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ON INTEGRAL FORMULAS OF ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES AND SOME RELATED PROBLEMS.

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1. Introduction.

It is well known that an analytic function of one complex variable can be represented by Cauchy's integral formula. The generalization of this formula to the case of several complex variables have been treated by S.Bergman, A.Weil, K.Oka, S.Bochner and many other authors. These expressions can be classified into two types: its integration manifolds are, in the one case, the "distinguished boundary surfaces", and in the other case, the whole boundary hypersurface. In this Note, we shall consider some relations between both types, and some related problems.

2. Integral Formula of Weil and Oka.

The integral formula of Weil [1] and Oka [1] belongs to the first type.

Let $m(\geq n)$ functions $X_j(z_1,\cdots,z_n)$ $(j=1,\ \ ,m)$ be regular in a bounded domain $\mathfrak{D}$ in the space of n complex variables and satisfy the following conditions:

(i) There exist functions $R(\zeta_1,\cdots,\zeta_n;z_1,\cdots,z_n)$ and $P_{j\nu}(\zeta_1,\ \ ,\zeta_n;z_1,\cdots,z_n)$ regular for $2n$ variables $z_1,\cdots,z_n$ in $(\zeta_1,\cdots,\zeta_n)\in\mathfrak{D}$, $\zeta_1,\ \ ,\zeta_n;z_1,\ \cdots,z_n$ $(z_1,\cdots,z_n)\in\mathfrak{D}$ such that

$$[X_j(\zeta_1,\cdots,\zeta_n)-X_j(z_1,\cdots,z_n)]\cdot\times R(\zeta_1,\cdots,\zeta_n;z_1,\cdots,z_n)\equiv\sum_{\nu=1}^{n}(\zeta_\nu-z_\nu)\cdot P_{j\nu}(\zeta_1,\cdots,\zeta_n;z_1,\cdots,z_n)\ .$$

and

$$R(\zeta_1,\cdots,\zeta_n;\zeta_1,\cdots,\zeta_n)\equiv 1\ .$$

(ii) For each j, there exists a region D_j bounded by finite number of analytic curves C_j in the range-plane of $X_j(z_1,\cdots,z_n)$. Put

$$\Delta\equiv\{(z_1,\cdots,z_n);\ X_j(z_1,\cdots,z_n)\in D_j+C_j\ ,\ (j=1,\cdots,m)\}\ ,$$
$$S_j\equiv\{(z_1,\cdots,z_n);X_j(z_1,\cdots,z_n)\in C_j\}\cap\Delta\ ,$$
$$\sigma^{(k)}_{k_1,\cdots,k_k}\equiv S_{k_1}\cap\cdots\cap S_{k_k}$$

Suppose that Δ is a closed domain completely contained in $\mathfrak{D}$, and that all the $\sigma^{(k)}_{k_1,\cdots,k_k}$ are at most $(2n-k)$ dimensional manifolds.

Then a function $f(z_1,\cdots,z_n)$ regular on Δ can be represented in Δ by the following expression due to Weil and Oka:

$$(1)\qquad f(z_1,\cdots,z_n)=\frac{1}{(2\pi i)^n}\sum_{(n)}\int_{\sigma^{(n)}_{j_1,\cdots,j_n}}f(\zeta_1,\cdots,\zeta_n)\cdot\times\frac{[R((\zeta);(z))]^{-n}\cdot\det\boxed{P_{j_\mu,\nu}((\zeta);(z))}_{\mu,\nu=1,\cdots,n}}{\prod_{\lambda=1}^{n}[X_{j_\lambda}((\zeta))-X_{j_\lambda}((z))]}\times d\zeta_1\cdots d\zeta_n\ ,$$

where the summation $\sum_{(n)}\int_{\sigma^{(n)}_{j_1,\cdots,j_n}}$ means that we take an n-combination $(j_1,\cdots,j_n)$ from $(1,\cdots,m)$, integrate over n-dimensional manifold $\sigma^{(n)}_{j_1,\cdots,j_n}$ and then sum up on all the possible n-combinations from $(1,\cdots,m)$. Since $R((\zeta);(\zeta))\equiv 1$, the factor $[R((\zeta);(z))]^{-n}$ can be replaced by 1.

3. Integral Formula of S.Bochner and S.Bergman.

These expressions belong to the second type.

