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SPECTRAL ZETA FUNCTIONS FOR COMPACT SYMMETRIC SPACES OF RANK ONE

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Abstract

We study the spectral zeta function $\zeta_M(s)$ associated with the spectrum of Laplacian acting on functions of a compact simply connected Riemannian symmetric space M of rank one and the spectral zeta function $\zeta_{S^n}^{p,\delta}(s)$ associated with the spectrum of Laplacian acting on p-forms of the sphere S^n We give the residues of $\zeta_M(s)$ and $\zeta_{S^n}^{p}(s)$ explicitly. For the odd dimensional sphere S^n, we show that $\zeta_{S^n}^{p,\delta}(s)$ vanishes at negative integers.

1. Introduction

Let M be a compact connected Riemannian manifold, Δ_M the Laplace-Beltram operator acting on smooth functions. It has a discrete specrtrum

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots, \tag{1}$$

where every λ_i is repeated with its multiplicity. The spectral zeta function of M which is well defined in $\mathrm{Re}(s) > \dim M/2$ is given by

$$\zeta_M(s) = \sum_{k=1}^{\infty} \lambda_k^{-s}. \tag{2}$$

In [9] Minakshisundaram-Pleijel proved $\zeta_M(s)$ has a meromorphic continuation on the whole complex plane $\boldsymbol{C}$ and analytic on $\boldsymbol{C}$ except at simple poles at $s = \dim M/2 - k$ $(k = 0, 1, 2, \ldots)$, and express the residues in terms of metric invariants of M. In [3], [4] Carletti and Monti Bragardin studied Dirichlet series $L(s) = \sum_{k=1}^{\infty} P(k)/((k+d_1)^2(k+d_2)^s)$ and applied to the spectral zeta functions for the standard spheres of constant curvature 1. They give the residues of the spectral zeta functions explicitly.

Let Δ_M^p be the Laplace-Beltrami operator acting on smooth p-forms $\Lambda^p(M)$. Denote the differential and codifferential of M by d and δ, respectively;

$$d : \Lambda^p(M) \to \Lambda^{p+1}(M),$$
$$\delta : \Lambda^p(M) \to \Lambda^{p-1}(M).$$

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Let $H^p(M)$ be the space of harmonic p-forms. We define the space of δ-closed forms by

(3) $$\Lambda_\delta^p(M) = \{\alpha \in \Lambda^p(M) \mid \delta\alpha = 0, \alpha \perp H^p(M)\},$$

and the space of d-closed forms by

(4) $$\Lambda_d^p(M) = \{\alpha \in \Lambda^p(M) \mid d\alpha = 0, \alpha \perp H^p(M)\}.$$

Since d and δ commute with Δ_M^p, Δ_M^p acts invariantly on $\Lambda_\delta^p(M)$. We say the eigenvalue λ to be δ-eigenvalue if there is an eigen p-form in $\Lambda_\delta^p(M)$ with the eigenvalue λ.

Let

(5) $$0 < \lambda_1^p \leq \lambda_2^p \leq \cdots \leq \lambda_n^p \leq \cdots,$$

be the set of all the δ-eigenvalues, where every λ_i^p is repeated with its multiplicity. We define the spectral zeta function $\zeta_M^{p,\delta}(s)$ by

(6) $$\zeta_M^{p,\delta}(s) = \sum_{i=1}^{\infty} (\lambda_i^p)^{-s}.$$

Note that

(7) $$\zeta_M^{0,\delta}(s) = \zeta_M(s).$$

In this paper, we study the spectral zeta function $\zeta_{S^n}^{p,\delta}(s)$ of the standard spheres of constant curvature 1 and $\zeta_M(s)$ of other compact simply connected Riemannian symmetric spaces of rank 1. We give the residues of their spectral zeta functions explicitly which have *much simpler forms* than Carletti and Monti Bragardin's results.

It is well know that the Riemann zeta function has trivial zeros at any negative even integer. In [10] Minakshisundaram proved that the spectral zeta function $\zeta_{S^n}(s)$ of the odd dimensional sphere vanishes at any negative integer. In the last section, we show the spectral zeta function $\zeta_{S^n}^{p,\delta}(s)$ of the odd dimensional sphere also vanishes at any negative integer.

