CONFORMAL MAPPING OF POLYGONAL DOMAINS.

By Yûsaku KOMATU

**1. Introduction.** A function which maps a circular disc or a half-plane conformally onto a rectilinear polygon is, as is well known, given by Schwarz-Christoffel formula. Let $w=f(z)$ be such a function, and let the interior angle at vertex $f(a_\mu)$ ($\mu=1,\cdots,m$) of the image-polygon, having $m$ vertices, be denoted by $\alpha_\mu\pi$, the formula may be written in the form:

$$(1.1)\quad f(z)=C\int^z\prod_{\mu=1}^{m}(a_\mu-z)^{\alpha_\mu-1}dz+C',$$

where $C$ and $C'$ are both constants depending only on position and magnitude of the image-polygon.

The present author(1) has previously shown that this formula can be generalized to the case of analogous mapping of doubly-connected domains. We may adopt, as a standard doubly-connected basic domain, an annular domain $q<|z|<1$, $-\lg q$ being a uniquely determined conformal invariant, i.e. the so-called modulus of given polygonal ring domain. Let the boundary components corresponding to circumferences $|z|=1$ and $|z|=q$ be rectilinear polygons with $m$ and $n$ vertices respectively. Let further $\alpha_\mu\pi$ and $\beta_\nu\pi$ denote the interior angles (with respect to each boundary polygon itself) at vertices $f(e^{i\varphi_\mu})$ and $f(qe^{i\psi_\nu})$ respectively. The mapping function $w=f(z)$ is then expressed in the form:

$$(1.2)\quad f(z)=C\int^z z^{ic^*-1}\Big\{\prod_{\mu=1}^{m}\sigma(i\lg z+\varphi_\mu)^{\alpha_\mu-1}\div\prod_{\nu=1}^{n}\sigma_3(i\lg z+\psi_\nu)^{\beta_\nu-1}\Big\}dz+C',$$

where the sigma-functions are those of Weierstrass with primitive periods $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$ and the constant $c^*$ is given by

$$(1.3)\quad c^*=\frac{\eta_1}{\pi}\Big(\sum_{\mu=1}^{m}(1-\alpha_\mu)\varphi_\mu-\sum_{\nu=1}^{n}(1-\beta_\nu)\psi_\nu\Big);$$

the constants $C$ and $C'$ having similar meanings as before. It can, moreover, be shown that the Schwarz-Christoffel formula (1.1), for basic domain $|z|<1$, may be regarded as being a limiting case of (1.2) when $q\longrightarrow 0$.

On the other hand, any function $w=f(z)$ which maps a circular disc or a half-plane conformally onto the interior of a circular polygonal domain, i.e. the interior of a polygon having circular arcs as its sides, is linear-polymorphic. A differential equation of the third order of the form:

$$(1.4)\quad \{f(z),\ z\}=R(z)$$

holds good always for such a function $f(z)$. The left member of this equation denotes, as usual, Schwarzian derivative of $f(z)$ with respect to $z$, i.e.

$$\{f(z),z\}\equiv\frac{d^2}{dz^2}\lg\frac{df(z)}{dz}-\frac{1}{2}\Big(\frac{d}{dz}\lg\frac{df}{dz}\Big)^2=\frac{f'''(z)}{f'(z)}-\frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2,$$

and $R(z)$ is a rational function which possesses, as poles of order at most two, the points $a_\mu$ ($\mu=1,\cdots,m$) corresponding to the vertices of image-polygon. More precisely, if we denote by $\alpha_\mu\pi$ the interior angle at $f(a_\mu)$ of the image-polygon, we have, at the pole in question, the relation:

$$(1.5)\quad \lim_{z\to a_\mu}(z-a_\mu)^2R(z)=\frac{1-\alpha_\mu^2}{2}.$$

The above mentioned results (1.1) and (1.4) are usually derived by making use of analytic continuability of mapping function, that is, by performing successive inversions with respect to boundary arcs. But the author of this paper(2) previously pointed out that Schwarz-Christoffel formula (1.1) can be deduced immediately from Poisson integral representation of functions analytic in a circular disc. He(1) also has derived the formula (1.2) by means of Villat's integral representation of functions analytic in an annular domain. It will, however, be shown that the formula (1.2) can also be derived by the classical method without particular difficulty.

