ON THE EXISTENCE OF ANALYTIC FUNCTIONS ON CLOSED ANALYTIC SURFACES.

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1. Introduction. The present short note is a preliminary report on an attempt to generalize the classical existence theorem of analytic functions on closed Riemann surfaces[1] to the case of the theory of functions of two complex variables. Let $\mathfrak{M}$ be a closed analytic surface, i.e. a 2-dimensional (topologically 4-dimensional) analytic manifold; the local (analytic) coordinates on $\mathfrak{M}$ will be denoted by z^1, z^2. The poles and zero-points of a one-valued meromorphic function $f(z^1, z^2)$ defined on $\mathfrak{M}$ constitute a 1-dimensional analytic submanifold of $\mathfrak{M}$ consisting of a finite number of irreducible closed analytic curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_k, \cdots\cdots$, each of which is a polar or a zero-point curve of $f(z^1, z^2)$ The formal sum

$$D = \sum m_k \Gamma_k$$

of these curves multiplied respectively by the multiplicity m_k of Γ_k is called the divisor of $f(z^1, z^2)$, where the multiplicities of the polar curves are to be associated with the negative sign. The divisor D of $f(z^1, z^2)$ can be also defined in case $f(z^1, z^2)$ is a many-valued meromorphic function, if the absolute value $|f(z^1, z^2)|$ is one-valued on $\mathfrak{M}$. Such a function is called multiplicative, since, if one prolongs $f(z^1, z^2)$ analytically along a closed continuous curve γ, then $f(z^1, z^2)$ is multiplied by a constant factor $\chi(\gamma)$ depending only on the homology class of γ on $\mathfrak{M}$. From the topological viewpoint, the divisor D is a 2-cycle on $\mathfrak{M}$. It can be readily proved that the divisor D of an arbitrary multiplicative meromorphic function on $\mathfrak{M}$ satisfies

$$D \approx 0 \text{ (homology with division allowed).}$$

Assume now that a cycle $D = \sum m_k \Gamma_k$ consisting of a finite number of irreducible closed analytic curves $\Gamma_1, \Gamma_2, \cdots\cdots$ is given. Then, does a multiplicative meromorphic function $f(z^1, z^2)$ on $\mathfrak{M}$ having D as its divisor exist? In what follows this fundamental question will be answered affirmatively under the assumption that a positive definite Hermitian metric

$$ds^2 = g_{\alpha\bar{\beta}}\, dz^\alpha d\bar{z}^\beta$$

without torsion is defined on $\mathfrak{M}$. In this note we shall give the main results and the outline of the proofs only. The detailed proofs and more systematic theory of analytic functions on analytic surfaces will be given elsewhere.

§2. Harmonic integrals. Putting

$$z^1 = x^1 + \sqrt{-1}\, x^2, \qquad z^2 = x^3 + \sqrt{-1}\, x^4,$$

we introduce the real coordinates x^1, x^2, x^3, x^4 on $\mathfrak{M}$. Then $\mathfrak{M}$ becomes a 4-dimensional closed Riemannian manifold with the positive definite metric

$$ds^2 = 2 g_{\alpha\bar{\beta}}\, dz^\alpha d\bar{z}^\beta = g_{jk}\, dx^j dx^k$$

(in what follows Latin subscripts j, k etc. take values ranging from 1 to 4 and Greek subscripts α, β denote 1 or 2).

Now we shall consider differential forms

$$\psi = \psi^p = \frac{1}{p!} \psi_{jk\cdots l} [dx^j dx^k \cdots dx^l]$$

defined on $\mathfrak{M}$, where p denotes the rank of ψ. The form ψ is said to be measurable, to have continuous derivatives or to be regular, if the coefficients $\psi_{jk\cdots l}$ are measurable, have continuous derivatives or regular analytic as functions of the local coordinates x^1, x^2, x^3, x^4. The derived form and the dual form of ψ will be denoted by $\mathfrak{r}^*\psi$ and $\mathfrak{d}\psi$, resp.; as is well known, they are defined by

$$\mathfrak{r}^*\psi^p = \frac{1}{p!} [d\psi_{jk\cdots l}\, dx^j dx^k \cdots dx^l],$$