2. Dirichlet series $L(s) = \sum_{k=1}^{\infty} P(k)/((k+d_1)^s(k+d_2)^s)$

Let $P(x)$ be a polynomial of degree N. For two real numbers d_1 and d_2 with $d_2 \geq d_1 > -1$, we consider the Dirichlet series

(8) $$L(s) = \sum_{k=1}^{\infty} \frac{P(k)}{(k+d_1)^s(k+d_2)^s}.$$

In this section, we review Carletti-Monti Bragardin's results for $L(s)$. We prepare some notations for later use. Let $P_{m-j}^m(\beta)$ be polynomials in β of degree $m-j$ defined by

$$P_0^0(\beta) = 1, \quad P_m^m(\beta) = \frac{1}{m!}\prod_{k=1}^{m}(\beta - k),$$

$$P_{m-j}^m(\beta) = \frac{1}{j!}\frac{d^j}{d\beta^j}P_m^m(\beta) \quad 0 \le j \le m.$$

Put

(9)
$$P(x) = \sum_{j=0}^{N} a_j x^j,$$

and define the constants a_p^m, $(m, p \ge 1)$ by

(10)
$$a_p^m = \sum_{\ell=1}^{p}(-1)^{p-\ell}\binom{m+1}{p-\ell}\ell^m.$$

For $a > 0$, the Hurewicz zeta function is defined by

(11)
$$\zeta(s,a) = \sum_{r=0}^{\infty}\frac{1}{(r+a)^s}.$$

Note that the Riemann zeta function $\zeta(s)$ is

(12)
$$\zeta(s) = \zeta(s,1).$$

Then we have the following formula (see [3]):

PROPOSITION 1.

$$\begin{aligned} L(s) = &\sum_{\ell=0}^{\infty}\left(\frac{d_2-d_1}{2}\right)^{2\ell}(-1)^\ell\binom{-s}{\ell} \\ &\times\left[a_0\zeta\left(2s+2\ell, 1+\frac{d_1+d_2}{2}\right)\right. \\ &\left.+\sum_{m=1}^{N}\sum_{j=0}^{m}\sum_{i=1}^{m}a_m a_i^m P_{m-j}^m\left(m+1-i-\frac{d_1+d_2}{2}\right)\zeta\left(2s+2\ell-j, 1+\frac{d_1+d_2}{2}\right)\right]. \end{aligned}$$

Note that if $d_1 = d_2$, then the above formula shoud be read as follows;

(13)
$$\begin{aligned} L(s) = &a_0\zeta(2s, 1+d_1) \\ &+\sum_{m=1}^{N}\sum_{j=0}^{m}\sum_{i=1}^{m}a_m a_i^m P_{m-j}^m(m+1-i-d_1)\zeta(2s-j, 1+d_1). \end{aligned}$$

From this formula, we can get easily the following corollary:

Corollary 1. 1. *If* $P(X) = 1$, *then*

$$L(s) = \sum_{\ell=0}^{\infty} \left(\frac{d_2 - d_1}{2}\right)^{2\ell} (-1)^{\ell} \binom{-s}{\ell} \zeta\left(2s + 2\ell, 1 + \frac{d_2 - d_1}{2}\right).$$

2. *If* $P(X) = X + (d_2 - d_1)/2$, *then*

$$L(s) = \sum_{\ell=0}^{\infty} \left(\frac{d_2 - d_1}{2}\right)^{2\ell} (-1)^{\ell} \binom{-s}{\ell} \zeta\left(2s + 2\ell - 1, 1 + \frac{d_2 - d_1}{2}\right).$$

In the above Corollary, if $d_1 = d_2$, then this formula should be read as follows

(14) $$L(s) = \zeta(2s, 1 + d_1),$$

and

(15) $$L(s) = \zeta(2s - 1, 1 + d_1),$$

according to $P(X) = 1$ or $P(X) = X + d_1$.