Let us change the variables from x_ν,y_ν $(z_\nu=x_\nu+iy_\nu)$ to $z_\nu,\bar{z}_\nu$ by the transformations $z_\nu=x_\nu+iy_\nu$, and $\bar{z}_\nu=x_\nu-iy_\nu$ (cf. for ex.S.Bochner [1]). Then from the Green's formula, it is easy to see that if a function

$f(z_1,\cdots,z_n)$ is regular on a closed domain D, whose boundary B consists of a finite number of smooth hypersurfaces, then $f(z_1,\cdots,z_n)$ can be represented by the following integral formula due to S.Bochner [1] :

$$(2)\quad f(z_1,\cdots,z_n) = \frac{(n-1)!}{(2\pi i)^n}\int_B \sum_{\nu=1}^{n} \frac{f(\zeta_1,\cdots,\zeta_n)\cdot(\bar\zeta_\nu-\bar z_\nu)}{[\sum_{\mu=1}^{n}(\zeta_\mu-z_\mu)(\bar\zeta_\mu-\bar z_\mu)]^n} \times d\bar\zeta_1\, d\zeta_1\cdots \widehat{d\bar\zeta_\nu}\, d\zeta_\nu\cdots d\bar\zeta_n\, d\zeta_n .$$

The symbol $\wedge$ over a letter indicates that this letter is to be omitted.

When the domain D is cylindrical, i.e. D has the structure of a direct product of D_j (D_j is a domain on the z_j-plane), this formula (2) can be reduced to the iterated Cauchy's formula.

An analogous formula have been shown by S.Bergman [1] . He considered only the case of two variables, and his expression is:

$$(3)\quad f(z_1,z_2) = \frac{1}{4\pi}\iiint_B f(\zeta_1,\zeta_2)\cdot \times\Big\{\frac{\partial(1/r^2)}{\partial\bar\zeta_1}d\eta_1 d\xi_2 d\eta_2 - i\,\frac{\partial(1/r^2)}{\partial\zeta_1}d\xi_1 d\xi_2 d\eta_2 + \frac{\partial(1/r^2)}{\partial\bar\zeta_2}d\xi_1 d\eta_1 d\eta_2 - i\,\frac{\partial(1/r^2)}{\partial\zeta_2}d\xi_1 d\eta_1 d\xi_2\Big\}.$$

where B means the three dimensional boundary hypersurface of D, and

$$\zeta_\kappa \equiv \xi_\kappa + i\eta_\kappa ,\quad \bar\zeta_\kappa \equiv \xi_\kappa - i\eta_\kappa \quad (\kappa = 1,2),$$

$$r^2 = (\zeta_1-z_1)(\bar\zeta_1-\bar z_1)+(\zeta_2-z_2)(\bar\zeta_2-\bar z_2).$$

This is quite the same one to (2) where $n=2$, but it seems that the coefficients $1/4\pi$ should be replaced by $1/2\pi^2$.

S.Bergman and his disciples have shown many other integral formulas most of which concerned to "distinguished boundary surfaces" (cf. for ex.S.Bergman [2] and its bibliography.) All these formulas can be reduced to (1) or (2), but here we omit the detail.

4. Relation between (1) and (2).

Now we show that when the domain D has the same structure as Δ described in § 2, the expression (1) can be reduced to (2). Using hypothesis (i) and expanding the "det" in the numerator, the integrand of (1) is equal to

$$\frac{[R((\zeta);(z))]^{-n+1}\cdot f((\zeta))}{K((\zeta-z);(\bar\zeta-\bar z))} \times\sum_{\rho=1}^{n}\sum_{\lambda=1}^{n}(-1)^{\rho+\lambda}\frac{\det\boxed{P_{j_\mu,\nu}((\zeta);(z))}^{\mu\neq\rho,\,\nu\neq\lambda}\cdot(\bar\zeta_\lambda-\bar z_\lambda)}{\prod_{\mu=1,\mu\neq\rho}^{n}[X_{j_\mu}((\zeta))-X_{j_\mu}((z))]}.$$

where

$$K((\zeta-z),(\bar\zeta-\bar z)) \equiv \sum_{\mu=1}^{n}(\zeta_\mu-z_\mu)(\bar\zeta_\mu-\bar z_\mu).$$