$$\mathfrak{d}\psi^p = \frac{1}{p!(4-p)!} \frac{1}{\sqrt{g}} \operatorname{sgn}\begin{pmatrix} i \cdots j\, k \cdots l \\ 1\, 2 \cdots\cdots 4 \end{pmatrix}$$

$$\times\, g_{ip} \cdots g_{jq} \psi_{k\cdots l} [dx^p \cdots dx^q],$$

where $g = \det(g_{jk})$. The dual derivation $\mathfrak{r}$ and the Laplacian Δ are defined by

$$\mathfrak{r} = \mathfrak{d}\mathfrak{r}^*\mathfrak{d},$$

$$\Delta = -\mathfrak{r}\mathfrak{r}^* - \mathfrak{r}^*\mathfrak{r}.$$

We introduce furthermore the product

$$\psi^{\rho}\cdot\varphi^{\sigma} = \varphi^{\sigma}\cdot\psi^{\rho} = \frac{1}{(\rho-\sigma)!\,\sigma!}\,\psi_{j\cdots k\ell\cdots m}\,\varphi^{\ell\cdots m}\,[dx^{j}\cdots dx^{k}], \qquad (\sigma\leqq\rho)$$

and the inner product

$$(\psi,\ \varphi)_{G} = \int_{G}\psi\cdot\varphi\sqrt{g}\,dx^1dx^2dx^3dx^4,$$

where G means an arbitrary open subset of $\mathfrak{M}$. Especially if $G=\mathfrak{M}$, we write $(\psi,\ \varphi)$ for $(\psi,\ \varphi)_{\mathfrak{M}}$. Again we introduce the "absolute value"

$$|\psi(p)| = \sqrt{\psi(p)\cdot\psi(p)} = \left[\frac{1}{\rho!}\psi_{jk\cdots\ell}(p)\,\psi^{jk\cdots\ell}(p)\right]^{\frac{1}{2}}$$

of ψ at a point p in $\mathfrak{M}$ and, by its means, define the norm $\|\psi\|_{G}$ by

$$\|\psi\|_{G} = \int_{G}|\psi(p)|\sqrt{g}\,dx^1dx^2dx^3dx^4.$$

In case $G=\mathfrak{M}$, we write $\|\psi\|$ for $\|\psi\|_{\mathfrak{M}}$. We shall mean by $\psi\subset G$ that the closure of the set $\{p\,;\ |\psi(p)|\neq 0\}$ is contained in G.

A differential form ψ is said to be regular harmonic in a open subset G of $\mathfrak{M}$, if ψ admits continuous first derivatives and satisfies

$$\mathfrak{r}^{*}\psi = 0, \quad \mathfrak{r}\psi = 0$$

everywhere in G. By a harmonic form we shall mean a form ψ which is regular harmonic in $\mathfrak{M}$ except for a nowhere dense compact subset S of $\mathfrak{M}$; then ψ is said to be regular in $\mathfrak{M}-S$ and singular on S. Considered as a linear functional of variable chains C, the integral

$$\int_{C}\psi$$

of a harmonic form ψ is called a harmonic integral. If ψ is regular harmonic everywhere in $\mathfrak{M}$, then $\int\psi$ is called a harmonic integral of the 1st kind. Besides the theorem of Hodge[4)] concerning the harmonic integrals of the 1st kind, we need following two theorems from the theory of harmonic forms:

Theorem 1 (Principle of Orthogonal Projections[5)]). Let G be an open subset of $\mathfrak{M}$. If a measurable form ψ defined in G with $\|\psi\|_{G}<+\infty$ satisfies the "integral equations"

$$\begin{cases}(\psi,\ \mathfrak{r}^{*}\lambda)_{G} = 0,\\ (\psi,\ \mathfrak{r}\eta)_{G} = 0\end{cases}$$

for arbitrary forms $\lambda,\ \eta\subset G$ having continuous second derivatives, then ψ is regular harmonic in G.

