Y. KAMIYAMA
KODAI MATH. J.
23 (2000), 380–390

CHERN NUMBERS OF THE MODULI SPACE OF SPATIAL POLYGONS

YASUHIKO KAMIYAMA

Abstract

Let $\mathscr{M}_n$ $(n \geq 3)$ be the moduli space of spatial polygons with n edges. We consider the case of odd n. First we establish a procedure to determine the Chern numbers of $\mathscr{M}_n$. Next we follow the procedure and get a description of $\mathscr{M}_n$ $(n \leq 9)$ in the complex cobordism group Ω_{2n-6}^U. Finally we determine some characteristic numbers of $\mathscr{M}_n$. In particular, we calculate the Todd genus of $\mathscr{M}_n$ by showing that $\mathscr{M}_n$ is birationally equivalent to $\boldsymbol{C}P^{n-3}$.

1. Introduction

Let $\mathscr{M}_n$ $(n \geq 3)$ be the moduli space of spatial polygons $P = (a_1, a_2, \dots, a_n)$ whose edges are vectors $a_i \in \boldsymbol{R}^3$ of length $|a_i| = 1$ $(1 \leq i \leq n)$. Two polygons are identified if they differ only by motions in $\boldsymbol{R}^3$. The sum of the vectors is assumed to be zero. Thus:

$$\mathscr{M}_n = \{P = (a_1, \dots, a_n) \in (S^2)^n : a_1 + \cdots + a_n = 0\}/SO(3). \tag{1.1}$$

It is known that $\mathscr{M}_n$ is a Kähler manifold of complex dimension $n-3$. For odd n or $n = 4$, $\mathscr{M}_n$ has no singular points. For even n with $n \geq 6$, $P = (a_1, a_2, \dots, a_n)$ is a singular point if and only if all the a_i $(1 \leq i \leq n)$ lie on a line in $\boldsymbol{R}^3$ through O. Such singular points are cone-like singularities and have neighborhoods $C(S^{n-3} \times_{S^1} S^{n-3})$, where C denotes the cone and S^1 acts on both copies of S^{n-3} by complex multiplication (see for example [8]).

For odd n, the module $H_*(\mathscr{M}_n; \boldsymbol{R})$ was determined by Kirwan and Klyachko [10], [12]. Later the cohomology ring $H^*(\mathscr{M}_n; \boldsymbol{R})$ was determined by Brion and Kirwan [1], [11] (cf. Theorem 2.2). In particular $H^*(\mathscr{M}_n; \boldsymbol{R})$ is generated by certain two dimensional cohomology classes.

In contrast to this, for even n, $H_*(\mathscr{M}_n; \boldsymbol{R})$ is complicated and is not generated by two dimensional cohomology classes nor does not obey Poincaré duality [5]. The cohomology ring $H^*(\mathscr{M}_n; \boldsymbol{R})$ is not yet known.

1991 *Mathematics Subject Classification*: 57R20, 57R77
Key words and phrases: spatial polygon, Chern number, complex cobordism.
Received September 3, 1999.

For the rest of this paper, we assume n to be odd and set $n = 2m+1$. In [6], we described $\mathscr{M}_n$ in the oriented cobordism group Ω^{SO}_{2n-6}. The result is that $\mathscr{M}_n$ is oriented cobordant to $(-1)^{m+1}\binom{2m-1}{m}\boldsymbol{C}P^{2m-2}$. Here $\binom{2m-1}{m}$ denotes the binomial coefficient, and we give an orientation to $\mathscr{M}_n$ which is induced from the complex structure. The proof of this fact is carried out by constructing an oriented manifold with boundary, which gives the desired cobordism. However, such a method seems difficult when we describe $\mathscr{M}_n$ in the complex cobordism group Ω^U_{2n-6}, where $2n-6$ denotes the real dimension.

The main topic of this paper is a description of $\mathscr{M}_n$ in Ω^U_{2n-6}. Since $\mathscr{M}_3 = \{\text{point}\}$, the problem is trivial for $\mathscr{M}_3$. Hence for the rest of this paper, we assume n to be odd ≥ 5. Recall that Ω^U_* is determined by the Chern numbers [14]. Hence the problem is essentially to determine the Chern numbers of $\mathscr{M}_n$. In fact, we have a procedure to determine such numbers:

THEOREM A. *We have a procedure to determine the Chern numbers of* $\mathscr{M}_n$.

For more details of Theorem A, see Section 2. The key theorems for Theorem A are as follows.