3. Spectral zeta functions for spheres S^n

Let S^n denote the n-dimensional standard sphere of constant curvature 1. By [7], all the positive δ-eigenvalues of $\Delta^p_{S^n}$ $(0 \le p \le n/2)$ are

(16) $$(k + p)(k + n - p - 1), \quad k \ge 1,$$

with their multiplicities

(17) $$\frac{(2k + n - 1)k(k + 1)(k + 2) \cdots (k + n - 1)}{p!(n - p - 1)!(k + p)(k + n - p - 1)}.$$

The spectral zeta function $\zeta^{p,\delta}_{S^n}(s)$ are given as follows;

(18) $$\zeta^{p,\delta}_{S^n}(s) = \frac{2}{p!(n - p - 1)!} \sum_{k=1}^{\infty} \frac{(k + m)k(k + 1)(k + 2) \cdots (k + n - 1)}{((k + p)(k + n - p - 1))^{s+1}}.$$

We treat with $\zeta^{p,\delta}_{S^n}(s)$ separately according to whether n is odd or even.

S^n, n odd $(n = 2m + 1)$.

Define the polynomial $F_{n,p}(x)$ by

(19) $$F_{n,p}(x) = \frac{2 \prod_{i=0}^{m} (x + (m - p)^2 - i^2)}{p!(n - p - 1)!} = \sum_{j=1}^{m+1} \alpha_{n,p,j} x^j.$$

Then we have

$$\zeta_{S^n}^{p,\delta}(s) = \sum_{k=1}^{\infty} \frac{F_{n,p}((k+p)(k+n-p-1))}{((k+p)(k+n-p-1))^{s+1}} \tag{20}$$
$$= \sum_{j=1}^{m+1} \alpha_{n,p,j} \sum_{k=1}^{\infty} \frac{1}{((k+p)(k+n-p-1))^{s+1-j}}.$$

Using Corollary 1, we have

$$\zeta_{S^n}^{p,\delta}(s) \tag{21}$$
$$= \sum_{j=0}^{m+1} \alpha_{n,p,j} \sum_{\ell=0}^{\infty} (m-p)^{2\ell} \frac{\Gamma(s+1-j+\ell)}{\ell!\Gamma(s+1-j)} \zeta(2(s+1-j)+2\ell-1, 1+m)$$
$$= \sum_{j=1}^{m+1} \alpha_{n,p,j} \sum_{\ell=0}^{\infty} (m-p)^{2\ell} \frac{\Gamma(s+1-j+\ell)}{\ell!\Gamma(s+1-j)} \zeta(2s+2(\ell-j+1), 1+m)$$
$$= \sum_{t=0}^{\infty} \sum_{\ell=t}^{m+t} \alpha_{n,p,\ell-t+1} (m-p)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-\ell)} \zeta(2s+2t, 1+m)$$
$$+ \sum_{t=-(n-1)}^{-1} \sum_{\ell=0}^{n-1+t} \alpha_{n,p,\ell-t+1} (m-p)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-\ell)} \zeta(2s+2t, 1+m).$$

From this formula, we have

THEOREM 1. *The spectral zeta function $\zeta_{S^n}^{p,\delta}(s)$ of the odd dimensional standard sphere S^{2m+1} has a meromorphic continuation on the whole complex plane with at most simple poles at $s = n/2 - k$ $(k = 0, 1, 2, \ldots)$. The residue at $s = n/2 - k$ is given as*

$$\begin{cases} \dfrac{1}{2} \displaystyle\sum_{\ell=0}^{k} \alpha_{n,p,\ell+m-k+1} \left(-\frac{(m-p)^2}{4}\right)^{\ell} \binom{2\ell}{\ell} & \text{if } m \geq k \geq 0 \\ \dfrac{1}{2} \displaystyle\sum_{\ell=m-k}^{2m-k} \alpha_{n,p,\ell-m+k+1} \left(-\frac{(m-p)^2}{4}\right)^{\ell} \binom{2\ell}{\ell} & \text{if } k > m \end{cases} \tag{22}$$

S^n, n even $(n = 2m)$.
Define the function $F_{n,p}(x)$ by

$$F_{n,p}(x) = \frac{2\prod_{i=1}^{m}(x + ((n-1)/2 - p)^2 - (i-1/2)^2)}{p!(n-p-1)!} \tag{23}$$
$$= \sum_{j=1}^{m} \alpha_{n,p,j} x^j.$$