Theorem 2 (Existence Theorem[5)]). For every ρ-chain C on $\mathfrak{M}$, there exists one and only one harmonic form $e[C]$ with $\|e[C]\|<+\infty$ satisfying the "integral equations"

$$(1)\qquad (e[C],\ \zeta) = \int_{C}\zeta, \qquad (\mathfrak{r}^{*}\zeta = 0),$$

$$(2)\qquad (e[C],\ \mathfrak{r}\eta) = 0,$$

where ζ is an arbitrary form with continuous first derivatives satisfying $\mathfrak{r}^{*}\zeta = 0$ and η an arbitrary form having continuous second derivatives. $e[C]$ is regular in $\mathfrak{M}$ except for the boundary of C. The period of the dual form

$$e^{*}[C] = \mathfrak{I}\,e[C]$$

on an arbitrary $(4-\rho)$-cycle Z is given by the formula

$$(3)\qquad \int_{Z}e^{*}[C] = I(Z,\ C),$$

where $I(Z,\ C)$ denotes the intersection number of Z and C.

For our purpose it is convenient to use the coordinates $z^1,\ \bar{z}^1,\ z^2,\ \bar{z}^2$ instead of $x^1,\ x^2,\ x^3,\ x^4$. Then the differential forms can be written as

$$\begin{cases}\psi^{1} = \psi_{\alpha}\,dz^{\alpha} + \psi_{\bar{\alpha}}\,d\bar{z}^{\alpha}\\ \psi^{2} = \frac{1}{2}\psi_{\alpha\beta}[dz^{\alpha}dz^{\beta}] + \psi_{\alpha\bar{\beta}}[dz^{\alpha}d\bar{z}^{\beta}] + \frac{1}{2}\psi_{\bar{\alpha}\bar{\beta}}[d\bar{z}^{\alpha}d\bar{z}^{\beta}]\\ \text{etc.}\end{cases}$$

Corresponding to this, the partial differentiations $\partial/\partial z^{\alpha}$, $\partial/\partial\bar{z}^{\alpha}$ are to be defined as

$$\begin{cases}\dfrac{\partial}{\partial z^1} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^1} - \sqrt{-1}\,\dfrac{\partial}{\partial x^2}\right),\\ \dfrac{\partial}{\partial\bar{z}^1} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x^1} + \sqrt{-1}\,\dfrac{\partial}{\partial x^2}\right),\\ \text{etc.}\end{cases}$$

Then, as one readily infers, a function $f(z^1,\bar{z}^1,z^2,\bar{z}^2)$ having continuous first derivatives is a regular analytic function of $z^1,\ z^2$ if and only if $\partial f/\partial\bar{z}^1$ and $\partial f/\partial\bar{z}^2$ vanish identically.

A differential form of the type $\Phi = \Phi_{\alpha}dz^{\alpha}$ will be called regular analytic in a subdomain G of $\mathfrak{M}$ if $\Phi_{\alpha} = \Phi_{\alpha}(z^1,\ z^2)$ are regular analytic functions of $z^1,\ z^2$ (whereas by a regular form we mean a differential form with coefficients which are regular analytic functions of real coordinates $x^1,\ x^2,\ x^3,\ x^4$). By using the hypothesis that the metric $ds^2 = 2g_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}$

has no torsion, it can be readily verified that every regular analytic form Φ satisfies automatically

$$\mathfrak{v}\Phi = 0.$$

Hence a regular analytic form Φ in G is regular harmonic if Φ satisfies $\mathfrak{v}^*\Phi = 0$. Thus the differential df of a regular analytic function $f = f(z^1, z^2)$ of z^1, z^2 is always a regular harmonic form. By an exact analytic form we shall mean a form $\Phi = \Phi_\alpha dz^\alpha$ which is regular analytic and satisfies $\mathfrak{v}^*\Phi = 0$ in $\mathfrak{M}$ except for a nowhere dense compact subset S of $\mathfrak{M}$ with $\dim S \leq 2$. The integral

$$\int \Phi_\alpha dz^\alpha$$

of an exact analytic form $\Phi_\alpha dz^\alpha$ is called a (simple) abelian integral. An abelian integral is said to be of the first, second or third kind according as it possesses no singularities, polar singularities only, or logarithmic singularities.