(i) First we give the ring structure of $H^*(\mathscr{M}_n;\boldsymbol{R})$ in Theorem 2.2. In particular, $H^*(\mathscr{M}_n;\boldsymbol{R})$ is generated by certain two dimensional cohomology classes $z_1,\dots,z_n \in H^2(\mathscr{M}_n;\boldsymbol{R})$.

(ii) Next for a sequence $(d_1,\dots,d_n)$ of nonnegative integers with $\sum_{i=1}^n d_i = n-3$, we give the intersection number $\langle z_1^{d_1}\cdots z_n^{d_n}, \mu_{\mathscr{M}_n}\rangle$ in Theorems 2.5 and 2.6, where $\mu_{\mathscr{M}_n}$ denotes the fundamental homology class determined by the orientation which is induced from the complex structure on $\mathscr{M}_n$.

(iii) Finally we describe $c(\mathscr{M}_n)$, the total Chern class of the tangent bundle of $\mathscr{M}_n$, in terms of $z_1,\dots,z_n \in H^2(\mathscr{M}_n;\boldsymbol{R})$ (cf. (i)) in Theorem 2.8.

From (iii), we can describe $c_i(\mathscr{M}_n)$ in terms of $z_1,\dots,z_n$ (cf. Theorems 2.8 and 2.12). Then for each partition $I = i_1,\dots,i_r$ of $n-3$, the I-th Chern number $c_I[\mathscr{M}_n] = c_{i_1,\dots,i_r}[\mathscr{M}_n]$ is determined from (ii). (As usual, we set $c_{i_1,\dots,i_r}[\mathscr{M}_n] = \langle c_{i_1}(\mathscr{M}_n)\cdots c_{i_r}(\mathscr{M}_n), \mu_{\mathscr{M}_n}\rangle$.)

Theorem A is effective for the calculations of the Chern numbers of $\mathscr{M}_n$. In fact, we give the results for $n \leq 9$ in Theorem 3.1. Using these results, we have the following:

THEOREM B. (i) *In* Ω^U_4, *we have*

$$[\mathscr{M}_5] = 4[\boldsymbol{C}P^1\times\boldsymbol{C}P^1] - 3[\boldsymbol{C}P^2].$$

(ii) *In* Ω^U_8, *we have*

$$[\mathscr{M}_7] = -9[(\boldsymbol{C}P^1)^4] + 33[(\boldsymbol{C}P^1)^2\times\boldsymbol{C}P^2] - 33[\boldsymbol{C}P^1\times\boldsymbol{C}P^3] + 0[(\boldsymbol{C}P^2)^2] + 10[\boldsymbol{C}P^4].$$

(iii) *In* $\Omega_{12}^U \otimes \boldsymbol{Q}$, *we have*

$$\begin{aligned}[\mathcal{M}_9] = {} & 43[(\boldsymbol{CP}^1)^6] - \frac{668}{3}[(\boldsymbol{CP}^1)^4 \times \boldsymbol{CP}^2] + 234[(\boldsymbol{CP}^1)^3 \times \boldsymbol{CP}^3] \\ & + 220[(\boldsymbol{CP}^1)^2 \times (\boldsymbol{CP}^2)^2] - 220[(\boldsymbol{CP}^1)^2 \times \boldsymbol{CP}^4] + \frac{440}{3}[\boldsymbol{CP}^1 \times \boldsymbol{CP}^5] \\ & - 220[\boldsymbol{CP}^1 \times \boldsymbol{CP}^2 \times \boldsymbol{CP}^3] + 0[(\boldsymbol{CP}^2)^3] + 0[\boldsymbol{CP}^2 \times \boldsymbol{CP}^4] \\ & + 55[(\boldsymbol{CP}^3)^2] - 35[\boldsymbol{CP}^6].\end{aligned}$$

Remark 1.2. The rational coefficients $-668/3$ and $440/3$ in Theorem B (iii) are due to the fact that $[\boldsymbol{CP}^5] \in \Omega_{10}^U$ is not a ring generator of Ω_*^U [14]. Instead of $[\boldsymbol{CP}^5]$, if we use x_5 defined by $x_5 = [\boldsymbol{CP}^5] + [H_{3,3}] - [H_{2,4}]$, then we obtain a description of $[\mathcal{M}_9]$ in Ω_{12}^U. For more details, see Remark 3.2.

Finally we consider the case of general odd n. Note that for each partition $I = i_1, \dots, i_r$ of $n-3$, the I-th Chern number $c_I[\mathcal{M}_n]$ is defined. In this paper, instead of giving all the $c_I[\mathcal{M}_n]$, we give some characteristic numbers of $\mathcal{M}_n$.