We can, on the other hand, consider the problem of generalization of (1.4) corresponding to that of (1.1) to (1.2).

In the present preliminary Note, we shall mention, from a more general standpoint, general relations corresponding to (1.1) and (1.4) in the case of multiply-connected domains, and then remark that, by specifying them to doubly-connected case, we can obtain the expression (1.2) again and the result generalizing (1.4) too.

Complete paper involving details and proofs will be published elsewhere.

**2.** Mapping onto circular polygonal domains. Consider, in $w$-plane, an $n$-ply connected domain $\Delta$ whose boundary consists of $n$ circular polygons $\Gamma_j$ ($j=1, \cdots, n$), each $\Gamma_j$ being formed by $m_j$ circular arcs. We can now take several types of domains as a standard $n$-ply connected basic domain. But we shall first take a domain $D$ bounded by $n$ full circles. This domain $D$ is uniquely determined for the given domain $\Delta$, except possible linear transformations[3]. Such a domain is defined in general by $3n$ real parameters denoting the coordinates of centers and the radii of $n$ boundary circles. But, since a linear transformation depends on 6 real parameters, essentially $3n-6$ real conformal invariants belong to an $n$-ply connected domain (with non-degenerating boundary components) as moduli, provided $n>2$. In an exceptional case $n=2$, there exists just one invariant, and in case $n=1$ there remains freedom corresponding to 3 real parameters.

Now, let the boundary circle of $D$ corresponding to $\Gamma_j$ be

(2.1) $$C_j: \quad |z-c_j|=r_j \qquad (j=1, \cdots, n),$$

and the mapping function be, as before, $w=f(z)$. Let further the points corresponding to vertices of $\Gamma_j$ be $a_{j\mu}$ ($\mu=1, \cdots, m_j$) and the interior angle of $\Gamma_j$ at its vertex $f(a_{j\mu})$ with respect to $\Delta$ be $\alpha_{j\mu}\pi$. The function $f(z)$ remains, of course, regular even on each interior part of $C_j$ divided by $a_{j\mu}$. If we denote the inversion $z \mid z_j^*$ with respect to $C_j$ by

$$z_j^* = \lambda_j(z) \equiv c_j + \frac{r_j^2}{\bar{z}-\bar{c}_j},$$

then $\lambda_j(z)$ being all linear in $\bar{z}$, the composed functions

(2.2) $$l_{jk}(z) \equiv \lambda_j(\lambda_k(z)) \quad (j, k=1, \cdots, n)$$

are also linear with respect to $z$. The transformation $z \mid l_{jk}(z)$ is composed of successive inversions with respect first to $C_k$ and next to $C_j$. Since operation of inversion is involutory, i.e. the identical relation $\lambda_j^{-1}(z) \equiv \lambda_j(z)$ holds, we have $l_{jj}(z)=z$ and

$$l_{jk}^{-1}(z) = \lambda_k^{-1}(\lambda_j^{-1}(z)) = \lambda_k(\lambda_j(z)) = l_{kj}(z).$$

The aggregate of all linear transformations corresponding to inversions repeated even times with respect to boundary circles (2.1) (and their successive transforms) forms a group $\mathfrak{G}$ generated thus by $\binom{n}{2}$ linear transformations $z \mid l_{jk}(z)$ ($j<k$).

