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HARNACK INEQUALITY AND REGULARITY OF p-LAPLACE EQUATION ON COMPLETE MANIFOLDS

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Abstract

In this paper, we will derive a mean value inequality and a Harnack inequality for nonnegative functions which satisfies the differential inequality

$$|\mathrm{div}(|f|^{p-2}\nabla f)| \le A \quad f^{p-1}$$

in the weak sence on complete manifolds, where constants $A \ge 0$, $p > 1$; as a consequence, we give a C^{α} estimate for weak solutions of the above differential inequality, then we generalize the results in [1], [2].

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1. Introduction

Let M be a complete Riemannian manfold, and f be a real C^2 function on M. Fix $p > 1$ and consider a compact domain $\Omega \subset M$. The p-energy of f on Ω, is defined to be,

$$E_p(\Omega, f) = \frac{1}{p}\int_{\Omega} |\nabla F|^p \, dv_g \tag{1.1}$$

The function f is said to be p-harmonic on M if f is a critical point of $E_p(\Omega, *)$ for every compact domain $\Omega \subset M$. Equaivalently, f satisfies the Euler-Lagrange equation.

$$\Delta_p f \equiv \mathrm{div}(|\nabla f|^{p-2}\nabla f) = 0 \tag{1.2}$$

Let $g \in H_{1,p}(\Omega)$ satisfies the equation (1.2) in the weak sence, it is:

$$\int_{\Omega} \langle |\nabla g|^{p-2} \cdot \nabla_g, \nabla\phi\rangle \, dv_g = 0 \tag{1.3}$$

for any $\phi \in C_0^{\infty}(\Omega)$, then g is said to be a weakly solution of eguation (1.2) on Ω.

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Definition. f is called a weakly p-harmonic if $f \in H_{1,p}^{loc}(M)$ is a weak solution of the Euler-Lagrange equation of the p-energy functional (1.2) as follows:

$$\int_M |f|^{p-2}\langle d\eta, df\rangle = 0$$

for all $\eta \in C_0^\infty(M)$.

Regularity estimates for elliptic systems on domain $\Omega \subset R^n$, in particular the Euler-Lagrange equation for p-energy, were first obtained by Uhlenbeck [3] for $p \geq 2$, and later by Tolksdorff [4] for $p > 1$. The aim of this paper is to obtain regularity estimates for a more general class of equations on complete manifolds. In section two and section three, by using the iteration procedure of Moser and discussing like that in [1], we derive a mean value inequality and a Harnack inequality for nonnegative functions which satisfies the differetial inequality of the following form:

$$|\mathrm{div}(|f|^{p-2}\nabla f)| \leq A \cdot f^{p-1} \tag{1.4}$$

in the weak sence for some constant $A \geq 0$. As a special case: $A = 0$, using the above Harnack inequality, we can derive a (global) Harnack inequality for weakly p-harmonic function which is similar to a result of M. Rigoli, M. Salvatori, and M. Vignati [2]. At the end of this paper, we will give a C^α etimate for solutions of above differential inequality. When $p = 2$, the above mean value inequality, Harnack inequality, and C^α estimate is just the results due to P. Li in [1]. On the other hand, using the above Harnack inequality, we can obtain a Liouville type theorem which can be see a generalization of the result in [2].

Theorem. *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Then there exists a constant $0 < \alpha \leq 1$ such that any p-harmonic function f defined on M satisfying the growth condition*

$$|f(x)| = \circ(\rho^\alpha(x)),$$

as $x \to 0$, where $\rho(x)$ denotes the geodesic distance from o to x; must be identically constant.

