ON GENERALIZED TRANSFINITE DIAMETER

By Yasuo NOZAKI.

(Communicated by Y. Komatu)

1. Introduction.

In the three dimensional Euclidean space Ω_3, let M be a bounded, closed set which contains infinitely many points, and let r_{pq} be the distance between the points p and q. G. Pólya and G. Szegö [1] defined the following quantities:

$$(I) \qquad R_n^{(\lambda)} = \underset{p_\nu\in\Omega_3}{Min}\ \underset{p\in M}{Max} \left\{ \frac{r_{pp_1}^{\lambda}+\cdots+r_{pp_n}^{\lambda}}{n} \right\}^{\frac{1}{\lambda}}$$

$$(I') \qquad \lim_{n\to+\infty} R_n^{(\lambda)} = R^{(\lambda)} \ ;$$

and

$$(II) \qquad D_n^{(\lambda)} = \underset{p_\nu\in M}{Max} \left\{ \left(\sum_{\mu<\nu}^{1,\dots,n} r_{p_\mu p_\nu}^{\lambda}\right) \Big/ \binom{n}{2} \right\}^{\frac{1}{\lambda}}$$

$$(II') \qquad \lim_{n\to+\infty} D_n^{(\lambda)} = D^{(\lambda)}$$

λ being an arbitrary real number. $D^{(\lambda)}$ is called the transfinite diameter of M, and $R^{(\lambda)}$ is the quantity which corresponds to the quantity $\lim_{n\to+\infty} \left\{ \underset{z\in M}{Max} \left| T_n(z) \right| \right\}^{\frac{1}{n}}$ defined in two dimensional Euclidean space Ω_2 where the $T_n(z)$ mean Tchebycheff's polynomials with respect to the set M. In Ω_2, and Ω_3, M. Fekete [2], G. Pólya and G. Szegö [1][3], and O. Frostman [4] have already proved that $D^{(\lambda)} = R^{(\lambda)}$ for $\lambda = -\alpha$ with $1 \leq \alpha < 3$

In this paper, replacing the functions r^λ by a more general one, $\Phi(r)$, as in the case of the generalized potential [4][5], we shall investigate the case where $D^{(\Phi)}$ and $R^{(\Phi)}$ coincide, and further relations between these quantities and the generalized potential.

2. Definitions.

We consider a function $\Phi(r)$ with following properties:

$$\Phi(r) \begin{cases} = +\infty & \text{for } r = 0 \ ; \\ > 0 & \text{and continuous, monotone decreasing in the strict sense, for } r>0 \ ; \\ \to 0 & \text{for } r \to +\infty \end{cases}$$

Let M be a bounded and closed set in an Euclidean space Ω, which contains infinitely many points. We define $R_n^{(\Phi)}$ and $D_n^{(\Phi)}$ as follows:

$$(A) \qquad \Phi(R_n^{(\Phi)}) = \underset{p_\nu\in\Omega}{Max}\ \underset{p\in M}{Min} \frac{\Phi(r_{pp_1})+\cdots+\Phi(r_{pp_n})}{n}$$

$$(B) \qquad \Phi(D_n^{(\Phi)}) = \underset{p_\nu\in M}{Min} \frac{\sum_{\mu<\nu}^{1,\dots,n} \Phi(r_{p_\mu p_\nu})}{\binom{n}{2}}$$

The class of functions $\Phi(r)$ contains some kind of the convex functions, for instance $\Phi(r) = \frac{1}{r}e^{-\lambda r}$ $\lambda > 0$ [6].

3. Existence of $\lim_{n\to+\infty} R_n^{(\Phi)}$

The following proof of the existence of $\lim_{n\to+\infty} R_n^{(\Phi)}$ and $\lim_{n\to+\infty} D_n^{(\Phi)}$ is due to the method of G. Pólya and G. Szegö [1]. Let p_ν be an arbitrary point of the space Ω, and $p\in M$, and let d denote the diameter of M. We describe the sphere S with radius $2d$ about a point q of M. If one of p_ν, say p_1, lies outside S, then we denote the intersecting point of the segment $p_1 p$ and the boundary of S by $\bar{p}_1$. Then we have $\Phi(r_{pp_1}) \leq \Phi(r_{p\bar{p}_1})$ and hence

$$(1) \qquad \underset{p_\nu\in\Omega}{Max}\ \underset{p\in M}{Min} \frac{\sum_{\nu=1}^{n} \Phi(r_{pp_\nu})}{n} \leq \underset{p_\nu\in M}{Max}\ \underset{p\in M}{Min} \frac{\Phi(r_{p\bar{p}_1})+\Phi(r_{pp_2})+\cdots+\Phi(r_{pp_n})}{n}$$

Therefore, we may replace the points which lie outside S by those of the spherical surface S, obtaining a relation analogous to (1). Now, we confine ourselves to the case where all the points p_ν belong to the closed sphere $\bar{S}$. Then we clearly have

$$\underset{p\in M}{Min} \frac{\Phi(r_{pp_1})+\cdots+\Phi(r_{pp_n})}{n} < +\infty$$

This minimum is the continuous function of the points $p_1,\dots,p_n$. Let $p_1,\dots,p_m$; $q_1,\dots,q_n$ be arbitrary points of $\bar{S}$, then

$$\underset{p\in M}{Min} \frac{\Phi(r_{pp_1})+\cdots+\Phi(r_{pp_m})+\Phi(r_{pq_1})+\cdots+\Phi(r_{pq_n})}{m+n} \leq \underset{p\in M}{Min} \frac{\sum_{\mu=1}^{m}\Phi(r_{pp_\mu})}{m+n} + \underset{p\in M}{Min} \frac{\sum_{\nu=1}^{n}\Phi(r_{pq_\nu})}{m+n}$$