After these preparatory considerations, we shall now state a result generalizing (1.4):

Theorem 1. Let $w=f(z)$ denote a mapping function from $D$ onto $\Delta$. Then

(2.3) $$\{f(z), z\}\, dz^2$$

is an automorphic differential belonging to the group $\mathfrak{G}$, whose fundamental domain may be composed of basic domain $D$ and its inverse with respect to any one of boundary circles of $D$ (speaking more exactly, the fundamental domain must be the open kernel of closure of the above mentioned one). The function $\{f(z), z\}$, being meromorphic in $\bar{D} \equiv D + \sum_{j=1}^{n} C_j$, is regular everywhere except possibly at $a_{j\mu}$ ($\mu=1, \cdots, m_j$; $j=1, \cdots, n$), where a pole of order at most two appears as shows the following relation:

(2.4) $$\lim_{z \to a_{j\mu}} (z-a_{j\mu})^2 \{f(z), z\} = \frac{1-\alpha_{j\mu}^2}{2}.$$

In a particular case, $n=1$, that is, when $\Delta$ is simply-connected, $\mathfrak{G}$ degenerates to a trivial group composed of a unique element, the identical transformation. In virtue of this degeneration, the automorphic property of (2.3) vanishes out, and the Schwarzian derivative $\{f(z), z\}$ becomes an analytic function possessing $a_\mu$ ($\equiv a_{1\mu}$)($\mu=1, \cdots, m$) as its poles of order at most two, and hence becomes a rational function.

If the image-domain $\Delta$ is particularly bounded by rectilinear polygons, more concrete properties of the mapping function $f(z)$ can be derived. In fact, we have the following theorem:

Theorem 2. If, in the theorem 1, the boundary components of $\Delta$ are all rectilinear polygons, then the differential expression

$$(2.5)\qquad d_2 \lg d_1 f(z) = \left( \frac{f''(z)}{f'(z)} + \frac{d_2 d_1 z}{d_2 z\, d_1 z} \right) d_2 z$$

possesses an automorphic property, $d_1$ and $d_2$ both denoting differentiation operators. The function $f''(z)/f'(z)$ meromorphic in $\bar{D}$ is regular except at the points $a_{j\mu}$ which are poles of order one with residue $\alpha_{j\mu} - 1$.

In the particular case $n = 1$, $\mathfrak{G}$ consists of the identical transformation alone. The automorphic property of (2.5) thus vanishes out, and $f''(z)/f'(z)$ becomes an analytic function in the entire plane possessing $a_\mu (\equiv a_{1\mu})$ $(\mu = 1, \cdots, m)$ as poles of order one. Furthermore, since $f(z)$ remains evidently regular at $z = \infty$ $(\neq a_\mu)$, we have

$$\frac{f''(z)}{f'(z)} = \sum_{\mu=1}^{m} \frac{\alpha_\mu - 1}{z - a_\mu},$$

which, by integration, leads us to the Schwarz-Christoffel formula (1.1).

**3.** Specialization to doubly-connected domains. In case of doubly-connected domains, we can take the annular domain $D:\ q < |z| < 1$ as a standard basic domain of modulus $-\lg q$. Two general theorems of the last section then take more clear and concrete forms. In the first place, by specializing theorem 1, we obtain the following result:

Theorem 3. Any function $w = f(z)$, mapping the annular domain $D$ conformally onto a ring domain $\Delta$ bounded by two circular polygons, satisfies the differential equation of the third order:

$$(3.1)\qquad \{f(z), z\} = \frac{E(i \lg z)}{z^2},$$

$E(Z)$ being an elliptic function with primitive periods $2\pi$ and $-2i \lg q$ (or being a constant). If we now denote by $e^{i\varphi_\mu}$ $(\mu = 1, \cdots, m)$ and $q e^{i\psi_\nu}$ $(\nu = 1, \cdots, n)$ the boundary points of $D$ corresponding to vertices of boundary polygons $\Gamma_1$ and $\Gamma_2$ of $\Delta$ respectively, and further by $\alpha_{1\mu}\pi$ and $\alpha_{2\nu}\pi$ the interior angles of $\Gamma_1$ and $\Gamma_2$ at vertices $f(e^{i\varphi_\mu})$ and $f(q e^{i\psi_\nu})$ respectively, then the function $E(Z)$ possesses at $Z = -\varphi_\mu$ and at $Z = -\psi_\nu + i \lg q$ its primitive poles of order at most two, and further