Then we have

$$\zeta_{S^n}^{p,\delta}(s) = \sum_{k=1}^{\infty} \frac{(k+(n-1)/2)F_{n,p}((k+p)(k+n-p-1))}{((k+p)(k+n-p-1))^{s+1}} \tag{24}$$
$$= \sum_{j=1}^{m} \alpha_{n,p,j} \sum_{k=1}^{\infty} \frac{(k+(n-1)/2)}{((k+p)(k+n-p-1))^{s+1-j}}.$$

Using Corollary 1, we have

(25) $\zeta_{S^n}^{p,\delta}(s)$

$$= \sum_{j=1}^{m} \alpha_{n,p,j} \sum_{\ell=0}^{\infty} \left(\frac{n-1}{2}-p\right)^{2\ell} \frac{\Gamma(s+1-j+\ell)}{\ell!\Gamma(s+1-j)} \zeta\left(2s+2(\ell-j+1), 1+\frac{n-1}{2}\right)$$
$$= \sum_{t=0}^{\infty} \sum_{\ell=t}^{m+t} \alpha_{n,p,\ell-t+1} \left(\frac{n-1}{2}-p\right)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-\ell)} \zeta\left(2s+2t-1, 1+\frac{n-1}{2}\right)$$
$$+ \sum_{t=-m}^{-1} \sum_{\ell=0}^{m-t} \alpha_{n,p,\ell+t+1} \left(\frac{n-1}{2}-p\right)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-\ell)} \zeta\left(2s+2t-1, 1+\frac{n-1}{2}\right).$$

From this formula, we have

THEOREM 2. *The spectral zeta function $\zeta_{S^n}^{p,\delta}(s)$ of the even dimensional sphere S^{2m} has a meromorphic continuation on the whole complex plane with at most simple poles at $s = k$ $(k = 1, 2, \ldots m)$. The residue at $s = k$ is $\alpha_{n,p,k}/2$.*

4. Spectral zeta functions for complex projective spaces $P^n(\boldsymbol{C})$

Let $P^n(\boldsymbol{C})$ denote the complex projective n-space with Riemannian metric induced by $(n+1) \times$ negative of the Cartan-Killing form of $SU(n+1)$. Then every geodesic on $P^n(\boldsymbol{C})$ is closed of length π. By [2], all the eigenvalues of $\Delta_{P^n(\boldsymbol{C})}^0$ are given by

$$k(k+n), \quad k \geq 0, \tag{26}$$

with their multiplicities

$$\frac{(2k+n)((k+1)(k+2)\cdots(k+n-1))^2}{n!(n-1)!}. \tag{27}$$

So that the spectral zeta function $\zeta_{P^n(\boldsymbol{C})}(s)$ of $P^n(\boldsymbol{C})$ is of the form:

$$\zeta_{P^n(\boldsymbol{C})}(s) = \frac{2}{n!(n-1)!} \sum_{k=1}^{\infty} \frac{(k+n/2)\{(k+1)(k+2)\cdots(k+n-1)\}^2}{(k(k+n))^s}. \tag{28}$$

We treat with $\zeta_{P^n(\boldsymbol{C})}(s)$, separately according to whether n is odd or even.

$P^n(\boldsymbol{C})$, n odd $(n = 2m+1)$.
Define the polynomial $G_n(x)$ by

$$G_n(x) = \frac{2\prod_{i=0}^{m}(x + (n/2)^2 - (i-1/2)^2)^2}{n!(n-1)!} = \sum_{j=0}^{n-1} \gamma_{n,j} x^j. \tag{29}$$

Then we have

$$\zeta_{P^n(\boldsymbol{C})}(s) = \sum_{k=1}^{\infty} \frac{(k+n/2)G_n(k(k+n))}{(k(k+n))^s} = \sum_{j=0}^{n-1} \gamma_{n,j} \sum_{k=1}^{\infty} \frac{(k+n/2)}{(k(k+n))^{s-j}}. \tag{30}$$