Recall that for a compact, complex k-dimensional manifold M, the Todd genus $T[M]$ and a certain, well-known integral combination of the Chern numbers $s_k[M]$ are defined as follows (see for example [13]). First let $\{T_k\}$ be the multiplicative sequence of polynomials belonging to the power series $f(x) = x/(1-e^{-x})$. Then the Todd genus $T[M]$ is defined to be the characteristic number $\langle T_k(c_1(M), \dots, c_k(M)), \mu_M \rangle$.

Next let σ_i denote the i-th elementary symmetric polynomial in variables $t_1, \dots, t_k$, and let $s_k(\sigma_1, \dots, \sigma_k)$ denote the polynomial in σ_i which express the sum $t_1^k + \cdots + t_k^k$. Then $s_k[M]$ is defined to be the characteristic number $\langle s_k(c_1(M), \dots, c_k(M)), \mu_M \rangle$. (The characteristic number $s_k[M]$ is important. For example if $s_k[M] \neq 0$, then M cannot be expressed non-trivially as a product of complex manifolds.)

Then we have the following results on characteristic numbers of $\mathcal{M}_n$. As before, we set $n = 2m+1$.

Theorem C. (i) *We have*

$$c_1^{n-3}[\mathcal{M}_n] = \sum_{j=0}^{m-1} (-1)^j \binom{2m}{j} (2m-1-2j)^{2m-2}.$$

(ii) *We have*

$$c_{n-3}[\mathcal{M}_n] = -2^{2m-1} + (2m+1)\binom{2m-1}{m}.$$

(iii) *We have*

$$T[\mathcal{M}_n] = 1.$$

(iv) *We have*

$$s_{n-3}[\mathcal{M}_n] = (-1)^{m+1}(2m-1)\binom{2m-1}{m}.$$

In fact we can deduce Theorem C (iii) from the following stronger assertion, which may be of interest in their own right (cf. Assertion 4.1): $\mathcal{M}_n$ is birationally equivalent to $\boldsymbol{C}P^{n-3}$. And as examples for other Chern numbers, we give $c_1c_{n-4}[\mathcal{M}_n]$ and $c_2c_{n-5}[\mathcal{M}_n]$ in Theorem 4.5.

Remark 1.3. (i) It is known that $\mathcal{M}_n$ admits a symplectic structure [8], [12]. Let ω_n be the symplectic form on $\mathcal{M}_n$. Then it is known that $[\omega_n] = c_1(\mathcal{M}_n) \in H^2(\mathcal{M}_n; \boldsymbol{R})$ [2] (cf. Remark 2.13). Thus Theorem C (i) gives the symplectic volume $\langle \omega_n^{n-3}, \mu_{\mathcal{M}_n} \rangle$ of $\mathcal{M}_n$.

(ii) From the fact that $\mathcal{M}_n$ is oriented cobordant to $(-1)^{m+1}\binom{2m-1}{m} \cdot \boldsymbol{C}P^{2m-2}$ [6], we can determine all the Pontrjagin numbers and the Stiefel-Whitney numbers of $\mathcal{M}_n$.

This paper is organized as follows. In Section 2, we study Theorem A in detail. We explain how to compute the Chern numbers of $\mathcal{M}_n$ according to the steps (i), (ii) and (iii) in this section. In Section 3 we prove Theorem B, and in Section 4 we prove Theorem C.

2. Procedure for the Chern numbers of $\mathcal{M}_n$

In this section, we study Theorem A in detail. First we recall the structure of $H^*(\mathcal{M}_n; \boldsymbol{R})$ for odd n, which was determined by Brion and Kirwan [1], [11]. For $i \in \{1, \dots, n\}$, we define $A_{n,i} \subset (\boldsymbol{R}^3)^n$ by

$$A_{n,i} = \left\{ P = (a_1, \dots, a_n) \in (S^2)^n : a_1 + \cdots + a_n = 0 \text{ and } a_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Let $SO(2)$ act on $\boldsymbol{R}^3$ by rotation about the z-axis. Then for odd n, the diagonal $SO(2)$-action on $(\boldsymbol{R}^3)^n$ is free on $A_{n,i}$ and we have $\mathcal{M}_n = A_{n,i}/SO(2)$ (cf. (1.1)). Therefore, $A_{n,i} \to \mathcal{M}_n$ is a principal $SO(2)$-bundle. Let $\xi_i \to \mathcal{M}_n$ be a complex line bundle associated with $A_{n,i} \to \mathcal{M}_n$:

$$\xi_i = (A_{n,i} \times \boldsymbol{C})/S^1,$$

where we identify $SO(2)$ with S^1 and let S^1 act on $A_{n,i} \times \boldsymbol{C}$ by

$$(P, \alpha) \cdot g = (Pg, \alpha g), \quad (P, \alpha) \in A_{n,i} \times \boldsymbol{C}, \quad g \in S^1.$$

Then we define $z_i \in H^2(\mathcal{M}_n; \boldsymbol{R})$ to be the Chern class of ξ_i:

$$z_i = c_1(\xi_i), \quad 1 \le i \le n. \tag{2.1}$$

Now we have the following theorem.

