) \quad (n = 2, 3, \cdots).$$

Integrating these equations with respect to $t$ from $t$ to $t_0$ with the boundary condition $Q_n(t_0) = 0$, we have the explicit formulae for $Q_n(t)$. For $P_n(t)$, it is sufficient to put $0$, $t$ in place of $t$, $t_0$ concerning to $Q_n(t)$.

**3.3. Estimation of coefficients.** In case $n = 1$, we have, from (25),

$$Q_1(t) = -2e^t \int_t^{t_0} e^{-\tau}\kappa(\tau)\,d\tau,$$

and hence, by the above-mentioned fact,

$$(27) \qquad P_1(t) = -2\int_0^t e^{-\tau}\kappa(\tau)\,d\tau$$

From the last relation, we can easily obtain

$$(28) \qquad |P_1(t)| \leqq 2\int_0^t e^{-\tau}\,d\tau = 2\,(1 - e^{-t}) \leqq 2.$$

This is, however, an immediate consequence of Koebe-Bieberbach's distortion theorem.

In case $n = 2$, we have

$$Q_2(t) = 4e^{2t}\left(\int_t^{t_0} e^{-\tau}\kappa(\tau)\,d\tau\right)^2 - 4e^t\int_t^{t_0} e^{-2\tau}\kappa(\tau)^2\,d\tau$$

and so

$$(29) \qquad P_2(t) = 4\left(\int_0^t e^{-\tau}\kappa(\tau)\,d\tau\right)^2 - 4\int_0^t e^{-2\tau}\kappa(\tau)^2\,d\tau.$$

Considering the real parts of (29) we get

$$\mathcal{R}\,P_2(t) = 4\left(\left(\int_0^t e^{-\tau}\cos\theta(\tau)\,d\tau\right)^2 - \left(\int_0^t e^{-\tau}\sin\theta(\tau)\,d\tau\right)^2\right) - 4\int_0^t e^{-2\tau}\cos 2\theta(\tau)\,d\tau,$$

and hence we obtain

$$
\begin{aligned}
\Re P_2(t) &\leqq 4\int_0^t (e^{-\tau}-e^{-2\tau})\cos^2\theta(\tau)\,d\tau + 1 - e^{-2t} \\
&\leqq 2 - 2e^{-t}(2-e^{-t}) \\
&\leqq 2
\end{aligned}
$$

In both relations (28) and (29), the equalities hold only for the limiting case $t \to \infty$ with $\kappa(t) \equiv 1$, that is, for the extremal function

$$f(z) = \frac{z}{(1-z)^2}$$

By virtue of (21), we easily have

$$|a_2| \leqq 2 \quad \text{and} \quad |a_3| \leqq 3,$$

which are the Bieberbach's and Löwner's coefficient theorems.

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(7) S.Nagura: Faber's polynomials. Kōdai Math. Sem. Rep. No.5. 5-6 (1949).
(8) Y.Komatu: Über einen Satz von Herrn Löwner. Proc.Imp.Acad. Tokyo. 16 (1940), 512-4.
(9) K.Löwner: Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I. Math. Ann. 89 (1923), 103-121.
(10) A.C.Schaeffer, M.Schiffer and D.C.Spencer: The coefficient regions of schlicht functions, Duke Math. Journ. Vol.16 (1949), 493-527.
On the other hand, above mentioned method can also be proceeded almost verbatin for Schaeffer-Spencer's differential equation. Cf. A.C.Schaeffer and Spencer, The coefficients of schlicht function, II. Duke Math. Journ. 12 (1945), 107-25.

(To be concluded)

Nagoya University.