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EXISTENCE THEOREM OF CONFORMAL MAPPING OF DOUBLY-CONNECTED DOMAINS.

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In this Note, we shall give a brief proof of mapping theorem of doubly-connected domains stating that any ring domain, i.e., doubly-connected domain with two continua as its boundary, can be mapped conformally (and schlicht) onto a standard domain, a concentric annular ring.(1)

Let B be a given ring domain on z -plane. By means of Riemann's mapping theorem with respect to simply-connected domains, we may suppose that B is bounded by $|z|=1$ and by a regular analytic Jordan curve lying in the interior of the unit circle and enclosing the origin. Denote by $\mathcal{F}$ the family of functions $F(z)$ regular analytic and schlicht in B , $|z|=1$ inclusive, which satisfy the following conditions:

$$0<|F(z)|<1 \quad (z \in B) \text{ and } |F(z)|=1 \quad (|z|=1).$$

Since the particular function z belongs to the family, $\mathcal{F}$ is surely not empty. Putting

$$m[F] = \text{fin}\inf_{z\in B} |F(z)|$$

and

$$q = \text{fin}\sup_{F\in\mathcal{F}} m[F],$$

q being evidently a positive quantity, we select a maximizing sequence $\{f_n(z)\}$:

$$f_n(z) \in \mathcal{F}, \qquad \lim_{n\to\infty} m[f_n] = q.$$

As $\mathcal{F}$ is a normal family, we can suppose without loss of generality that this sequence $\{f_n(z)\}$ itself converges uniformly in the wider sense in B and, by means of analytic continuability, also in its reflected domain with respect to $|z|=1$; the convergence on $|z|=1$ is, of course, uniform. Let the limit function be $f(z)$.

Since every $f_n(z)$ does not vanish in B and the origin lies outside B each branch of $\frac{1}{z}\lg\frac{f_n(z)}{z}$ is regular and one-valued in B . On the other hand, the region $r \leqq |z| < 1$ with a positive r near to unity being contained in B , the value of integral

$$\frac{1}{2\pi i}\int_{|z|=\rho} \lg\frac{f_n(z)}{z}\,\frac{dz}{z}$$

is independent of ρ belonging to the interval $r \leqq \rho \leqq 1$. Comparing the real parts of the values of this integral for $\rho=r$ and $\rho=1$, we get the relation

$$\frac{1}{2\pi}\int_0^{2\pi}\lg\frac{|f_n(re^{i\theta})|}{r}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\lg|f_n(e^{i\theta})|\,d\theta = 0,$$

which leads us to the inequality

$$\lg \lambda_r[f_n] \equiv \lg \operatorname*{Min}_{|z|=r} |f_n(z)| \leqq \lg r.$$

But, since $\{f_n(z)\}$ converges to $f(z)$ uniformly on $|z|=r$ and $\lambda_r[F]$ is a continuous functional, we get

$$\lambda_r[f] = \lim_{n\to\infty}\lambda_r[f_n] \leqq r < 1 = |f(1)|.$$

Thus, the limit function $f(z)$ does not reduce to a constant and is hence regular and schlicht in B . Evidently it holds moreover

$$0 < q = m[f] \leqq r < 1.$$

The image-domain $\Delta = f(B)$ of B by the mapping $w=f(z)$ is contained in the annular ring $q<|w|<1$ and possesses the circumference $|w|=1$ as its outer boundary component. We shall now prove that the another boundary component Γ of the domain Δ coincides with the circumference $|w|=q$ and hence the original domain B is mapped by $w=f(z)$ just onto the annular ring $q<|w|<1$.

Assuming that it were not the case, let now

$$q < \operatorname*{Max}_{w\in\Gamma}|w| = |w^*| \equiv p < 1, \quad w^* \in \Gamma.$$

Denote by $w=g(\zeta)$ a mapping function between the exterior of Γ and $|\zeta|>p$, under the condition $g(\infty)=\infty$. The function $h(\omega)=g(\omega^{-1})^{-1}$ is then regular in $|\omega|<p^{-1}$ and vanishes at $\omega=0$. Since it satisfies the inequalities $|h(\omega)|<q^{-1}$ $(|\omega|<p^{-1})$ and $h(\omega)/\omega \not\equiv$ const, we

obtain, by Schwarz's lemma,

$$|h(\omega)| < \frac{q^{-1}}{p^{-1}}|\omega| \qquad (|\omega|<p^{-1}),$$

i.e.,

$$|g(\zeta)| > \frac{q}{p}|\zeta| \qquad (|\zeta|>p).$$

Let L be the curve on ζ -plane corresponding to $|w|=1$ by the mapping $w=g(\zeta)$ and put

$$\underset{\zeta\in L}{\mathrm{Max}}\,|\zeta| = |\zeta^*| \equiv P, \qquad \zeta^*\in L.$$

As $|g(\zeta^*)|=1$, we have, by the above-mentioned inequality,

$$1 > \frac{q}{p}P, \qquad \text{i.e.,} \qquad q < \frac{p}{P}.$$

Next, let $\zeta = G(W)$ be a mapping function between $|W|<1$ and the interior of L, under the condition $G(0)=0$. The function $G(W)$ is regular in $|W|<1$, vanishes at $W=0$ and satisfies the inequality $|G(W)|<P$ $(|W|<1)$. Hence we have, again by Schwarz s lemma.

$$|G(W)| \leqq P|W| \qquad (|W|<1).$$

(It will also be easily seen that the equality sign here never appears for $0<|W|<1$.) Denote by C the curve on W -plane corresponding to $|\zeta|=p$ by the mapping $\zeta=G(W)$ and put

$$\underset{W\in C}{\mathrm{Min}}\,|W| = |W^*| \equiv Q, \qquad W^*\in C.$$

As $|G(W^*)|=p$, we have

$$p \leqq PQ, \qquad \text{i.e.,} \qquad \frac{p}{P} \leqq Q$$

Thus, we obtain the inequalities

$$q < \frac{p}{P} \leqq Q.$$

On the other hand, the composed function $w=f^*(z)\equiv G^{-1}(g^{-1}(f(z)))$ is admissible for the family $\mathfrak{F}$, and satisfies the relation

$$m[f^*] = Q > q.$$

This contradicts to the defining maximum-property of q. Hence, $w=f(z)$ must be a mapping function from B onto the annular ring $q<|w|<1$, and the proposed mapping theorem is thus completely proved.

In the above-stated proof, we have, essentially, made use of Riemann's mapping theorem with respect to simply-connected domains alone. That the mapping function is uniquely determined except any rotation about $w=0$, can easily be established also by a similar argument.[2]

(*) Received September 30, 1949.

(1) Cf. also Y.Komatu, Ein alternierendes Approximationsverfahren für konforme Abbildung von einem Ringgebiete auf einen Kreisring. Proc. Imp. Acad. Tokyo 21(1945), 146-155.

(2) See also loc. cit. (1) .

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