THEOREM 2.2 [1], [11]. *When* $n = 2m+1$, *the algebra* $H^*(\mathcal{M}_n; \boldsymbol{R})$ *is generated by* $z_1, \dots, z_n$ *with the relations*:

(i) $z_1^2 = \cdots = z_n^2$.

(ii) $\prod_{j \in J}(z_i + z_j) = 0$, *for all* $1 \le i \le n$ *and* $J \subset \{1, \dots, n\}$ *such that* $i \notin J$ *and* card $(J) = m$, *where card denotes the cardinal.*

Next we study the intersection numbers. For a sequence $(d_1, \dots, d_n)$ of nonnegative integers with $\sum_{i=1}^n d_i = n-3$, we define $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ by

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle z_1^{d_1} \cdots z_n^{d_n}, \mu_{\mathcal{M}_n} \rangle, \tag{2.3}$$

where $z_i \in H^2(\mathcal{M}_n; \boldsymbol{R})$ $(1 \le i \le n)$ is defined in (2.1), and $\mu_{\mathcal{M}_n}$ denotes the fundamental homology class of $\mathcal{M}_n$. Thus we need to determine $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ for all $(d_1, \dots, d_n)$. To do this, we consider the following types of $(d_1, \dots, d_n)$. As before, we set $n = 2m+1$.

(i) $d_1 = \cdots = d_{n-3} = 1$ and $d_{n-2} = d_{n-1} = d_n = 0$.

(ii) $d_1 = 2k$, $d_2 = \cdots = d_{n-2k-2} = 1$ and $d_{n-2k-1} = \cdots = d_n = 0$, where $1 \le k \le m-1$ and $n = 2m+1$.

If $(d_1, \dots, d_n)$ is of the type (i), then we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ by $\langle \rho_{n,0} \rangle$. On the other hand, if $(d_1, \dots, d_n)$ is of the type (ii), then we write $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ by $\langle \rho_{n,2k} \rangle$. Thus:

$$\begin{cases} \langle \rho_{n,0} \rangle = \langle z_1 \cdots z_{n-3}, \mu_{\mathcal{M}_n} \rangle \\ \langle \rho_{n,2k} \rangle = \langle z_1^{2k} z_2 \cdots z_{n-2k-2}, \mu_{\mathcal{M}_n} \rangle \quad (1 \le k \le m-1). \end{cases} \tag{2.4}$$

For a sequence $(d_1, \dots, d_n)$ of nonnegative integers with $\sum_{i=1}^n d_i = n-3$, we set $d_i = 2\alpha_i + \varepsilon_i$ $(1 \le i \le n)$, where $\varepsilon_i = 0$ or 1. Then we have the following:

THEOREM 2.5 [7]. *We have the following relations in* $H^*(\mathcal{M}_n; \boldsymbol{R})$.

(i) *If* $\alpha_i = 0$ *for* $1 \le i \le n$, *then we have*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \rho_{n,0} \rangle.$$

(ii) *If* $\alpha_i \ne 0$ *for some* i, *then we have*

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \langle \rho_{n,2(\alpha_1 + \cdots + \alpha_n)} \rangle.$$

Thus it suffices to determine $\langle \rho_{n,2k} \rangle$ $(0 \le k \le m-1)$ in order to determine the intersection numbers. About this, we have the following theorem.

THEOREM 2.6 [7]. *When* $n = 2m+1$, *the number* $\langle \rho_{n,2k} \rangle$ $(0 \le k \le m-1)$ *is given as follows.*

$$\langle \rho_{n,2k} \rangle = (-1)^k \frac{\binom{m-1}{k}\binom{2m-1}{m}}{\binom{2m-1}{2k+1}}.$$

Example 2.7. We have the following examples:
(i) $\mathscr{M}_5 : \langle \rho_{5,0} \rangle = 1$ and $\langle \rho_{5,2} \rangle = -3$.
(ii) $\mathscr{M}_7 : \langle \rho_{7,0} \rangle = 2$, $\langle \rho_{7,2} \rangle = -2$ and $\langle \rho_{7,4} \rangle = 10$.
(iii) $\mathscr{M}_9 : \langle \rho_{9,0} \rangle = 5$, $\langle \rho_{9,2} \rangle = -3$, $\langle \rho_{9,4} \rangle = 5$ and $\langle \rho_{9,6} \rangle = -35$.