Using Corollary 1, we have

$$\begin{aligned}
\zeta_{P^n(\boldsymbol{C})}(s) &= \sum_{j=0}^{n-1} \gamma_{n,j} \sum_{\ell=0}^{\infty} \left(\frac{n}{2}\right)^{2\ell} \frac{\Gamma(s-j+\ell)}{\ell!\Gamma(s-j)} \zeta\left(2(s-j)+2\ell-1, 1+\frac{n}{2}\right) \\
&= \sum_{j=0}^{n-1} \gamma_{n,j} \sum_{\ell=0}^{\infty} \left(\frac{n}{2}\right)^{2\ell} \frac{\Gamma(s-j+\ell)}{\ell!\Gamma(s-j)} \zeta\left(2s+2(\ell-j)-1, 1+\frac{n}{2}\right) \\
&= \sum_{t=0}^{\infty} \sum_{\ell=t}^{n-1+t} \gamma_{n,\ell-t} \left(\frac{n}{2}\right)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)} \zeta\left(2s+2t-1, 1+\frac{n}{2}\right) \\
&\quad + \sum_{t=-(n-1)}^{-1} \sum_{\ell=0}^{n-1+t} \gamma_{n,\ell-t} \left(\frac{n}{2}\right)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)} \zeta\left(2s+2t-1, 1+\frac{n}{2}\right).
\end{aligned}$$

$P^n(\boldsymbol{C})$, n even $(n = 2m)$.
Define the polynomial $G_n(x)$ by

$$G_n(x) = \frac{2(x+m^2)\prod_{i=1}^{m-1}(x+m^2-i^2)^2}{n!(n-1)!} = \sum_{j=0}^{n-1} \gamma_{n,j} x^j. \tag{31}$$

Then we have

$$\zeta_{P^n(\boldsymbol{C})}(s) = \sum_{k=1}^{\infty} \frac{(k+m)G_n(k(k+n))}{(k(k+n))^s} = \sum_{j=0}^{n-1} \gamma_{n,j} \sum_{k=1}^{\infty} \frac{(k+m)}{(k(k+n))^{s-j}}. \tag{32}$$

Using Corollary 1, we have

$$\begin{aligned}\zeta_{P^n(\boldsymbol{C})}(s) &= \sum_{j=0}^{n-1}\gamma_{n,j}\sum_{\ell=0}^{\infty} m^{2\ell}\frac{\Gamma(s-j+\ell)}{\ell!\Gamma(s-j)}\zeta(2(s-j)+2\ell-1,1+m)\\ &= \sum_{j=0}^{n-1}\gamma_{n,j}\sum_{\ell=0}^{\infty} m^{2\ell}\frac{\Gamma(s-j+\ell)}{\ell!\Gamma(s-j)}\zeta(2s+2(\ell-j)-1,1+m)\\ &= \sum_{t=0}^{\infty}\sum_{\ell=t}^{n-1+t}\gamma_{n,\ell-t}m^{2\ell}\frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)}\zeta(2s+2t-1,1+m)\\ &\quad+\sum_{t=-(n-1)}^{-1}\sum_{\ell=0}^{n-1+t}\gamma_{n,\ell-t}m^{2\ell}\frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)}\zeta(2s+2t-1,1+m).\end{aligned}$$

From these formulae, we have

THEOREM 3. *The spectral zeta function* $\zeta_{P^n(\boldsymbol{C})}$ *has a meromorphic continuation on the whole complex plane with at most simple poles at* $s=k$ $(k=1,2,\dots n)$. *The residue at* $s=k$ *is* $\gamma_{n,k-1}/2$.

5. Spectral zeta functions for quaternionic projective spaces $P^n(\boldsymbol{H})$

Let $P^n(\boldsymbol{H})$ denote the quaternionic projective n-space with Riemannian metric induced by $2(n+2)\times$ negative of the Cartan-Killing form of $Sp(n+1)$. Then every geodesic on $P^n(\boldsymbol{H})$ is closed of length π. By [2], all the eigenvalues of $\Delta^0_{P^n(\boldsymbol{H})}$ are

$$k(k+2n+1),\quad k\geq 0, \tag{33}$$

with multiplicities

$$\frac{2(k+n+1/2)(k+1)\{(k+2)\cdots(k+2n-1)\}^2(k+2n)}{(2n+1)!(2n-1)!}. \tag{34}$$