By taking the maximum with respect to p_μ, p_ν, we obtain

$$\underset{(p_\mu, q_\nu)}{Max}\ \underset{p\in M}{Min} \frac{\Phi(r_{pp_1})+\cdots+\Phi(r_{pp_m})+\Phi(r_{pq_1})+\cdots+\Phi(r_{pq_n})}{m+n}$$

$$\leqq \underset{(p_i)}{\mathrm{Max}}\ \underset{p\in M}{\mathrm{Min}}\ \frac{\Phi(r_{pp_1})+\cdots+\Phi(r_{pp_m})}{m+n}$$

$$+\ \underset{(q_i)}{\mathrm{Max}}\ \underset{p\in M}{\mathrm{Min}}\ \frac{\Phi(r_{pq_1})+\cdots+\Phi(r_{pq_n})}{m+n},$$

i.e.

(2) $\quad (m+n)\,\Phi(R^{(\Phi)}_{m+n}) \leqq m\,\Phi(R^{(\Phi)}_m)+n\,\Phi(R^{(\Phi)}_n)$

Since $\Phi(R^{(\Phi)}_n) \geqq 0$, by the lemma below, there exists the limit

(3) $\quad \lim_{n\to+\infty} \Phi(R^{(\Phi)}_n) = A \geqq 0.$

i) If $+\infty > A > 0$, we get, by the continuity of Φ, $\lim_{n\to+\infty}\Phi(R^{(\Phi)}_n)=\Phi(\lim_{n\to+\infty}R^{(\Phi)}_n)=A$;

ii) if $A=0$, then $\lim_{n\to+\infty}R^{(\Phi)}_n=+\infty$;

iii) if $A=+\infty$, then $\lim_{n\to+\infty}R^{(\Phi)}_n=0$

In every case, we write $\lim_{n\to+\infty}R^{(\Phi)}_n = R^{(\Phi)}$.

Lemma.**) Let $\{a_n\}$ be a sequence of real numbers which satisfies the condition

$$a_{m+n} \leqq a_m + a_n \;;\; m, n = 1,2,\cdots\cdots.$$

Then the sequence $\left\{\frac{a_n}{n}\right\}$ is either convergent or divergent to $-\infty$.

4. Existence of $\lim_{n\to+\infty} D^{(\Phi)}_n$.

We consider the identity

(1) $\quad \sum_{\mu<\nu}^{1,\cdots,n}\Phi(r_{p_\mu p_\nu}) = \frac{1}{n-2}\sum_{k=1}^{n}\sum_{\mu<\nu}^{(k)}\Phi(r_{p_\mu p_\nu}),\ p\in M,$

where $\sum^{(k)}$ indicates the sum with respect to p_k except the case when $\mu = k$. Since

$$\binom{n-1}{2}\Phi(D^{(\Phi)}_{n-1}) \leqq \binom{n-1}{2}\sum_{\mu<\nu}^{(k)}\Phi(r_{p_\mu p_\nu}),$$

(1) becomes

$$\sum_{\mu<\nu}^{1,\cdots,n}\Phi(r_{p_\mu p_\nu}) \geqq \frac{1}{n-2}\sum_{k=1}^{n}\binom{n-1}{2}\Phi(D^{(\Phi)}_{n-1})$$

$$= \binom{n}{2}\Phi(D^{(\Phi)}_{n-1}),$$

i.e.

(2) $\quad \sum_{\mu<\nu}^{1,\cdots,n}\frac{\Phi(r_{p_\mu p_\nu})}{\binom{n}{2}} \geqq \Phi(D^{(\Phi)}_{n-1})$

By taking here the minimum of the first term, we obtain,

$$\Phi(D^{(\Phi)}_n) \geqq \Phi(D^{(\Phi)}_{n-1}),\ \text{i.e.}\ D^{(\Phi)}_{n-1} \geqq D^{(\Phi)}_n$$

Since $D^{(\Phi)}_n \geqq 0$, we obtain

(3) $\quad \lim_{n\to+\infty} D^{(\Phi)}_n = D^{(\Phi)}$

5. Relations between $D^{(\Phi)}$ and $R^{(\Phi)}$.

We consider the points $p_i, p_j \in M$, which satisfies the equalities:

$$\binom{n+1}{2}\Phi(D^{(\Phi)}_{n+1}) = \sum_{\mu<\nu}^{1,\cdots,n+1}\Phi(r_{p_\mu p_\nu}) = \frac{1}{2}\sum_{k=1}^{n+1}\sum_{\nu=1}^{n+1\,(k)}\Phi(r_{p_k p_\nu})$$

$$= \underset{p_i,p_j\in M}{\mathrm{Min}}\ \sum_{i<j}^{1,\cdots,n+1}\Phi(r_{p_i p_j}).$$