Lemma 1. A differential form of the type $\Phi = \Phi_\alpha dz^\alpha$ having continuous first derivatives is regular analytic in G if Φ satisfies $\mathfrak{v}^*\Phi = 0$ in G, G being an open subset of $\mathfrak{M}$.

Proof. From $\mathfrak{v}^*\Phi = 0$ follows $\partial\Phi_\alpha/\partial\bar{z}^\beta = 0$ for $\alpha, \beta = 1, 2$. Hence Φ is regular analytic, q.e.d.

Lemma 2. If a differential form ψ admits continuous second derivatives and satisfies $\Delta\psi = 0$ everywhere in $\mathfrak{M}$, then ψ is regular harmonic everywhere in $\mathfrak{M}$.

Lemma 3[7]. If $\varphi = \varphi_\alpha dz^\alpha + \varphi_{\dot\alpha} d\bar{z}^\alpha$ is regular harmonic in a subdomain G of $\mathfrak{M}$, then $\Phi = \varphi_\alpha dz^\alpha$ satisfies $\Delta\Phi = 0$ in G.

Combining this with Lemma 1 and Lemma 2, we infer the following theorem of W.V.D. Hodge:

Theorem 3. If $\varphi = \varphi_\alpha dz^\alpha + \varphi_{\dot\alpha} d\bar{z}^\alpha$ is regular harmonic everywhere in $\mathfrak{M}$, then the integral

$$\int \varphi_\alpha dz^\alpha$$

is an abelian integral of the first kind.

§3. Abelian integrals of the third kind. A compact subset Γ of $\mathfrak{M}$ will be called a closed analytic curve, if, for every point $\mathfrak{p} \in \Gamma$, there exists a regular analytic function $f_\mathfrak{p}(z^1, z^2)$ defined in a neighbourhood $N(\mathfrak{p})$ of $\mathfrak{p}$ such that Γ coincides in $N(\mathfrak{p})$ with the zero-point manifold of $f_\mathfrak{p}(z^1, z^2)$; then

$$f_\mathfrak{p}(z^1, z^2) = 0$$

will be called the local equation of Γ at $\mathfrak{p}$. Choose the local coordinates z^1, z^2 so that $\mathfrak{p}$ coincides with the origin $(0, 0)$. Then the set of all holomorphic functions

$$h(z^1, z^2) = c_0 + \sum c_\alpha z^\alpha + \frac{1}{2}\sum c_{\alpha\beta} z^\alpha z^\beta + \cdots$$

defined in some neighbourhoods of $\mathfrak{p} = (0, 0)$ constitutes a ring $\mathfrak{o}_\mathfrak{p}$ without null divisor, in which every $h(z^1, z^2)$ with $h(0, 0) \neq 0$ is considered as a unit. As an element of $\mathfrak{o}_\mathfrak{p}$, $f_\mathfrak{p}(z^1, z^2)$ is decomposed into the power product

$$f_\mathfrak{p}(z^1, z^2) = U(z^1, z^2) \cdot \prod \{f^{(j)}(z^1, z^2)\}^{m_j}, \quad (U(0, 0) \neq 0)$$

of irreducible factors $f^{(j)}(z^1, z^2)$, where the decomposition is unique up to the unit $U(z^1, z^2)$ in $\mathfrak{o}_\mathfrak{p}$ [9]. In accordance with this, Γ is decomposed in $N(\mathfrak{p})$ into the sum

$$\Gamma'_\mathfrak{p} \cup \Gamma''_\mathfrak{p} \cup \cdots\cdots \cup \Gamma^{(j)}_\mathfrak{p} \cup \cdots\cdots$$

of the branches $\Gamma^{(j)}_\mathfrak{p}$ which are the zero-point manifolds of the factors $f^{(j)}(z^1, z^2)$. Obviously the local equation $f_\mathfrak{p}(z^1, z^2) = 0$ of Γ at $\mathfrak{p}$ is equivalent to

$$f^*_\mathfrak{p}(z^1, z^2) \equiv \prod f^{(j)}(z^1, z^2) = 0,$$

which will be called the minimal local equation of Γ at $\mathfrak{p}$. The minimal local equation is characterized by the following property: If $f^*_\mathfrak{p}(z^1, z^2)$ is represented as the product