Finally we give $c(\mathscr{M}_n)$, the total Chern class of the tangent bundle of $\mathscr{M}_n$.

THEOREM 2.8 [2]. *We have*

$$c(\mathscr{M}_n) = (1 - z_1^2)^{-1} \prod_{i=1}^{n} (1 + z_i).$$

Note that we have $z_1^2 = \cdots = z_n^2$ (cf. Theorem 2.2 (i)). Hence we can replace $(1 - z_1^2)^{-1}$ in Theorem 2.8 by $(1 - z_j^2)^{-1}$ for any j with $2 \le j \le n$.

Proof of Theorem 2.8. This theorem is essentially [2, p. 307]. But in [2], the result is stated in terms of other generators R, V_i $(1 \le i \le n-1) \in H^2(\mathscr{M}_n; \boldsymbol{R})$. So we summarize how to deduce Theorem 2.8 from [2].

In [2, p. 296], two dimensional cohomology classes R, V_i $(1 \le i \le n-1)$, which are the generators of $H^*(\mathscr{M}_n; \boldsymbol{R})$, are defined. Then in [2, Proposition 7.3], it is shown that

$$z_i = \begin{cases} R + 2V_i & 1 \le i \le n-1 \\ -R & i = n. \end{cases} \tag{2.9}$$

(Note that [2] uses the symbol c_i to denote z_i in this paper.)

Finally in [2, p. 307], the following result is proved.

$$c(\mathscr{M}_n) = (1 + R)^{-1} \prod_{i=1}^{n-1} (1 + V_i + R) \prod_{j=1}^{n-1} (1 + V_j). \tag{2.10}$$

Then using (2.9) and the fact $z_1^2 = \cdots = z_n^2$ (cf. Theorem 2.2), we see that (2.10) is equivalent to Theorem 2.8. □

Let $\sigma_i(z_1, \ldots, z_n) \in H^{2i}(\mathscr{M}_n; \boldsymbol{R})$ be the elementary symmetric polynomial on $z_1, \ldots, z_n \in H^2(\mathscr{M}_n; \boldsymbol{R})$. Recall that $z_1^2 = \cdots = z_n^2$ (cf. Theorem 2.2 (i)). We define $D^2 \in H^4(\mathscr{M}_n; \boldsymbol{R})$ by

$$z_1^2 = \cdots = z_n^2 = D^2. \tag{2.11}$$

(Note that we shall not define $D \in H^2(\mathscr{M}_n; \boldsymbol{R})$, but just define D^2.) Then Theorem 2.8 implies the following:

THEOREM 2.12. *For* $0 \le k \le n-3$, *we have*

$$c_k(\mathscr{M}_n) = \sum_{i=0}^{[k/2]} D^{2i} \sigma_{k-2i}(z_1, \ldots, z_n).$$

Remark 2.13. From Theorem 2.8, we see that $c_1(\mathcal{M}_n) = z_1 + \cdots + z_n$. On the other hand, if we write the symplectic form on $\mathcal{M}_n$ by ω_n, then a theorem of [2] tells us that $[\omega_n] = z_1 + \cdots + z_n \in H^2(\mathcal{M}_n; \boldsymbol{R})$. Hence we have $c_1(\mathcal{M}_n) = [\omega_n]$.

3. Proof of Theorem B

For $n = 5, 7$ and 9, the Chern numbers of $\mathcal{M}_n$ are given by the following:

THEOREM 3.1. (i) *$c_1^2[\mathcal{M}_5] = 5$ and $c_2[\mathcal{M}_5] = 7$.*
(ii) *For $\mathcal{M}_7$, we have the following table of the Chern numbers.*

	Chern number
$c_1^4[\mathcal{M}_7]$	154
$c_1^2c_2[\mathcal{M}_7]$	112
$c_1c_3[\mathcal{M}_7]$	56
$c_2^2[\mathcal{M}_7]$	136
$c_4[\mathcal{M}_7]$	38

(iii) *For $\mathcal{M}_9$, we have the following table of the Chern numbers.*

	Chern number
$c_1^6[\mathcal{M}_9]$	13005
$c_1^4c_2[\mathcal{M}_9]$	7857
$c_1^3c_3[\mathcal{M}_9]$	3393
$c_1^2c_2^2[\mathcal{M}_9]$	5157
$c_1^2c_4[\mathcal{M}_9]$	1287
$c_1c_2c_3[\mathcal{M}_9]$	2421
$c_1c_5[\mathcal{M}_9]$	423
$c_2^3[\mathcal{M}_9]$	4969
$c_2c_4[\mathcal{M}_9]$	1459
$c_3^2[\mathcal{M}_9]$	1221
$c_6[\mathcal{M}_9]$	187