Since

$$\sum_{\mu=1}^{n+1\,(k)}\Phi(r_{p_k p_\nu}) = \underset{p_k\in M}{\mathrm{Min}}\sum_{\nu=1}^{n+1\,(k)}\Phi(r_{p_k p_\nu}) \leqq n\,\Phi(R^{(\Phi)}_n),$$

we get

$$\binom{n+1}{2}\Phi(D^{(\Phi)}_{n+1}) \leqq \frac{1}{2}\sum_{k=1}^{n+1} n\,\Phi(R^{(\Phi)}_n)$$

i.e

$$\Phi(D^{(\Phi)}_{n+1}) \leqq \Phi(R^{(\Phi)}_n)$$

By the monotony of $\Phi(r)$, we obtain

$$D^{(\Phi)}_n \geqq D^{(\Phi)}_{n+1} \geqq R^{(\Phi)}_n$$

and hence it follows

(1) $\quad D^{(\Phi)}_n \geqq R^{(\Phi)}_n$

Letting $n \to +\infty$, we have

(2) $\quad D^{(\Phi)} \geqq R^{(\Phi)}$

6. The preliminary remarks on the generalized potential.(4)

Let $\mathcal{B}$ be the Borel's "Mengenkoerper", and μ denote a completely additive set function defined for the sets measurable in the Borel sense which we call the mass-distribution. We say that μ is a positive mass-distribution, if $\mu(e) \geqq 0$, $e \subset \mathcal{B}$. The closed set F is called the kernel of the mass with respect to μ, when F consists of points which bear the mass actually. In the following section, the integrals are considered in the sense of Stieltjes-Lebesge-Radon. We now introduce the generalized potential by the integral of the form

(1) $\quad u(p) = \int_\Omega \Phi(r_{pq})\,d\mu(q)$,

μ denoting a positive mass-distribution. Then the well-known properties of the potential are as follows:

(i) $u(p)$ is lower semi-continuous,

(ii) If $\Phi(r)$ is a convex function of r, and F is the kernel of the mass, then

(2) $\quad \Delta\Phi(r) = \Phi''(r) - \frac{2}{r}\Phi'(r) \geqq 0$, for $r>0$,

and hence, in $\Omega - F$,

(3) $\Delta u(p) \geq 0$

That is, $u(p)$ is subharmonic in $\Omega - F$. Consequently, by the maximum principle, if $u(p)$ is continuous on $\Omega - F$, the maximum of $u(P)$ is taken at a boundary point of $\Omega - F$, namely on the kernel F.

(iii) If $\{\mu_n\}$ converges to the distribution μ, then we have

(5) $u(p) \leqq \lim_{n\to+\infty} u_n(p)$,

(6) $I(\mu) \leqq \lim_{n\to+\infty} I(\mu_n)$, $\quad I(\mu) = \iint_{\Omega} \Phi(r_{pq})\, d\mu(p)\, d\mu(q)$

where $u(p)$, $u_n(p)$ are the potentials due to μ, μ_n respectively, and $I(\mu)$, $I(\mu_n)$ the energy integrals corresponding to μ, μ_n.

7. Lemmas.

For the function $\Phi(r)$ we consider the several conditions (α) $\Phi(r)$ is convex function of r, (β) $\lim_{r\to 0} \frac{\Phi(cr)}{\Phi(r)} = k > 0$ and (γ) $\lim_{r\to+\infty} \frac{\Phi(r+c)}{\Phi(r)} = \ell > 0$,

where c is an arbitrary positive constant, and k, ℓ are constants depending on $\Phi(r)$ only. We shall prove a lemma analogous to that of O. Frostman (4).

Lemma 1. The necessary and sufficient conditions that the potential $u(p)$ is continuous on the bounded and closed set F are as follows: for any positive ε there corresponds a positive number δ such that the value of the potential u_δ at P due to the mass within the sphere S_δ whose centre is at a point P of F and its radius δ is less than ε.

Necessity. We denote by u', u'' the potentials due to the mass interior and exterior to the sphere S, then u'' is continuous and evidently satisfies the conditions of the lemma. Hence, it suffice to show the lemma only for the potential is u'. We consider the sphere S_a with radius a about $P \in F$ and denote by $u(p)$ the potential

$$u(p) = \int_{S_a F} \Phi(r_{pq})\, d\mu(q) = \lim_{N\to+\infty} \int_{S_a F} \Phi_N(r_{pq})\, d\mu(q)$$

where Φ_N denotes the function such that $\Phi_N = \Phi$, if $\Phi < N$ and $\Phi_N = N$, if $\Phi \geqq N$.

As $u(p)$ is continuous on the closed and bounded set $S_a F$, $u(p)$ is bounded there. Hence, for any positive ε, we can take a constant N_0 depending only on ε and not on p, such that

$$(1) \quad \left| u(p) - \int_{S_a F} \Phi_N(r_{pq})\, d\mu(q) \right| < \frac{\varepsilon}{5},$$

for $N \geqq N_0$, $p \in S_a F$.

Let $\Phi(\delta) = N_0$. If we take M, N, such that, $M > N \geqq N_0$, by (1), we get

$$(2) \quad \begin{aligned} &\left| u(p) - \int_{S_a F} \Phi_N(r_{pq})\, d\mu(q) \right| < \frac{\varepsilon}{5}, \\ &\left| u(p) - \int_{S_a F} \Phi_M(r_{pq})\, d\mu(q) \right| < \frac{\varepsilon}{5} \end{aligned}$$

Now let $\Phi(a) = \ell_0 < \ell_1 < \cdots < \ell_n = N$, and let e_i be the set of points satisfying the inequalities $\ell_{i-1} \leqq \Phi(r) < \ell_i$ ($i = 1, 2, \ldots, n$) and put $\Delta\mu_i = \mu(e_i)$, then, for sufficiently large n, we have

$$(3) \quad \left| \int_{S_a F} \Phi_N(r_{pq})\, d\mu(q) - \sum_1^n \ell_i \Delta\mu_i \right| < \frac{\varepsilon}{5}.$$