$$f^*_\mathfrak{p}(z^1, z^2) = g(z^1, z^2) \cdot h(z^1, z^2)$$

of two functions g, h in $\mathfrak{o}_\mathfrak{p}$ and $g(z^1, z^2)$ vanishes on Γ, then h is a unit in $\mathfrak{o}_\mathfrak{p}$, i.e. $h(0, 0) \neq 0$. By a suitable choice of the local coordinates z^1, z^2, each branch $\Gamma^{(j)}_\mathfrak{p}$ can be represented as

$$(4) \quad \begin{cases} z^1 = t^\nu \\ z^2 = c_0 t^\mu + c_1 t^{\mu'} + c_2 t^{\mu''} + \cdots, \quad (c_k \neq 0) \end{cases}$$

where t means the local uniformization variable on $\Gamma^{(j)}_\mathfrak{p}$ and $\nu, \mu, \mu', \mu'', \cdots\cdots$ are positive integers having no common divisor such that $0 < \nu < \mu < \mu' < \mu'' < \cdots$. The exponent ν in (4) is called the multiplicity of $\mathfrak{p}$ with respect to the branch $\Gamma^{(j)}_\mathfrak{p}$. If, in $N(\mathfrak{p})$, Γ consists of a single branch $\Gamma'_\mathfrak{p}$ and $\mathfrak{p}$ has the multiplicity 1 with respect to $\Gamma'_\mathfrak{p}$, then $\mathfrak{p}$ is called a simple point of Γ; otherwise $\mathfrak{p}$ is a singular point. Obviously Γ has only a finite number of singular points. A closed analytic curve Γ is said to be irreducible, if Γ can not be decomposed into the sum of two closed analytic curves $\Gamma', \Gamma'' \neq \Gamma$. An irreducible closed analytic curve Γ can be considered as a closed Riemann surface with the local uniformization variables t introduced in (4). Evidently a closed analytic curve is decomposed into the sum of irreducible closed analytic curves.

Let $D = \sum m_k \Gamma_k$ be a cycle with integral coefficients in $\mathfrak{M}$ consisting of a finite number of irreducible closed analytic curves Γ_k, and let

$$f_{kp}(z^1, z^2) = 0$$

be the minimal local equations of Γ_k at p (if p does not lie on Γ_k, we have to put $f_{kp} \equiv 1$). The analytic curve composed of $\Gamma_1, \Gamma_2, \cdots, \Gamma_k, \cdots$ will be denoted by $|D|$. Then we have

Theorem 4. If $D = \sum m_k \Gamma_k \approx 0$ (homology with division allowed), then there exists on $\mathfrak{M}$ one and only one exact analytic form $\varphi^D_\alpha dz^\alpha$ such that, for every $p \in \mathfrak{M}$, $\varphi^D_\alpha dz^\alpha$ is represented as

$$(5) \quad \varphi^D_\alpha dz^\alpha = \sum m_k \, d \log f_{kp}$$
$$+ \text{ regular analytic form}$$

in some neighbourhood $N(p)$ of p and that the integral

$$(6) \qquad \mathcal{R}e \int \varphi^D_\alpha dz^\alpha$$

is one-valued on $\mathfrak{M}$.

Proof. By hypothesis, D is the boundary of a 3-chain C on $\mathfrak{M}$:

$$D = \mathfrak{v} C.$$

Consider now the harmonic form $e^*[C] = \mathcal{Y} e[C]$ introduced in Theorem 2. Then it follows from (1) and (2) that $e^*[C]$ satisfies

$$(1)^* \quad (e^*[C], \ \mathfrak{v}\eta) = \int_D \mathcal{Y}\eta,$$

$$(2)^* \quad (e^*[C], \ \mathfrak{v}^*\lambda) = 0$$

for arbitrary differential forms $\eta = \eta^2$, $\lambda = \lambda^0$. In order to determine the "singularities" of $e^*[C]$ on $|D|$, we fix an arbitrary point p on $|D|$ and compare $e^*[C]$ with the harmonic form