Proof. This theorem follows from Theorems 2.2, 2.5, 2.6 and 2.12. As an example, we show $c_1c_3[\mathcal{M}_7]$. From Theorem 2.12, we have $c_1(\mathcal{M}_7) = \sigma_1(z_1, \dots, z_7)$ and $c_3(\mathcal{M}_7) = \sigma_3(z_1, \dots, z_7) + D^2\sigma_1(z_1, \dots, z_7)$. Hence

$$c_1c_3[\mathcal{M}_7] = \langle \sigma_1(z_1, \dots, z_7)\sigma_3(z_1, \dots, z_7), \mu_{\mathcal{M}_7}\rangle + \langle D^2\sigma_1(z_1, \dots, z_7)^2, \mu_{\mathcal{M}_7}\rangle.$$

Using Theorem 2.5 and Example 2.7 (ii), we have

$$\begin{aligned}
&\langle \sigma_1(z_1, \dots, z_7)\sigma_3(z_1, \dots, z_7), \mu_{\mathcal{M}_7}\rangle \\
&= 7\langle z_1\sigma_3(z_1, \dots, z_7), \mu_{\mathcal{M}_7}\rangle \\
&= 7\left(\binom{6}{2}\langle \rho_{7,2}\rangle + \binom{6}{3}\langle \rho_{7,0}\rangle\right) \\
&= 70.
\end{aligned}$$

Similarly, we have

$$\langle D^2\sigma_1(z_1, \dots, z_7)^2, \mu_{\mathcal{M}_7}\rangle = -14.$$

Hence we have $c_1c_3[\mathcal{M}_7] = 56$. □

Now we complete the proof of Theorem B. It is known that Ω_*^U is the integral polynomial ring on classes x_i of dimension $2i$ for each integer i. Ω_*^U is determined by the Chern numbers. Moreover, $\Omega_*^U \otimes \boldsymbol{Q}$ is the rational polynomial ring on the cobordism classes of complex projective spaces [14].

We consider $\mathcal{M}_7$. The above fact tells us that in $\Omega_8^U \otimes \boldsymbol{Q}$, $[\mathcal{M}_7]$ is a linear combination of $[(\boldsymbol{CP}^1)^4]$, $[(\boldsymbol{CP}^1)^2 \times \boldsymbol{CP}^2]$, $[\boldsymbol{CP}^1 \times \boldsymbol{CP}^3]$, $[(\boldsymbol{CP}^2)^2]$ and $[\boldsymbol{CP}^4]$. The coefficients are determined completely since we know all Chern numbers of $\mathcal{M}_7$ in Theorem 3.1 (ii). Thus we get a description of $[\mathcal{M}_7]$ in $\Omega_8^U \otimes \boldsymbol{Q}$, and the result is given in Theorem B (ii). Since the coefficients of the description are integers, this is also a description in Ω_8^U.

Similarly, we can prove Theorem B (i) and (iii).

Remark 3.2. In order to get a description of $\mathcal{M}_9$ in Ω_{12}^U, we define an element x_5 of Ω_{10}^U by

$$x_5 = [\boldsymbol{CP}^5] + [H_{3,3}] - [H_{2,4}],$$

where $H_{a,b}$ denotes a non-singular hypersurface of degree $(1,1)$ in $\boldsymbol{CP}^a \times \boldsymbol{CP}^b$. Since $s_5(x_5) = 1$, we see that $x_5 \in \Omega_{10}^U$ is a ring generator. Then it is easy to see that in Ω_{10}^U, we have

$$\begin{aligned}
(3.3)\quad [\boldsymbol{CP}^5] = {} & 21[(\boldsymbol{CP}^1)^5] - 68[(\boldsymbol{CP}^1)^3 \times \boldsymbol{CP}^2] + 27[(\boldsymbol{CP}^1)^2 \times \boldsymbol{CP}^3] \\
& + 51[\boldsymbol{CP}^1 \times (\boldsymbol{CP}^2)^2] - 6[\boldsymbol{CP}^1 \times \boldsymbol{CP}^4] - 30[\boldsymbol{CP}^2 \times \boldsymbol{CP}^3] + 6x_5.
\end{aligned}$$

If we put (3.3) into Theorem B (iii), then we get the following description of $[\mathcal{M}_9]$ in Ω_{12}^U.