Arrange this for $(n-1)$-combinations, and use the Stokes' theorem, (1) is equal to

$$\frac{1}{(2\pi i)^n}\sum_{(n-1)}\int_{\sigma^{(n-1)}_{l_1,\cdots,l_{n-1}}} f(\zeta_1,\cdots,\zeta_n) \times\frac{[R((\zeta),(z))]^{-(n-1)}}{[K((\zeta-z),(\bar\zeta-\bar z))]^2}\cdot\frac{1}{\prod_{\nu=1}^{n-1}[X_{l_\nu}((\zeta))-X_{l_\nu}((z))]}$$

$$\times\sum_{\lambda=1}^{n}\sum_{\rho=1}^{n}\Big[(-1)^\lambda\det\boxed{P_{l_\mu,\nu}((\zeta);(z))}^{\mu=1,\cdots,n-1}_{\nu\neq\lambda}\cdot(\zeta_\rho-z_\rho) - (-1)^\rho\det\boxed{P_{l_\mu,\nu}((\zeta);(z))}^{\mu=1,\cdots,n-1}_{\nu\neq\rho}\cdot(\zeta_\lambda-z_\lambda)\Big]\cdot \times(\bar\zeta_\lambda-\bar z_\lambda)\, d\bar\zeta_\rho\quad d\zeta_1\cdots d\zeta_n .$$

By induction, it is easily seen that (1) is equal to

$$(4)\quad \frac{k!}{(2\pi i)^n}\sum_{(n-k)}\int_{\sigma^{(n-k)}_{h_1,\cdots,h_{n-k}}} f(\zeta_1,\cdots,\zeta_n)\cdot \times\frac{[R((\zeta);(z))]^{-(n-k)}}{[K((\zeta-z),(\bar\zeta-\bar z))]^{k+1}}\cdot\frac{1}{\prod_{\nu=1}^{n-k}[X_{h_\nu}((\zeta))-X_{h_\nu}((z))]}$$

$$\times\sum_{\lambda=1}^{n}\sum_{\rho_1<\cdots<\rho_k}(-1)^{\lambda+\rho_1+\cdots+\rho_k} \times\Big[(-1)^\lambda\det\boxed{P_{h_\mu,\nu}((\zeta);(z))}^{\mu=1,\cdots,n-k}_{\nu\neq\rho_1,\cdots,\rho_k}(\zeta_\lambda-z_\lambda) + \sum_{i=1}^{k}(-1)^{\rho_i+i}\det\boxed{P_{h_\mu,\nu}((\zeta),(z))}^{\mu=1,\cdots,n-k}_{\nu\neq\rho_1,\cdots\hat\rho_i\cdots\rho_k,\lambda} \times(\zeta_{\rho_i}-z_{\rho_i})\Big]\cdot$$

$$\times(\bar\zeta_\lambda-\bar z_\lambda)\,d\bar\zeta_{\rho_1}\cdots d\bar\zeta_{\rho_k}\,d\zeta_1\cdots d\zeta_n .$$

Now putting $k=n-1$, (4) becomes

$$\frac{(n-1)!}{(2\pi i)^n}\sum_{j=1}^{m}\int_{S_j} f(\zeta_1,\cdots,\zeta_n)\cdot \times\frac{[R((\zeta),(z))]^{-1}}{[K((\zeta-z),(\bar\zeta-\bar z))]^n}\cdot\frac{1}{[X_j((\zeta))-X_j((z))]}\cdot$$

$$\times\sum_{\lambda=1}^{n}\sum_{\rho=1}^{n}(-1)^{n(n+1)/2}\cdot P_{j\rho}((\zeta);(z))\cdot(\zeta_\rho-z_\rho)\cdot \times(\bar\zeta_\lambda-\bar z_\lambda)\,d\bar\zeta_1\cdots\widehat{d\bar\zeta_\rho}\cdots d\bar\zeta_n\cdot d\zeta_1\cdots d\zeta_n .$$

Remember that $B=\sum_{j=1}^{m} S_j$, and the hypothesis (i), then this expression, after changing the orders of $d\bar\zeta$'s and $d\zeta$'s , is nothing else the expression (2).

5. Application to "Continuation Theorem" of Oka.

Mr. K.Oka proved the following "continuation theorem" (Oka [2]):

Theorem 1. 'In the space of two complex variables, there exist a finite, univalent domain D and two parallel hyperplanes α and β . Let $S(\alpha;\beta)$ denote the half-space separated by α and including β . If $D_1 \overset{\leftarrow}{=} D \cap S(\beta;\alpha)$ and $D_2 \overset{\leftarrow}{=} D \cap S(\alpha;\beta)$ are both domains of regularity, i.e. existence domains of some analytic functions, then D itself is also a domain of regularity.'

This contains that every pseudoconvex domain is a domain of regularity (Oka [2]).