$$\sigma_p = \frac{1}{2\pi} \mathfrak{I} \sum m_k \, d \log f_{kp}$$

defined in $N(p)$. Assume first that p is a simple point of $|D|$. Then it can be readily verified by a simple calculation that σ_p satisfies

$$\begin{cases} (\sigma_p, \ \mathfrak{v}\eta) = \int_D \mathcal{Y}\eta, \\ (\sigma_p, \ \mathfrak{v}^*\lambda) = 0 \end{cases}$$

for arbitrary differential forms η, λ with $\eta \subset N(p)$, $\lambda \subset N(p)$. We get therefore

$$(e^*[C] - \sigma_p, \ \mathfrak{v}\eta) = 0,$$

$$(e^*[C] - \sigma_p, \ \mathfrak{v}^*\lambda) = 0.$$

Hence, by Theorem 1, $e^*[C] - \sigma_p$ is regular harmonic in $N(p)$.

Secondly, we consider the case that p is a singular point of $|D|$. Then, since, for a sufficiently small neighbourhood $N(p)$ of p, every point $q \in |D| \wedge N(p)$, $q \neq p$, is a simple point of $|D|$, it follows from above result that $e^*[C] - \sigma_p$ is regular harmonic in $N(p) - p$. To prove that $e^*[C] - \sigma_p$ is regular harmonic also in p, it is sufficient to consider the case that p lies on a single branch Γ'_p of $|D|$. In $N(p)$, $e[C]$ is represented as

$$e[C] = \frac{1}{2} \int_{\Gamma'_p} \mathfrak{v}^* \Xi_{pq}(x, \xi) [d\xi^p d\xi^q]$$
$$+ \text{ regular form},$$

where $\Xi_{pq}(x, \xi) = \frac{1}{2} \Xi_{jk;pq}(x, \xi) [dx^j dx^k]$ means the elementary solution of Laplace's equation $\Delta \Xi = 0$ [10]. Choose the local coordinates z^1, z^2 with the origin p so that the parametric representation of Γ'_p takes the form (4). Then, using the explicit expression of $e[C]$ mentioned above, we infer readily

$$|e[C]| = O\left(\frac{1}{|z^2|}\right), \qquad \text{for } z^1 = 0,$$

while it is obvious that

$$|\sigma_p| = O\left(\frac{1}{|z^2|}\right), \qquad \text{for } z^1 = 0.$$

The difference

$$\delta = e^*[C] - \sigma_p$$

satisfies therefore also the inequality

$$(7) \qquad |\delta| = O\left(\frac{1}{|z^2|}\right), \qquad \text{for } z^1 = 0.$$

On the other hand, δ is regular harmonic in $N(p) - p$ and satisfies

$$\|\delta\|_{N(p)} < +\infty.$$

δ is therefore represented as

$$\delta(x) = c_0 \cdot d\Xi(0, x) + \text{ regular form},$$

where $\Xi(x, \xi)$ means the elementary scalar solution of $\Delta \Xi = 0$. Now (7) shows that the coefficient c_0 in this expression must vanish, since $|\Xi(0, x)| \sim \{|z^1|^2 + |z^2|^2\}^{-1}$. Consequently $\sigma = e^*[C] - \sigma_p$ is regular harmonic also in p. Thus we conclude that, for every point $p \in \mathfrak{M}$, $e^*[C]$ is represented as

$$(8) \quad e^*[C] = \sigma_p + \text{regular harmonic form}$$

in a neighbourhood $N(p)$ of p.