$$[\mathcal{M}_9] = 3123[(CP^1)^6] - 10196[(CP^1)^4 \times CP^2] + 4194[(CP^1)^3 \times CP^3] + 7700[(CP^1)^2 \times (CP^2)^2] - 1100[(CP^1)^2 \times CP^4] - 4620[CP^1 \times CP^2 \times CP^3] + 0[(CP^2)^3] + 0[CP^2 \times CP^4] + 55[(CP^3)^2] - 35[CP^6] + 880[CP^1] \cdot x_5.$$

4. Proof of Theorem C

Proof of Theorem C (i). From Remark 2.13, we have $c_1(\mathcal{M}_n) = [\omega_n]$. In [7], the symplectic volume $\langle \omega_n^{n-3}, \mu_{\mathcal{M}_n} \rangle$ is determined. Hence Theorem C (i) follows.

Proof of Theorem C (ii). Note that $c_{n-3}[\mathcal{M}_n] = \chi(\mathcal{M}_n)$, the Euler characteristic of $\mathcal{M}_n$. For odd n, $H_*(\mathcal{M}_n; \boldsymbol{R})$ is determined in [10], [12]. Hence Theorem C (ii) follows.

Proof of Theorem C (iii). We can deduce this theorem from direct calculations using Theorems 2.5, 2.6 and 2.8, or from the fact $h^{p,q}(\mathcal{M}_n) = 0$ for $p \neq q$ [10], [12].

We can also deduce this theorem from the following stronger assertion. (Recall that the Todd genus is birational invariant [3].)

ASSERTION 4.1. *$\mathcal{M}_n$ is birationally equivalent to CP^{n-3}.*

Proof. In order to construct a birational map $f : \mathcal{M}_n \dashrightarrow CP^{n-3}$, it is convenient to substitute $\mathcal{M}_n$ by a space $\mathcal{N}_n$, which is biholomorphic to $\mathcal{M}_n$. Recall that an element $P = (x_1, \dots, x_n)$ of $(CP^1)^n$ is semistable if and only if P contains no point of CP^1 with multiplicity strictly greater than $n/2$. Let $\mathcal{N}_n$ be the orbit space of semistable points in $(CP^1)^n$ with respect to the natural action of the group $PSL(2, \boldsymbol{C})$. Thus:

$$(4.2) \qquad \mathcal{N}_n = \{P = (x_1, \dots, x_n) \in (CP^1)^n : P \text{ is semistable}\}/PSL(2, \boldsymbol{C}).$$

Then it is known that $\mathcal{M}_n$ is biholomorphic to $\mathcal{N}_n$ [8], [10], [12]:

$$\mathcal{M}_n \cong \mathcal{N}_n.$$

Now we construct a rational map $\phi : \mathcal{N}_n \dashrightarrow CP^{n-3}$ in the same way as in [9, p. 134]. (The inverse rational map $CP^{n-3} \dashrightarrow \mathcal{N}_n$ is constructed similarly.) Let $P = (x_1, \dots, x_n) \in \mathcal{N}_n$. By the $PSL(2, \boldsymbol{C})$-action, we can assume that $x_1 = \infty$. Thus:

$$(4.3) \quad \mathcal{N}_n = \{P = (x_1, \dots, x_n) \in (CP^1)^n : P \text{ is semistable and } x_1 = \infty\}/G,$$

where G is a subgroup of $PSL(2, \boldsymbol{C})$ defined by

$$G = \left\{ \begin{pmatrix} z & 0 \\ \xi & z^{-1} \end{pmatrix} : z \in \boldsymbol{C}^*, \xi \in \boldsymbol{C} \right\}.$$

Let $\mathcal{N}_n'$ be the subspace of $\mathcal{N}_n$ defined by

$$(4.4) \qquad \mathcal{N}_n' = \{P = (x_1, \dots, x_n) \in (\boldsymbol{CP}^1)^n : P \text{ is semistable and } x_1 = \infty, \\ x_i \neq \infty \ (2 \leq i \leq n)\}/G.$$

Then we define a map $\phi : \mathcal{N}_n' \to \boldsymbol{CP}^{n-3}$ as follows. Let $P = (x_1, \dots, x_n) \in \mathcal{N}_n'$. Note that $\boldsymbol{CP}^1 - \{\infty\}$ is isomorphic to $\boldsymbol{C}$. Since $x_i \neq \infty$ $(2 \leq i \leq n)$, we can regard that $x_i \in \boldsymbol{C}$ $(2 \leq i \leq n)$. There is exactly one point $z = z(x_1, \dots, x_n) \in \boldsymbol{CP}^1 - \{\infty\} = \boldsymbol{C}$ such that $\sum_{i=2}^n (z - x_i) = 0$. Then the point $\phi(x_1, \dots, x_n)$ is defined to be point with homogeneous coordinates $(z - x_2, \dots, z - x_n)$. This completes the proof of Assertion 4.1. □