Similarly,

$$(4) \quad \left| \int_{S_a F} \Phi_M(r_{pq})\, d\mu(q) - \sum_1^m \bar{\ell}_i \Delta\bar{\mu}_i \right| < \frac{\varepsilon}{5},$$

where $\bar{\ell}_i$, $\overline{\Delta\mu_i}$ have the meaning analogous to ℓ_i, $\Delta\mu_i$ respectively. Put $\Phi(r_1) = N$, $\Phi(r_2) = M$, $r_1 > r_2$. Denoting the ring domain R whose centre is at P and whose radii are r_1, r_2, we have

$$(5) \quad \left| \int_{RF} \Phi(r_{pq})\, d\mu(q) \right| \leqq \left| \sum_1^m \bar{\ell}_j \overline{\Delta\mu_j} - \sum_1^n \ell_i \Delta\mu_i \right| + \frac{\varepsilon}{5}.$$

from which, by (2), (3), and (4), we finally get

$$\left| \int_{RF} \Phi(r_{pq})\, d\mu(q) \right| < \frac{4\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon$$

This inequality holds for any $M, N \geqq N_0$ which implies that the value of the potential at P due to the mass within the sphere about P with radius δ, is less than ε. q.e.d.

Sufficiency. That the condition is sufficient is clear.

Corollary. Let $\Phi(r)$ satisfy the condition (β), and let F denote the kernel of the mass. If $u(p)$ is continuous on F, then it is continuous throughout the space Ω.

Proof. By the continuity of $u(p)$ on F, for a given ε, we can take $\delta > 0$ such that the value of the potential at p, due to the mass within $S_{2\delta}$, is less than $\frac{k\varepsilon}{2}$. In the case when dist $(P, F) > \delta$, let the point q is the one of the nearest points of F from p, and we describe the sphere S_δ about q. Then as $r_{pm} \geqq \delta$, $m \in F$, we have $u(p) \leqq \int \Phi(\delta)\, d\mu(m) < +\infty$. Therefore $u(p)$ is continuous at p.

In the case where dist $(P, F) \leqq \delta$, since $2 r_{pm} \geqq r_{pm} + r_{pq} \geqq r_{qm}$, $\Phi(r_{pq}) < \Phi(r_{qm})$ remembering the condition (β), we have

$$u_\delta(p) = \int_{S_\delta F} \Phi(r_{pm})\, d\mu(m) \leqq \int_{S_{2\delta} F} \Phi(r_{pm})\, d\mu(m)$$

$$\leqq \int_{S_{2\delta}\cdot F} \Phi\left(r_{qm}, \frac{\Phi(r_{pm})}{\Phi(r_{qm})}\right) d\mu(m) \leqq \int_{S_{2\delta}\cdot F} \Phi(r_{qm}) \frac{\Phi(r_{pm})}{\Phi(r_{qm})} d\mu(m)$$

$$\leqq \frac{1}{k} \int_{S_{2\delta}\cdot F} \Phi(r_{qm})\, d\mu(m) + \frac{\varepsilon}{2} < \frac{1}{k}\cdot k\,\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where k is the same constant as the one appearing in (β).

As ε is arbitrary, $u(p)$ is continuous at p, and hence also in the whole space.

Lemma 2. If $\Phi(r)$ satisfies the condition (γ), then

$$m_S(p) < A\cdot u(p),$$

where $m_S(p)$ denotes the mean value of $u(q)$ with respect to the sphere S about p, and A is a positive constant depending on the function $\Phi(r)$ only.

Proof.

$$(1) \qquad m_S(p) = \frac{1}{v} \int_{S_a} d\tau_m \int_{\Omega} \Phi(r_{qm})\, d\mu(q)$$

$$= \int_{\Omega} d\mu(q) \int_{S_a} \frac{1}{v} \Phi(r_{qm})\, d\tau_m$$

$$= \int_{\Omega} \Phi(r_{pq})\, d\mu(q) \int_{S_a} \frac{1}{v} \frac{\Phi(r_{qm})}{\Phi(r_{pq})}\, d\tau_m$$

where v is the volume of S_a and $d\tau_m$ is the volume element at m. The integral $I_1 = \int_{S_a} \frac{1}{v} \frac{\Phi(r_{qm})}{\Phi(r_{pq})} d\tau_m$ is the function of r_{pq} only, and if we change the integral region S_a to the unit sphere and put $r_{pq} = r$, then I_1 is continuous for $r > 0$ and tends to 0 with r. Now from the inequalities

$$\frac{\Phi(r_{pq}-a)}{\Phi(r_{pq})} \geqq \frac{\Phi(r_{qm})}{\Phi(r_{pq})} \geqq \frac{\Phi(r_{pq}+a)}{\Phi(r_{pq})},$$

we obtain, by (γ), $\frac{\Phi(r_{qm})}{\Phi(r_{pq})} \to l$ for $q \to \infty$. Therefore I_1 takes the positive maximum A for a value of r in $0 < r \leqq +\infty$, and hence (1) becomes

$$(2) \qquad m_S(p) \leqq A\cdot \int_{\Omega} \Phi(r_{pq})\, d\mu(q) = A\cdot u(p).$$

By the methods used in the above proofs, the conditions (β), (γ) are necessary. In the case where $\Phi(r)$ is assumed merely to be convex, I cannot ensure that corollary to Lemma 1, and Lemma 2 are holds or not.