$$\lim_{Z \to -\varphi_\mu} (Z + \varphi_\mu)^2 E(Z) = \frac{1 - \alpha_{1\mu}^2}{2},$$

$$(3.2)\qquad \lim_{Z \to -\psi_\nu + i \lg q} (Z + \psi_\nu - i \lg q)^2 E(Z) = \frac{1 - \alpha_{2\nu}^2}{2}.$$

As was already stated in the section **1**, if the boundary of doubly-connected domain $\Delta$ consists of two rectilinear polygons, the explicit integral representation (1.2) is valid. This result has previously been obtained by the present author by means of Villat's formula, but the general theorem 2 may also be specified in this case to derive the same result which is stated as follows:

Theorem 4. Any function which maps the annular domain $q < |z| < 1$ conformally onto a ring domain bounded by rectilinear polygons, is expressed by formula (1.2), the constant $c^*$ being given by (1.3).

**4.** Another basic domains. As a standard multiply-connected basic domain, we can take any one of various possible types other than that used in the section **2**. For instance, as is often used, parallel slit domain obtained from entire plane by cutting along parallel segments, circular slit domain or radial slit domain which is obtained from either entire plane, circular disc or annular ring by cutting along circular arcs or radial segments(4). For such a basic domain, a group $\mathfrak{G}$ with analogous fundamental domain can also be constructed in quite similar manner as in theorems 1 and 2. These theorems themselves remain to hold in almost the same form. We have only to carry out a few modifications by considering that the regularity of boundary curves is lost at end points of the slits.

Theorem 5. In any case of such a basic domain of above-mentioned type, the conclusion of theorem 1 remains to hold with following modifications. If an end point of a slit coincides with a point $a_{j\mu}$, the relation (2.4) is replaced by

$$(4.1)\qquad \lim_{z \to a_{j\mu}} (z - a_{j\mu})^2 \{f(z), z\} = \frac{4 - \alpha_{j\mu}^2}{8},$$

and if an end point, say $p$, of a slit coincides with none of $a_{j\mu}$, the Schwarzian derivative possesses it as a pole of the second order and satisfies the relation:

$$(4.2)\qquad \lim_{z \to p} (z - p)^2 \{f(z), z\} = \frac{3}{8}.$$

Theorem 6. If $\Delta$ is bounded merely by rectilinear polygons, the conclusion of theorem 2 remains to be true, in any case of the above-mentioned basic domains, with following modifications. If an end point of a slit coincides with $a_{j\mu}$, the residue of $f''(z)/f'(z)$ at this point becomes

(4.3) $\lim_{z \to a_{j\mu}} (z - a_{j\mu}) \frac{f''(z)}{f'(z)} = \frac{\alpha_{j\mu} - 2}{2}$,

and if an end point $p$ of a slit coincides with none of $a_{j\mu}$, then $f''(z)/f'(z)$ has the point $p$ as a pole also of the first order with residue $-1/2$ ; that is,

$$(4.4) \quad \lim_{z \to p} (z - p) \frac{f''(z)}{f'(z)} = -\frac{1}{2}$$

In conclusion, we remark that a circular disc with $n$ sheets may also be taken as a standard type of $n$-ply connected domains(5). The group $\mathfrak{G}$ considered in theorem 1 then consists of a unique transformation $z|z$, all inversions $z|\lambda_j(z)$ referring to a common circumference. Hence, the group degenerates to a trivial one, while the mapping function becomes $n$-valued one on the disc. In this case a corresponding theorem may be stated as follows:

Theorem 7. Let $w = f(z)$ be a function which maps a circular disc $D$ with $n$ sheets covering a circle $D_0$ on $z$-plane conformally onto an $n$-ply connected circular polygonal domain $\Delta$. Then, each branch $f_j(z)$ $(j = 1, \cdots, n)$ of $f(z)$ satisfies a differential equation of the third order of the form:

$$(4.5) \quad \{f_j(z), z\} = M_j(z),$$

where $M_j(z)$ is a one-valued meromorphic function. Denoting by $\Gamma_j$ a boundary polygon of $\Delta$ mapped from boundary circle $C_j$ of $D$ by $w = f(z)$, i.e. by $w = f_j(z)$, and by $a_{j\mu}$ a point lying on $C_j$ and corresponding to a vertex of $\Gamma_j$, the function $M_j(z)$ possesses at $a_{j\mu}$ a pole of order at most two and satisfies the relation:

$$(4.6) \quad \lim_{z \to a_{j\mu}} (z - a_{j\mu})^2 M_j(z) = \frac{1 - \alpha_{j\mu}^2}{2},$$

where $\alpha_{j\mu}$ denotes, as before, the interior angle at $f_j(a_{j\mu})$ with respect to $\Delta$. Let further $t_\kappa$ be a branch point of $D$ of order $\tau_\kappa - 1$, then, for all the branches $f_j(z)$ relating to this branch point, the function $M_j(z)$ possesses there a pole of order at most two and satisfies the relation

$$(4.7) \quad \lim_{z \to t_\kappa} (z - t_\kappa)^2 M_j(z) = \frac{\tau_\kappa^2 - 1}{2\tau_\kappa^2}.$$

Excepting those points, $M_j(z)$ is regular everywhere.

Theorem 8. If, in the previous theorem, $\Delta$ is bounded particularly by rectilinear polygons, then, we have, for each branch of the mapping function, an explicit expression of the form:

$$(4.8) \quad f_j(z) = C\int^z \left(\exp \int^z N_j(z)\,dz\right) dz + C',$$

where $N_j(z)$ is a one-valued meromorphic function. Corresponding to (4.6) and (4.7), we have, at its pole $a_{j\mu}$ and a branch point $t_\kappa$, the relations

$$(4.9) \quad \lim_{z \to a_{j\mu}} (z - a_{j\mu}) N_j(z) = \alpha_{j\mu} - 1,$$

$$(4.10) \quad \lim_{z \to t_\kappa} (z - t_\kappa) N_j(z) = \frac{1 - \tau_\kappa}{\tau_\kappa}$$

respectively. $N_j(z)$ is, except those points, regular everywhere.

(*) Received July 13, 1949.

(1) Y.Komatu, Darstellungen der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf konforme Abbildung ueber Polygonalringgebiete. Jap.Journ.Math. 19(1945), 203-215.

(2) Y.Komatu, Einige Darstellungen analytischer Funktionen und ihre Anwendungen auf konforme Abbildung. Proc.Imp.Acad.Tokyo 20(1944), 536-541.

(3) As to the possibility of taking such a domain, see, e.g. L.R.Ford, Automorphic Functions, New York (1929), p.279 et seq.; or R.Courant, Plateau's problem and Dirichlet's principle. Ann. of Math. 38(1937), 679-724.

(4) Cf. R.de Possel, Zum Parallelschlitztheorem unendlich vielfach zusammenhaengender Gebiete. Goettinger Nachr. (1931). 192-202; E.Rengel, Existenzbeweise fuer schlichte Abbildungen mehrfach zusammenhaengender Bereiche auf gewisse Normalbereiche. Jahresber.Deutsch.Math.-Vereinig. 44(1934), 51-55.

(5) Cf. L.Bieberbach, Ueber einen Riemannschen Satz aus der Lehre von der konformen Abbildung. Sitzungsber. Berliner Math.Ges. 24 (1925), 6-9; H.Grunsky, Ueber die konforme Abbildung mehrfach zusammenhaengender Bereiche auf mehrblaettrige Kreise. Sitzungsber. preuss.Akad.Wiss. (1937), 1-9.

Tokyo Institute of Technology.