Proof of Theorem C (iv). Recall that $s_{n-3}[\mathcal{M}_n]$ is a characteristic number defined from the Chern classes. Let $n = 2m+1$ and let $s_{m-1}(p)[\mathcal{M}_n]$ be the characteristic number which is defined in the same way as in $s_{n-3}[\mathcal{M}_n]$ but using the Pontrjagin classes instead of the Chern classes. Then it is known that $s_{n-3}[\mathcal{M}_n] = s_{m-1}(p)[\mathcal{M}_n]$ (see for example [13]).

Since $\mathcal{M}_n$ is oriented cobordant to $(-1)^{m+1}\binom{2m-1}{m}\boldsymbol{CP}^{2m-2}$ [6] and $s_{m-1}(p)[\boldsymbol{CP}^{2m-2}] = 2m-1$, Theorem C (iv) follows. □

Finally we give some more results on the Chern numbers.

THEOREM 4.5. *We have the following formulae for* $n = 2m+1$.

(i) *We have*

$$c_1 c_{n-4}[\mathcal{M}_n] = -(2m+1)2^{2m-1} + (2m+1)(m+1)\binom{2m-1}{m}.$$

(ii) *We have*

$$c_2 c_{n-5}[\mathcal{M}_n] = -(2m^2+m+1)2^{2m-1} + \frac{(2m+1)(3m^2+2m+3)}{3}\binom{2m-1}{m}.$$

Recall Theorem C (ii). In general, it is easy to see that $c_i c_{n-3-i}[\mathcal{M}_n]$ $(i \leq (n-3)/2)$ is of the form

$$c_i c_{n-3-i}[\mathcal{M}_n] = -f_i(m)2^{2m-1} + (2m+1)g_i(m)\binom{2m-1}{m},$$

where $f_i(m)$ and $g_i(m)$ are polynomials of degree i with variable m.

We can prove Theorem 4.5 in the same way as in the proof of Theorem 3.1 using Theorems 2.5, 2.6 and 2.12.

References

[1] M. Brion, Cohomologie équivariante des points semi-stables, J. Reine Angew. Math., **421** (1991), 125–140.

[2] J-C. Hausmann and A. Knutson, The cohomology ring of polygon spaces, Ann. Inst. Fourier (Grenoble), **48** (1998), 281–321.

[3] F Hirzebruch, Topological Methods in Algebraic Geometry 3rd ed., translated, Springer, Berlin, 1966.

[4] Y Kamiyama, Topology of equilateral polygon linkages, Topology Appl., **68** (1996), 13–31.

[5] Y. Kamiyama, The homology of singular polygon spaces, Canad. J. Math., **50** (1998), 581–594.

[6] Y Kamiyama, Remarks on the topology of spatial polygon spaces, Bull. Austral. Math. Soc., **58** (1998), 373–382.

[7] Y. Kamiyama and M. Tezuka, Symplectic volume of the moduli space of spatial polygons, J. Math. Kyoto Univ., **39** (1999), 557–575.

[8] M. Kapovich and J. Millson, The symplectic geometry of polygons in Euclidean space, J. Differential Geom., **44** (1996), 479–513.

[9] M. Kapranov, The permutoassociahedron, MacLane's coherence theorem and asymptotic zones for the *KZ* equation, J. Pure Appl. Algebra, **85** (1993), 119–142.

[10] F Kirwan, Cohomology of Quotients in Symplectic and Algebraic Geometry, Mathematical Notes **31**, Princeton University Press, Princeton, 1984.

[11] F Kirwan, The cohomology rings of moduli spaces of bundles over Riemann surfaces, J. Amer. Math. Soc., **5** (1992), 853–906.

[12] A. Klyachko, Spatial polygons and stable configurations of points in the projective line, Algebraic Geometry and its Applications Yaroslavl' 1992 (A. Tikhomirov and A. Tyurin eds.), Aspects Math. E **25**, Vieweg, Braunschweig, 1994, pp. 67–84.

[13] J. Milnor and J. Stasheff, Characteristic Classes, Princeton University Press, Princeton, 1974.

[14] R. Stong, Notes on Cobordism Theory, Princeton University Press, Princeton, 1968.

Department of Mathematics
University of the Ryukyus
Nishihara-Cho, Okinawa
903-01, Japan
E-mail: kamiyama@sci.u-ryukyu.ac.jp