Lemma 3. Let $\Phi(r)$ satisfy the conditions (α), (β), (γ) and let M be a bounded, closed set whose boundary satisfies the condition of Poincaré. By μ we denote an arbitrary positive unit mass-distribution on M and put $I(\mu) = \iint_M \Phi(r_{pq})\, d\mu(p)\, d\mu(q)$. If there exists a positive mass-distribution $\bar{\mu}$ which minimizes $I(\mu)$, that is if, for all admissible μ,

$$I(\bar{\mu}) \leqq I(\mu),$$

then $\bar{\mu}$ is an equilibrium-distribution.

Proof. Now we put $I(\bar{\mu}) = V$, and follow the method of O.Frostman [4]. We procede according to the next four steps. Let F denote the kernel of the mass with respect to $\bar{\mu}$.

i) $\bar{u}(p) = \int_M \Phi(r_{pq})\, d\bar{\mu}(q) \geqq V$ for all points of M except the points of the set whose spatial measure is zero.

Now

$$I(\bar{\mu}) = \int_{\Omega} \bar{u}(p)\, d\bar{\mu}(p) = V,$$

and it cannot always be $\bar{u}(p) \leqq V - \varepsilon$, by the semi-continuity of $\bar{u}(p)$, for any $\varepsilon > 0$. Assume that $\bar{u}(p) \leqq V - 2\varepsilon$ on the set E whose spatial measure is positive. We transport the mass m of $O(p_0)$ on E, $O(p_0)$ being a neighbourhood of p_0 where we have $\bar{u}(p) > V - \varepsilon$.

In such a transportation of the mass, we can make the potential due to the mass-distribution to be bounded. For example, we may take a new distribution σ such as:

$$\sigma = -\mu \quad in \quad O(p_0);$$

$$\sigma > 0 \quad on \ E \ and \ \sigma(E) = \mu[O(p_0)] = m;$$

$$\sigma = 0 \quad outside \ O(p_0) + E,$$

$$I(\sigma) = \iint_M \Phi(r_{pq})\, d\sigma(p)\, d\sigma(q) < +\infty.$$

For all positive number $k < 1$, the distribution $\bar{\mu} + k\sigma$ is non-negative and represent the positive unit mass-distribution on M. By the hypothesis

$$\delta I = I(\bar{\mu} + k\sigma) - I(\bar{\mu}) > 0.$$

But on the other hand, we have

$$\delta I = 2k \int_M u(p)\, d\sigma(p) + k^2 I(\sigma)$$

$$< -k\,[\,2m\varepsilon - k I(\sigma)\,].$$

If we take k so small that $\delta I \leqq 0$, this is absurd. Therefore letting $\varepsilon \to 0$ and we obtained the results mentioned above.

ii) $\bar{u}(p) \geqq V$ for all the points of M without exception.

Let P be the point of M (inner or boundary point). By the hypothesis we can take the cone c with vertex p and lies within M. Let the volume ratio between sphere about p and the cone c be $0 < \rho < 1$. Let S, s denote the sphere about p with radius R, r respectively. Now we can proceed under 1°, 2°.

$1°$ $\bar{u}'$ is the potential due to the mass $\bar{\mu}'$ within S, and take the radius R so small that $\bar{u}'(p) < \frac{\rho\varepsilon}{2A}$ holds.

$2°$ $\bar{u}''$ denotes the potential due to the mass $\bar{\mu}''$ outside S, and take δ such that $q \in \delta$, $\bar{u}''(q) < \bar{u}''(p) + \frac{\varepsilon}{2}$. In fact, this is true, for by the continuity of $\bar{u}''$ in δ, R being fixed and we have must only to take r small enough. Let $m_{c\delta}$ denote the mean of $\bar{u}$ on $c\delta$. Then as, except the point set of measure zero in $c\delta$, we have $\bar{u} \geqq V$,

$$(1) \quad V \leqq m_{c\delta} = m'_{c\delta} + m''_{c\delta} < m'_{c\delta} + \bar{u}''(p) + \frac{\varepsilon}{2}.$$

Clearly, it holds

$$(2) \quad m'_{c\delta} \leqq \frac{1}{\rho} m'_{\delta}$$

By Lemma 2 and the hypothesis $2°$,

$$(3) \quad m'_{c\delta} \leqq \bar{u}'(p) < \frac{\varepsilon}{2},$$

therefore

$$V \leqq \bar{u}''(p) + \varepsilon, \text{ i.e. } \bar{u}(p) \geqq \bar{u}''(p) > V - \varepsilon.$$

As ε is arbitrary, letting $\varepsilon \to 0$ we have without exception

$$(4) \quad \bar{u}(p) \geqq V, \quad p \in M.$$

iii) $\bar{u}(p) > V$ is never hold at every point of F.

In fact, if $\bar{u}(p_1) > V$, $p_1 \in F$ holds, then there exists a neighbourhood $O(p_1)$ of p_1 such that $p \in O(p_1)$, and $\bar{u}(p) > V$ and hence $I(\bar{\mu}) > V$. But this is absurd. Therefore we must have $\bar{u}(p) = V$, $p \in F$.

iv) $\bar{u}(p) = V$ for all points of M without exception.

Since $\Phi(r)$ is convex, the maximum principle of subharmonic functions holds good. As $\bar{u}(p)$ is continuous on F, by the corollary to the Lemma 1, it is continuous throughout the space Ω. By the maximum principle, the maximum of $\bar{u}(p)$ is attained on F. Therefore we have $\bar{u}(p) \leqq V$, $p \in \Omega$. Remembering the results of (ii) we have $\bar{u}(p) = V$ for all points of M without exception.