We shall now try to generalize this theorem to the case of n variables. By a linear transformation, we can choose α and β to be the form $\{(z_1,\dots,z_n);\ \Im z_n = a\}$ and $\{(z_1,\dots,z_n);\ \Im z_n = b\}$ respectively, where a , b are real constants and $a > 0 > b$.

To prove theorem 1, the following "principal problem" is essential:

Problem. 'Under suitable conditions (cf. Oka [2] , § 2), given a function $f(z_1,\dots,z_n)$ regular in a neighbourhood of the common boundary of $\Delta_1 \overset{\leftarrow}{=} \Delta \cap \{(z_1,\dots,z_n);\ \Im z_n \geqq 0\}$ and $\Delta_2 \overset{\leftarrow}{=} \{(z_1,\dots,z_n);\ \Im z_n \leqq 0\}$, where Δ is some closed domain in D , construct such pair of functions $F_1(z_1,\dots,z_n)$ and $F_2(z_1,\dots,z_n)$ that they are regular in the closed domains Δ_1 and Δ_2 respectively, and identically satisfy the relation

$$F_1(z_1,\dots,z_n) - F_2(z_1,\dots,z_n) \equiv f(z_1,\dots,z_n) \text{ ?}$$

Have been established this problem, theorem 1 for the case of n variables can be proved quite analogously to the case of two variables (Oka [2] chap. 2 §§ 5-9).

To solve the above problem for $n=2$, he used the Cauchy's integral formula for one variable. In the case of n variables, we may, in fact, use the integral formula of Weil and Oka, and treat it quite analogously; but this process seems too complicated. Because, since the functions $X_j(z_1,\dots,z_n)$ are defined only in $D_3 \overset{\leftarrow}{=} D_1 \cap D_2$ in this case, F_1 and F_2 can be defined only in D_3 by this method. In order to define them in Δ_1 or Δ_2 , we must approximate the integrand by functions regular in D_1 or D_2 , and cancel the differences by some integral equations.

To avoid these difficulties, it is better to use the integral expression of S.Bochner. First we define a function $I(z_1,\dots,z_n)$ by the integral

$$I(z_1,\dots,z_n) \overset{\leftarrow}{=} \frac{(n-2)!}{(2\pi i)^n}\int_L \sum_{\nu=1}^{n-1} \frac{f(\zeta_1,\dots,\zeta_n)(\bar\zeta_\nu - \bar z_\nu)}{\left[\sum_{\mu=1}^{n-1}(\zeta_\mu - z_\mu)\cdot(\bar\zeta_\mu - \bar z_\mu)\right]^{n-1}} \times \frac{1}{\zeta_n - z_n}\cdot d\bar\zeta_1\, d\zeta_1 \cdots \widehat{d\bar\zeta_\nu}\, d\zeta_\nu \cdots d\bar\zeta_{n-1}\, d\zeta_{n-1}\cdot d\zeta_n$$

where L is a $(2n-2)$ -dimensional manifold such that

$$L \overset{\leftarrow}{=} \{\text{boundary of } \Delta\} \cap \{(\zeta_1,\dots,\zeta_n);\ \Im \zeta_n = 0\}.$$

This function $I(z_1,\dots,z_n)$ is analytic for $z_1,\dots,z_n$ except on $\{(z_1,\dots,z_n);\ \Im z_n = 0\}$. Put $F_1(z_1,\dots,z_n) \overset{\leftarrow}{=} I(z_1,\dots,z_n)$ in Δ_1 and consider its continuation which is denoted also by $F_1(z_1,\dots,z_n)$. This function $F_1(z_1,\dots,z_n)$ is analytic on Δ_1 including its boundary. We can also define the function $F_2(z_1,\dots,z_n)$ as the continuation of a functional element which is identically equal to $I(z_1,\dots,z_n)$ in Δ_2 .

It is easy to show that these two functions $F_1(z_1,\dots,z_n)$ and $F_2(z_1,\dots,z_n)$ satisfy the conditions of the "problem". By this method, the hypotheses 2° and 3° in § 2 of Oka [2] (p.18) and the processes in §§ 3, 4 (pp.19-27) and § 7 (pp.32-34) in that paper are not needed. Consequently, the proof of theorem 1 becomes very simplified.

From this solution of problem, we can easily show the following:

Theorem 2. 'The notations are quite the same as in the theorem 1, but the number of variables may be n . If D_1 and D_2 are first Cousin domains, i.e. domains in which the first Cousin problem (cf. for ex. H.Cartan [1]) has always its solution, then D itself is also a first Cousin domain.' This theorem is also valid, when the word "first" is replaced by the word "second".

(*) Received Dec. 17, 1949.

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