So that the spectral zeta function $\zeta_{P^n(\boldsymbol{H})}(s)$ of $P^n(\boldsymbol{H})$ is of the form:

$$\zeta_{P^n(\boldsymbol{H})}(s)=\sum_{k=1}^{\infty}2\frac{(k+n+1/2)(k+1)\{(k+2)\cdots(k+2n-1)\}^2(k+2n)}{(2n+1)!(2n-1)!(k(k+2n+1))^s}.$$

Define the polynomial $H_n(x)$ by

$$\begin{aligned}H_n(x) &= \frac{2}{(2n+1)!(2n-1)!}(x+2n)\prod_{i=1}^{n-1}\left(x+\left(n+\frac{1}{2}\right)^2-\left(i-\frac{1}{2}\right)^2\right)^2\\ &= \sum_{j=0}^{2n-1}\delta_{n,j}x^j.\end{aligned}$$

Then we have

$$\zeta_{P^n(\boldsymbol{H})}(s) = \sum_{k=1}^{\infty} \frac{(k+n+1/2)H_n(k(k+2n+1))}{(k(k+2n+1))^s} \tag{35}$$
$$= \sum_{j=0}^{2n-1} \delta_{n,j} \sum_{k=1}^{\infty} \frac{(k+n+1/2)}{(k(k+2n+1))^{s-j}}.$$

Using Corollary 1, we have

$$\zeta_{P^n(\boldsymbol{H})}(s) = \sum_{j=0}^{2n-1} \delta_{n,j} \sum_{\ell=0}^{\infty} \left(n+\frac{1}{2}\right)^{2\ell} \frac{\Gamma(s-j+\ell)}{\ell!\Gamma(s-j)} \zeta\left(2(s-j)+2\ell-1, \frac{3}{2}+n\right)$$
$$= \sum_{j=0}^{2n-1} \delta_{n,j} \sum_{\ell=0}^{\infty} \left(n+\frac{1}{2}\right)^{2\ell} \frac{\Gamma(s-j+\ell)}{\ell!\gamma(s-j)} \zeta\left(2s+2(\ell-j)-1, \frac{3}{2}+n\right)$$
$$= \sum_{t=0}^{\infty} \sum_{\ell=t}^{2n-1+t} \delta_{n,\ell-t} \left(n+\frac{1}{2}\right)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)} \zeta\left(2s+2t-1, \frac{3}{2}+n\right)$$
$$+ \sum_{t=-(2n-1)}^{-1} \sum_{\ell=0}^{2n-1+t} \delta_{n,\ell-t} \left(n+\frac{1}{2}\right)^{2\ell} \frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)} \zeta\left(2s+2t-1, \frac{3}{2}+n\right).$$

From this formula, we have

THEOREM 4. *The spectral zeta function $\zeta_{P^n(\boldsymbol{H})}(s)$ has a meromorphic continuation on the whole complex plane with at most simple poles at $s=k$ $(k=1,2,\dots 2n)$. The residue at $s=k$ is $\delta_{n,k-1}/2$.*

6. Spectral zeta functions for Cayley projective plane $P^2(\boldsymbol{O})$

Let $P^2(\boldsymbol{O})$ denote the Cayley projective plane with Riemannian metric induced by $18\times$ negative of the Cartan-Killing form of F_4. Then every geodesic on $P^2(\boldsymbol{O})$ is closed of length π. By [2], all the eigenvalues of $\Delta^0_{P^2(\boldsymbol{O})}$ are

$$k(k+11), \quad k \geq 0, \tag{36}$$

with their multiplicities

$$\frac{(2k+11)3!(k+7)!(k+10)!}{11!7!k!(k+3)!}. \tag{37}$$

So that the spectral zeta function $\zeta_{P^2(\boldsymbol{O})}(s)$ of $P^2(\boldsymbol{O})$ is of the form:

$$\zeta_{P^2(\boldsymbol{O})}(s) = \sum_{k=1}^{\infty} \frac{(2k+11)3!(k+7)!(k+10)!}{11!7!k!(k+3)!(k(k+11))^s}. \tag{38}$$