Remark. I cannot yet determine whether the equilibrium-distribution is always unique or not under our assumptions.

8. Relations between $R^{(\Phi)}$ and the potential.

Let the function $\Phi(r)$ satisfy the conditions (α), (β), and (γ), and let the set M satisfy the condition of Poincaré. It is clear that $\Phi(r)$ is measurable in the sense of Lebesgue for $0 < r < +\infty$. Under μ we mean an arbitrary positive mass-distribution of unit mass on the set M. Then we have

$$\min_{p \in M} \frac{\Phi(r_{pp_1}) + \cdots + \Phi(r_{pp_n})}{n} \leqq \int_M \frac{\Phi(r_{pp_1}) + \cdots + \Phi(r_{pp_n})}{n} d\mu(p)$$

$$\leqq \underset{q \in \Omega}{l.u.b} \int_M \Phi(r_{pq}) d\mu(p)$$

By taking the minimum of the first member, we get

$$\Phi(R_n^{(\Phi)}) \leqq \underset{q \in \Omega}{l.u.b} \int_M \Phi(r_{pq}) d\mu(p).$$

Letting $n \to +\infty$, we have

$$(1) \quad \Phi(R^{(\Phi)}) \leqq \underset{q \in \Omega}{l.u.b} \int_M \Phi(r_{pq}) d\mu(p).$$

Under the condition (α), we can apply the maximum principle to the last member, and hence we get

$$(2) \quad \Phi(R^{(\Phi)}) \leqq \underset{q \in M}{l.u.b.} \int_M \Phi(r_{pq}) d\mu(p)$$

9. Relations between $D^{(\Phi)}$ and energy-integral.

Suppose that $\Phi(r)$ and μ are as in No.8.

At first, it is clear that for any μ

$$(1) \quad \Phi(D^{(\Phi)}) = \min_{p_\mu \in M} \frac{\sum_{\mu<\nu}^{1,\dots,n} \Phi(r_{p_\mu p_\nu})}{\binom{n}{2}}$$

$$\leqq \frac{1}{\binom{n}{2}} \sum_{\mu<\nu}^{1,\dots,n} \iint_M \Phi(r_{p_\mu p_\nu}) d\mu(p_\mu) d\mu(p_\nu)$$

$$= \iint_M \Phi(r_{pq}) d\mu(p) d\mu(q)$$

Considering the lower limit of the last member, we have

$$\Phi(D_n^{(\Phi)}) \leqq \underset{\mu}{g.l.b.} [I(\mu)],$$

where

$$I(\mu) = \iint_M \Phi(r_{pq}) d\mu(p) d\mu(q)$$

Letting $n \to +\infty$, it follows

$$(2) \quad \Phi(D^{(\Phi)}) \leqq \underset{\mu}{g.l.b.} [I(\mu)],$$

or

$$D^{(\Phi)} \geqq \Phi^{-1} [\underset{\mu}{g.l.b} \; I(\mu)]$$

Now by O.Frostman's method we proceed as follows: take the points $p_1, \cdots, p_n$ in such a manner that

$$\Phi(D_n^{(\Phi)}) = \min_{p_\mu \in M} \frac{\sum_{\mu<\nu}^{1,\dots,n} \Phi(r_{p_\mu p_\nu})}{\binom{n}{2}}$$

And put the mass $\frac{1}{n}$ on each point p_μ. Such a distribution on M is clearly a positive unit mass distribution, which we denote by μ_n. Then we have;

$$(3)\quad \frac{1}{n^2}\sum_{\mu\neq\nu}^{1,\dots,n}\Phi(r_{p_\mu p_\nu}) = \sum_{\mu=1}^{n}\Big\{\sum_{\substack{\mu\neq\nu\\ \nu=1}}^{n}\Phi(r_{p_\mu p_\nu})\frac{1}{n}\Big\}\frac{1}{n}$$
$$= \sum_{\mu=1}^{n}\sum_{\substack{\mu\neq\nu\\ \nu=1}}^{n}\Phi(r_{p_\mu p_\nu})\,d\mu_n(p_\mu)\,d\mu_n(p_\nu)$$
$$= \sum_{\mu\neq\nu}^{1,\dots,n}\Phi(r_{p_\mu p_\nu})\,d\mu_n(p_\mu)\,d\mu_n(p_\nu)$$
$$\geqq \sum_{\mu\neq\nu}^{1,\dots,n}\Phi_N(r_{p_\mu p_\nu})\,d\mu_n(p_\mu)\,d\mu_n(p_\nu)$$
$$= \sum_{\mu,\nu}\Phi(r_{p_\mu p_\nu})\,d\mu_n(p_\mu)\,d\mu_n(p_\nu) - \frac{N}{n}$$