Now we write $4\pi\sqrt{-1}\, e^*[C]$ as

$$4\pi\sqrt{-1}\, e^*[C] = \varphi_\alpha dz^\alpha - \bar{\varphi}_\alpha d\bar{z}^\alpha$$

and put

$$\Phi = \varphi_\alpha dz^\alpha.$$

Then, since

$$4\pi\sqrt{-1}\,\sigma_p = \sum m_k\, d\log f_{kp} - \sum m_k\, \overline{d\log f_{kp}},$$

it follows from (8) that

$$(9) \qquad \Phi - \sum m_k\, d\log f_{kp}$$

is regular (with respect to real coordinates x^1, x^2, x^3, x^4) in $N(p)$. This shows that $\mathfrak{r}^*\Phi$ is regular everywhere in $\mathfrak{M}$. On the other hand, Φ satisfies, by Lemma 3, $\Delta\Phi = 0$ in $\mathfrak{M} - |D|$, whence we get

$$\Delta \mathfrak{r}^*\Phi = \mathfrak{r}^*\Delta\Phi = 0.$$

Thus $\mathfrak{r}^*\Phi$ satisfies $\Delta\mathfrak{r}^*\Phi = 0$ everywhere in $\mathfrak{M}$. Consequently, by Lemma 2, $\mathfrak{r}^*\Phi$ is a harmonic form of the first kind. To prove $\mathfrak{r}^*\Phi = 0$, it is sufficient therefore to show

$$(10) \qquad \int_Z \mathfrak{r}^*\Phi = 0$$

for every 2-cycle Z. Assuming that Z meets with $|D|$ only in a finite number of simple points of $|D|$, we infer from the fact that the difference (9) is regular in $N(p)$ the equality

$$\int_Z \mathfrak{r}^*\Phi = 2\pi\sqrt{-1}\, I(Z, D),$$

where $I(Z, D)$ means the intersection number of Z and D. Now, since $D \approx 0$, we have $I(Z, D) = 0$, proving (10). Thus we get $\mathfrak{r}^*\Phi = 0$, and consequently, by Lemma 1, $\Phi = \varphi_\alpha dz^\alpha$ is an exact analytic form. It is obvious that the exact analytic form $\varphi_\alpha dz^\alpha$ thus obtained can be represented as (5), but the integral $\mathfrak{Re}\int \varphi_\alpha dz^\alpha$ is not necessarily one-valued.

Since

$$\mathfrak{Re}\, \varphi_\alpha dz^\alpha = \frac{1}{2\pi} \sum m_k\, d\log|f_{kp}| + \text{regular harmonic form}$$

in $N(p)$, the harmonic integral $\mathfrak{Re}\int \varphi_\alpha dz^\alpha$ is locally one-valued. The period

$$\mathfrak{Re}\int_\gamma \varphi_\alpha dz^\alpha$$

on a 1-cycle γ depends only on the homology class of γ. Hence, by a theorem of Hodge, there exists a real harmonic form

$$q = \kappa_\alpha dz^\alpha + \bar{\kappa}_\alpha d\bar{z}^\alpha$$

of the first kind such that

$$(11) \qquad 2\mathfrak{Re}\int_\gamma \varphi_\alpha dz^\alpha = \int_\gamma q$$

for all 1-cycle γ. On the other hand, by Theorem 3, $\kappa_\alpha dz^\alpha$ is an everywhere regular, exact analytic form. Hence, putting

$$\varphi_\alpha^D dz^\alpha = \varphi_\alpha dz^\alpha - \kappa_\alpha dz^\alpha$$

we obtain an exact analytic form $\varphi_\alpha^D dz^\alpha$ having the properties mentioned in Theorem 4, while the uniqueness of such $\varphi_\alpha^D dz^\alpha$ is obvious, q.e.d.

Now we shall evaluate the integral $\mathfrak{I}\int_\gamma \varphi_\alpha^D dz^\alpha$. First from (3) follows

$$(12) \qquad \mathfrak{I}\int_\gamma \varphi_\alpha dz^\alpha = 2\pi\, I(\gamma, C).$$

To calculate $\mathfrak{I}\int_\gamma \kappa_\alpha dz^\alpha$, we introduce the harmonic form

$$\omega = g_{\alpha\bar\beta}[dz^\alpha d\bar{z}^\beta].^{11)}$$

Then, using the identity

$$\frac{1}{2}\omega \cdot \mathcal{J}(\psi_\alpha dz^\alpha - \bar\psi_\alpha d\bar{z}^\alpha) = \psi_\alpha dz^\alpha + \bar\psi_\alpha d\bar{z}^\alpha,$$

we get

$$(13) \qquad \mathfrak{I}\int_\gamma \kappa_\alpha dz^\alpha = \frac{1}{4\sqrt{-1}}\int_\gamma \omega\cdot\mathcal{J}q,$$

$$(14) \qquad \mathfrak{Re}\int_\gamma \varphi_\alpha dz^\alpha = -\pi\sqrt{-1}\int_\gamma \omega\cdot e[C]$$