Define the polynomial $I(x)$ by

$$\begin{aligned} I(x) &= \frac{12}{11!7!}\prod_{i=1}^{5}\left(x+\left(\frac{11}{2}\right)^2-\left(i-\frac{1}{2}\right)^2\right)\prod_{i=1}^{2}\left(x+\left(\frac{11}{2}\right)^2-\left(i-\frac{1}{2}\right)^2\right) \\ &= \frac{12}{11!7!}(x+10)(x+18)(x+24)(x+28)^2(x+30)^2 \\ &= \sum_{j=0}^{7}\eta_j x^j. \end{aligned}$$

Then we have

$$(39) \qquad \begin{aligned} \zeta_{P^2(\boldsymbol{O})} &= \sum_{k=1}^{\infty}\frac{(k+11/2)I(k(k+11))}{(k(k+11))^s} \\ &= \sum_{j=0}^{7}\eta_j\sum_{k=1}^{\infty}\frac{(k+11/2)}{(k(k+11))^{s-j}}. \end{aligned}$$

Using Corollary 1, we have

$$\begin{aligned} \zeta_{P^n(\boldsymbol{O})}(s) &= \sum_{j=0}^{7}\eta_j\sum_{\ell=0}^{\infty}\left(\frac{11}{2}\right)^{2\ell}\frac{\Gamma(s-j+\ell)}{\ell!\Gamma(s-j)}\zeta\left(2(s-j)+2\ell-1,\frac{13}{2}\right) \\ &= \sum_{j=0}^{7}\eta_j\sum_{\ell=0}^{\infty}\left(\frac{11}{2}\right)^{2\ell}\frac{\Gamma(s-j+\ell)}{\ell!\Gamma(s-j)}\zeta\left(2s+2(\ell-j)-1,\frac{13}{2}\right) \\ &= \sum_{t=0}^{\infty}\sum_{\ell=t}^{7+t}\eta_{\ell-t}\left(\frac{11}{2}\right)^{2\ell}\frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)}\zeta\left(2s+2t-1,\frac{3}{2}\right) \\ &\quad+\sum_{t=-7}^{-1}\sum_{\ell=0}^{7+t}\eta_{\ell-t}\left(\frac{11}{2}\right)^{2\ell}\frac{\Gamma(s+t)}{\ell!\Gamma(s+t-j)}\zeta\left(2s+2t-1,\frac{13}{2}\right). \end{aligned}$$

From this formula, we have

Theorem 5. *The spectral zeta function $\zeta_{P^2(\boldsymbol{O})}(s)$ has a meromorphic continuation on the whole complex plane with at most simple poles at $s=k$ $(k=1,2,\dots 8)$. The residue at $s=k$ is $\eta_{k-1}/2$.*

7. Trivial zeros of $\zeta^{p,\delta}_{S^{2m+1}}(s)$

It is well known that the Riemann zeta function $\zeta(s)$ vanishes at even negative integers. In this section, we give analogous results for the spectral zeta function $\zeta^{p,\delta}_{S^{2m+1}}(s)$.

Theorem 6. *The spectral zeta function $\zeta^{p,\delta}_{S^{2m+1}}(s)$ vanishes at negative integers.*

Proof. Using the formula (21), we have for a positive integer k;