Since the sequence $\{\mu_n\}$ is bounded, we can select, if necessary, a convergent subsequence, which we denote also by $\{\mu_n\}$ and we denote its limiting distribution by μ^*. First by $n\to+\infty$, we get from (1)

$$\Phi(D^{(\Phi)}) \geqq \iint_M \Phi_N(r_{pq})\,d\mu^*(p)\,d\mu^*(q).$$

Then by $N\to+\infty$, we get the relation

$$(4)\qquad \Phi(D^{(\Phi)}) \geqq \iint_M \Phi(r_{pq})\,d\mu^*(p)\,d\mu^*(q)$$

or

$$(4')\qquad D^{(\Phi)} \leqq \Phi^{-1}[I(\mu^*)]$$

From (2) and (3), we see that μ^* is the one that minimizes the energy-integral, so that by Lemma 3 of No.7, μ^* becomes one of the equilibrium-distribution $\bar\mu$ of the unit mass. Therefore we can write

$$(5)\qquad \Phi(D^{(\Phi)}) \geqq I(\mu^*) = I(\bar\mu)$$

In (1) by substituting μ by $\bar\mu$, we get

$$\Phi(D^{(\Phi)}) \leqq I(\bar\mu)$$

and hence

$$(6)\ \ \Phi(D^{(\Phi)}) = I(\bar\mu) = \bar V \quad \text{or} \quad (7)\ \ D^{(\Phi)} = \Phi^{-1}[\bar V].$$

Therefore, we obtain

Theorem 1. If the set M satisfies the conditions of Poincaré, and if $\Phi(r)$ satisfies the conditions (α), (β), and (γ), then

$$\Phi(D^{(\Phi)}) = I(\bar\mu) = \bar V.$$

10. Theorem 2. If the set M and $\Phi(r)$ satisfy the conditions of the Theorem 1, then it holds

$$D^{(\Phi)} = R^{(\Phi)}$$

Proof. By the definition

$$\operatorname*{Min}_{p\in M}\frac{\Phi(r_{pp_1})+\dots+\Phi(r_{pp_n})}{n} \leqq \frac{1}{n}\sum_{\nu=1}^{n}\int_M \Phi(r_{pp_\nu})\,d\mu(p),$$

for any unit mass-distribution μ. Considering the maximum of the first member we get

$$\Phi(R_n^{(\Phi)}) \leqq \operatorname*{l.u.b.}_{q\in\Omega}\int_M \Phi(r_{pq})\,d\mu(p),$$

and, by $n\to+\infty$,

$$(1)\qquad \Phi(R^{(\Phi)}) \leqq \operatorname*{l.u.b.}_{q\in\Omega}\int_M \Phi(r_{pq})\,d\mu(p)$$

By the condition (α), we can apply the maximum principle to the second member, and (1) becomes

$$(2)\qquad \Phi(R^{(\Phi)}) \leqq \operatorname*{l.u.b.}_{q\in M}\int_M \Phi(r_{pq})\,d\mu(p) = \operatorname*{l.u.b.}_{q\in M} u(q).$$

By (2) of No.5 and (5) of No.9, we have, for any μ,

$$(3)\qquad \operatorname*{l.u.b.}_{q\in M}[u(q)] \geqq \Phi(R^{(\Phi)}) \geqq \Phi(D^{(\Phi)}) \geqq I(\bar\mu)$$

Using here $\bar\mu$ in the place of μ and remembering the relation

$$\operatorname*{l.u.b.}_{q\in M}\int_M \Phi(r_{pq})\,d\bar\mu(p) = \bar V = I(\bar\mu),$$

we get

$$(4)\qquad \Phi(R^{(\Phi)}) = \Phi(D^{(\Phi)}) = I(\bar\mu),$$

i.e.,

$$(5)\qquad R^{(\Phi)} = D^{(\Phi)} = \Phi^{-1}[\bar V].$$

11. Now we consider a closed and bounded set M, and denote by T the component of the complementary domain of M which contains the points at infinity. We approximate T by such regular regions T_n that $\Omega - T_n = F_n$ satisfy the condition of Poincaré. As in the case of No.9, there exists a unit mass-distribution μ_n on F_n such

that $\Phi[R^{(\Phi)}(F_n)] = \Phi[D^{(\Phi)}(F_n)] = \bar V_{F_n} = W_{F_n}$.

Then it is evident that $\mu_n(F_n)=1,\ \mu_n(\Omega-F_n)=0$. The sequence $\{\mu_n\}$ converges to μ (if necessary we apply the selection theorem), and μ becomes the unit mass-distribution on M in the following sense, i.e., for n large enough, the points outside F_n also lie outside M. Therefore we have $\mu(M)=1$ and $\mu(\Omega-M)=0$. We denote $\operatorname*{l.u.b.}_{(p)}[u(p)]$ and $I(\mu)$ by V_M and W_M respectively. As

$$F_1 \supset F_2 \supset \dots \supset F_n \supset \dots \supset M,$$

we have by the property of the equilibrium-potential

(1) $V_{F_1} \leqq V_{F_2} \leqq \cdots \leqq V_{F_n} \leqq \cdots \leqq V_M$,

(2) $W_{F_1} \leqq W_{F_2} \leqq \cdots \leqq W_{F_n} \leqq \cdots \leqq W_M$,

with $V_{F_n} = W_{F_n}$.

Then according to the relations

(3) $V_M \leqq \underset{(p)}{l.u.b.}[u(p)] \leqq \lim_{n\to+\infty} u_n(p)$

and

(3') $W_M \leqq I(\mu) \leqq \lim_{n\to+\infty} I(\mu_n)$.

we get

(4) $\lim_{n\to+\infty} V_{F_n} = V_M$ and $\lim_{n\to+\infty} W_{F_n} = W_M$

Therefore we have

(5) $$W_M = V_M$$

On one hand, by the properties of $D^{(\Phi)}$ and $R^{(\Phi)}$, we have

(6) $\Phi[R^{(\Phi)}(M)] \leqq V_M$, $\Phi[D^{(\Phi)}(M)] \leqq W_M$.