Combined with (11), (14) yields

$$\int_\gamma \{2\pi\sqrt{-1}\,\omega\cdot e[C] + q\} = 0,$$

proving that q is represented as

$$(15) \qquad q = -2\pi\sqrt{-1}\,\omega\cdot e[C] + \mathfrak{r}^*\theta,$$

where θ is a scalar harmonic function having logarithmic singularities on $|D|$. Now, using (1), (13) and (15), we get readily

$$\mathfrak{J}\int_{\gamma} \kappa_{\alpha}\, dz^{\alpha} = -2\pi \int_{C} e^{*}[\gamma].$$

Thus we conclude:

Theorem 5. For an arbitrary 1-cycle γ on $\mathfrak{M}$, we have

$$(16)\qquad \int_{\gamma} \varphi^{D}_{\alpha}\, dz^{\alpha} = 2\pi\sqrt{-1}\left\{ I(\gamma, C) + \int_{C} e^{*}[\gamma] \right\},$$

where C means a 3-chain whose boundary is D : $\partial C = D$.

Obviously $\int \varphi^{D}_{\alpha}\, dz^{\alpha}$ is an abelian integral of the third kind.

§4. Main theorem. The existence of the multiplicative meromorphic function having the given divisor D follows now immediately from Theorem 4. Indeed, putting

$$F^{D}(z^1, z^2) = \exp\left[\int^{(z^1, z^2)} \varphi^{D}_{\alpha}\, dz^{\alpha} \right]$$

we obtain from Theorem 4 the following

Theorem 6 (Main Theorem). Let $D = \sum m_k \Gamma_k$ be a cycle ≈ 0 on $\mathfrak{M}$ consisting of irreducible closed analytic curves Γ_k with the minimal local equations

$$f_{kp}(z^1, z^2) = 0.$$

Then there exists one and only one multiplicative meromorphic function $F^{D}(z^1, z^2)$ on $\mathfrak{M}$ such that, for every point $p \in \mathfrak{M}$, $F^{D}(z^1, z^2)$ is represented in a neighbourhood $N(p)$ of p as

$$F^{D}(z^1, z^2) = U_{p}(z^1, z^2) \cdot \prod_{k} \{ f_{kp}(z^1, z^2) \}^{m_k},$$

$$(U_{p}(z^1, z^2) \neq 0),$$

where $U_{p}(z^1, z^2)$ is a holomorphic function defined in $N(p)$ not vanishing at p.

Again, from Theorem 5 follows the following

Theorem 7. If one prolongs $F^{D}(z^1, z^2)$ along an arbitrary closed continuous curve γ, then $F^{D}(z^1, z^2)$ is multiplied by the factor

$$\chi^{D}(\gamma) = \exp 2\pi\sqrt{-1}\left\{ I(\gamma, C) + \int_{C} e^{*}[\gamma] \right\},$$

$$(\partial C = D).$$

Corollary. In order that a cycle $D = \sum m_k \Gamma_k \approx 0$ is the divisor of a one-valued meromorphic function on $\mathfrak{M}$, it is necessary and sufficient that

$$I(\gamma, C) + \int_{C} e^{*}[\gamma] \equiv 0 \pmod 1$$

for all 1-cycle γ.

(※) Received March 1, 1949.

(1) Weyl [7] Numbers in bracket refer to the bibliography at the end of the paper.
(2) Behnke and Thullen [1], Kap. V.
(3) Hodge [2], Kodaira [4], [5].
(4) Hodge [2].
(5) Kodaira [4], Theorem 9.
(6) Kodaira [4], Theorem 16
(7) Hodge [2], [3].
(8) Hodge [2], [3].
(9) Behnke and Thullen [1], Kap. V, Osgood [6], Kap. II.
(10) Kodaira [4], Chap. III.
(11) Hodge [2].

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