$$\begin{aligned}
\zeta^{p,\delta}_{S^n}(-k) &= \sum_{t=0}^{\infty}\sum_{\ell=t}^{m+t}\alpha_{n,p,\ell-t+1}(m-p)^{2\ell}(-1)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m) \\
&\quad + \sum_{t=-m}^{-1}\sum_{\ell=0}^{m+t}\alpha_{n,p,\ell-t+1}(m-p)^{2\ell}(-1)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m) \\
&= \sum_{t=0}^{k}\sum_{\ell=t}^{m+t}\alpha_{n,p,\ell-t+1}(-(m-p)^2)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m) \\
&\quad + \sum_{t=-m}^{-1}\sum_{\ell=0}^{m+t}\alpha_{n,p,\ell-t+1}(-(m-p)^2)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m) \\
&= \sum_{\ell=k}^{m+k}\alpha_{n,p,\ell-k+1}(-(m-p)^2)^{\ell}\binom{\ell}{0}\zeta(0,1+m) \\
&\quad + \sum_{t=0}^{k-1}\sum_{\ell=t}^{m+t}\alpha_{n,p,\ell-t+1}(-(m-p)^2)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m) \\
&\quad + \sum_{t=-m}^{-1}\sum_{\ell=0}^{m+t}\alpha_{n,p,\ell-t+1}(-(m-p)^2)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m).
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
&\sum_{\ell=k}^{m+k}\alpha_{n,p,\ell-k+1}(-(m-p)^2)^{\ell}\binom{\ell}{0}\zeta(0,1+m) \\
&\quad = \sum_{\ell=0}^{m}\alpha_{n,p,\ell+1}(-(m-p)^2)^{\ell+k}\zeta(0,1+m) \\
&\quad = \zeta(0,1+m)(-(m-p)^2)^{k-1}\sum_{\ell=0}^{m}\alpha_{n,p,\ell+1}(-(m-p)^2)^{\ell+1} \\
&\quad = \zeta(0,1+m)(-(m-p)^2)^{k-1}F_{n,p}(-(m-p)^2) \\
&\quad = 0.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
&\zeta_{S^n}^{p,\delta}(-k)\\
&= \sum_{t=0}^{k-1}\sum_{\ell=t}^{m+t}\alpha_{n,p,\ell-t+1}(-(m-p)^2)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m)\\
&\quad+\sum_{t=-m}^{-1}\sum_{\ell=0}^{m+t}\alpha_{n,p,\ell-t+1}(-(m-p)^2)^{\ell}\binom{k-t+\ell}{k-t}\zeta(-2(k-t),1+m)\\
&= \sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=0}^{k-1}(-(m-p)^2)^{\ell+t-1}\binom{k+\ell-1}{k-t}\zeta(-2(k-t),1+m)\\
&\quad+\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=-(\ell-1)}^{-1}(-(m-p)^2)^{\ell+t-1}\binom{k+\ell-1}{k-t}\zeta(-2(k-t),1+m)\\
&= -\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=0}^{k-1}(-(m-p)^2)^{\ell+t-1}\binom{k+\ell-1}{k-t}\sum_{\iota=1}^{m}\iota^{2(k-t)}\\
&\quad-\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=-(\ell-1)}^{-1}(-(m-p)^2)^{\ell+t-1}\binom{k+\ell-1}{k-t}\sum_{\iota=1}^{m}\iota^{2(k-t)}\\
&= -\sum_{\iota=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=0}^{k-1}(-(m-p)^2)^{\ell+t-1}\binom{k+\ell-1}{k-t}\iota^{2(k-t)}\\
&\quad-\sum_{\iota=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=-(\ell-1)}^{-1}(-(m-p)^2)^{\ell+t-1}\binom{k+\ell-1}{k-t}\iota^{2(k-t)}\\
&= -\sum_{\iota=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=-(\ell-1)}^{k-1}(-(m-p)^2)^{\ell+t-1}\binom{k+\ell-1}{k-t}\iota^{2(k-t)}\\
&= -\sum_{\iota=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=0}^{k+\ell-2}(-(m-p)^2)^{t}\binom{k+\ell-1}{k+\ell-1-t}\iota^{2(k+\ell-1-t)}\\
&= -\sum_{\iota=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\sum_{t=0}^{k+\ell-1}\binom{k+\ell-1}{k+\ell-1-t}(-(m-p)^2)^{t}\iota^{2(k+\ell-1-t)}\\
&\quad+\sum_{\iota=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}\binom{k+\ell-1}{0}(-(m-p)^2)^{k+\ell-1}\iota^{0}
\end{aligned}
$$

$$= -\sum_{i=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}(-(m-p)^2+i^2)^{k+\ell-1}+\sum_{i=1}^{m}\sum_{\ell=1}^{m+1}\alpha_{n,p,\ell}(-(m-p)^2)^{k+\ell-1}$$

$$= \sum_{i=1}^{m} -(-(m-p)^2+i^2)^{k-1}F_{n,p}(-(m-p)^2+i^2)$$

$$+ (-(m-p)^2)^{k-1}\sum_{i=1}^{m}F_{n,p}(-(m-p)^2)$$

$$= 0. \qquad \square$$

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