On the other, considering the relation (3), we obtain

(7) $V_M = \lim_{n\to+\infty} V_{F_n} = \lim_{n\to+\infty} \Phi[R^{(\Phi)}(F_n)] \leqq \Phi[R^{(\Phi)}(M)]$,

and

(7') $W_M = \lim_{n\to+\infty} W_{F_n} = \lim_{n\to+\infty} \Phi[D^{(\Phi)}(F_n)] \leqq \Phi[D^{(\Phi)}(M)]$

By (6), (7) and (7'), we have

(8) $\Phi[R^{(\Phi)}(M)] = V_M$, $\Phi[D^{(\Phi)}(M)] = W_M$

By (5) and (8),

$$\Phi[R^{(\Phi)}(M)] = \Phi[D^{(\Phi)}(M)] \quad i.e. \quad R^{(\Phi)}(M) = D^{(\Phi)}(M) .$$

Therefore, we obtain the

Theorem 3. If $\Phi(r)$ satisfies the conditions (α), (β), and (γ), and M is a bounded and closed set, then we have

$$R^{(\Phi)}(M) = D^{(\Phi)}(M).$$

Remark. V_M and W_M are independent on the manner of approximation of the closed domain by F_n . And the distribution μ in Theorem 3 is the one which minimizes the energy-integral among all the positive distributions of the unit mass.

12. Relations between $D^{(\Phi)}$, $R^{(\Phi)}$ and the capacity.

Concerning the relation between $D^{(\Phi)}$, $R^{(\Phi)}$ and the capacity $C^{(\Phi)}$ of the bounded, closed set M , (i) M.Fekete[2] and G.Szegö[3] proved that $D^{(\Phi)} = R^{(\Phi)} = C^{(\Phi)}$ in the case where $\Phi(r) = \log\frac{1}{r}$; (ii) G.Pólya and G.Szegö[1] showed the same relations in the case where $\Phi(r) = \frac{1}{r}$; (iii) O.Frostman[4] proved the relation $D^{(\Phi)} = R^{(\Phi)}$ in the case where $\Phi(r) = \frac{1}{r^\alpha}$, $1 \leqq \alpha < 3$. O.Frostman defined the capacity of M , when $\Phi(r)$ is a more general one, as follows: let $\bar{\mu}$ and $\bar{u}(p)$ be the equilibrium-distribution of unit mass on M and its potential respectively, and put $V_M = \underset{(p)}{l.u.b.}\,\bar{u}(p)$, $W_M = \underset{(\mu)}{g.l.b.}\, I(\mu) = I(\bar{\mu})$ then the capacity of M is defined by

(1) $$C^{(\Phi)} = \Phi^{-1}(V_M).$$

In the case (iii) we have also $D^{(\alpha)} = R^{(\alpha)} = C^{(\alpha)}$. Now we have demonstrated that $D^{(\Phi)} = R^{(\Phi)}$ in the case where $\Phi(r)$ satisfies the conditions (α), (β), and (γ). Therefore, O.Frostman's definition of the capacity in the case (iii) is natural in the sense mentioned above. But in this case, it is inconvenient that the distribution which gives equilibrium-potential does not uniquely determined. From this point of view, the Vallée Poussin's definition of capacity has also the same inconvenience.

Now, we consider the Theorem 3 again. First, we have clearly $u(p) \leqq V_M$. Secondly, we have $I(\mu) = \int_M u(p)\,d\mu(p) = V_M$ If there exists a mass-point p_0 of M such that $u(p_0) < V_M$, then by the lower semi-continuity of $u(p)$, we can take some neighbourhood $O(p_0)$ of p_0 where $u(p)$ is less than V_M . Then we have $I(\mu) < V_M$, this is absurd. Hence, except a subset E of M where $\mu = 0$, it must be $u(p) = V_M$. If the capacity of E is positive, we can distribute the positive mass on E whose energy integral is finite. And by the same method used in (i), Lemma 3, No.7, we can construct a unit mass-distribution ν such as $I(\nu) < I(\mu)$. This contradicts clearly with the definition of $I(\mu)$. Therefore, the capacity of E must be zero. Thus we have:

Theorem 4. If the capacity of a bounded and closed set M is positive and if $\Phi(r)$ satisfies the conditions (α), (β) and (γ), then we have

$$u(p) = V_M \quad , \quad p \in M$$

except the point-set E whose capacity is 0 .

Remark. The potential $u(p)$ of the Theorem 4 is the equilibrium-potential on M . In all cases we have the fundamental relations;

$$D^{(\Phi)} = R^{(\Phi)} = C^{(\Phi)}$$

*) Received Feb. 11, 1950.

**) G.Pólya and G.Szegö: Aufgaben und Lehrsaetze aus der Analysis I, Berlin, 1925, p.17.

(1) G.Pólya and G.Szegö: Ueber den transfiniten Druchmesser von ebenen und raeumlichen Punktmengen; Crelles Journ. 165 (1931).

(2) M.Fekte: Ueber die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. Math.Zeits. 17 (1923).

(3) G.Szegö: Bemerkungen zu einer Arbeit von Herrn M.Fekete. Math.Zeits. 21 (1924).

(4) O.Frostman: Potentiel d'équilibre et capacité des ensembles. Lund, 1935.

(5) S.Kametani and T.Ugaeri: On the Theorem of G.C.Evans in the generalized potentials, Sugaku, Vol.1, 1948.

(6) K.Kunugui: On the generalized potential, Sugaku, Vol.1, 1948.

Gakushuin College, Tokyo.