2. Mean-value inequality

Lemma 2.1. *Let M be a complete Riemannian manifold, and geodesic ball $B_o(R)$ satisfies: $B_o(R) \cap \partial M = \emptyset$. If $f \in H_{1,p}(B_o(R))$ is a nonnegative function, and satisfies the following inequality in weak sence:*

$$\mathrm{div}(|\nabla f|^{p-2}\nabla f) \geq -A \cdot f^{p-1} \tag{2.1}$$

where constants $A \geq 0$; $p > 1$, then for any $0 < r \leq R$, $\tilde{q} \geq p$ and nonnegative function $\eta \in C_0^\infty(B_o(r))$, there:

$$\int_{B_o(r)} \eta^p \cdot f^{\tilde{q}-p} \cdot |\nabla f|^p \le \frac{2^p \cdot (p-1)^{p-1}}{(\tilde{q}-p+1)^p} \cdot \int_{B_o(r)} f^{\tilde{q}} \cdot |\nabla \eta|^p + \frac{2A}{\tilde{q}-p+1} \cdot \int_{B_o(r)} \eta^p \cdot f^{\tilde{q}} \tag{2.2}$$

Proof. Multiplying $\eta^p \cdot f^{\tilde{q}-p+1}$ to (2.1), and integrating yields

$$\int_{B_o(r)} \eta^p \cdot f^{\tilde{q}-p+1} \cdot \operatorname{div}(|\nabla f|^{p-2}\nabla f) \ge -A \cdot \int_{B_o(r)} \eta^p \cdot f^{\tilde{q}}$$

Using Green's formula; Schwartz inequality; and Young inequality, we have:

$$\begin{aligned}
(\tilde{q}-p+1) \cdot \int_{B_o(r)} \eta^p \cdot f^{\tilde{q}-p} \cdot |\nabla f|^p \\
&\le A \cdot \int_{B_o(r)} \eta^p \cdot f^{\tilde{q}} - p \cdot \int_{B_o(r)} |\nabla f|^{p-2} \cdot f^{\tilde{q}-p+1} \cdot \eta^{p-1} \cdot \langle \nabla f, \nabla \eta \rangle \\
&\le A \cdot \int_{B_o(r)} \eta^p \cdot f^{\tilde{q}} + p \cdot \int_{B_o(r)} |\nabla f|^{p-1} \cdot f^{\tilde{q}-p+1} \cdot \eta^{p-1} \cdot |\nabla \eta| \\
&\le A \cdot \int_{B_o(r)} \eta^p \cdot f^{\tilde{q}} + \frac{\tilde{q}-p+1}{2} \cdot \int_{B_o(r)} |\nabla f|^p \cdot f^{\tilde{q}-p} \cdot \eta^p \\
&\quad + \left[\frac{2p-2}{\tilde{q}-p+1}\right]^{p-1} \int_{B_o(r)} f^{\tilde{q}} \cdot |\nabla \eta|^p
\end{aligned}$$

then

$$\int_{B_o(r)} \eta^p \cdot f^{\tilde{q}-p} \cdot |\nabla f|^p \le \frac{2^p \cdot (p-1)^{p-1}}{(\tilde{q}-p+1)^p} \int_{B_o(r)} f^{\tilde{q}} \cdot |\nabla \eta|^p + \frac{2A}{\tilde{q}-p+1} \cdot \int_{B_o(r)} \eta^p \cdot f^{\tilde{q}} \qquad \square$$

PROPOSITION 2.2. *Let M be a complete Riemannian manifold, and a geodesic ball $B_o(R)$ satisfies: $B_o(R) \cap \partial M = \emptyset$. If there exists a sobolev inequality of the following form:*

$$\left(\int_{B_o(r)} \phi^{p\mu/(\mu-p)}\right)^{(\mu-p)/p\mu} \le C_s \cdot V(B_o(r))^{-1/\mu} \cdot r \cdot \left\{\left(\int_{B_o(r)} |\nabla \phi|^p\right)^{1/p} + r^{-1} \cdot \left(\int_{B_o(r)} |\phi|^p\right)^{1/p}\right\} \tag{2.3}$$

for any $\phi \in H^c_{1,p}(B_o(r))$ *and* $0 < r < R$. *Where constants* $\mu > p$, $C_s > 0$, *and* $V(B_o(r))$ *denotes the volume of geodesic ball* $B_o(r)$. *Assuming* $f \in H_{1,p}(B_o(R))$ *is a nonnegative function, and satisfies the following inequality in weak sence,*

$$\operatorname{div}(|\nabla f|^{p-2}\nabla f) \geq -A \cdot f^{p-1}$$

for some constant $A \geq 0$; *then for any* $q > 0$, $0 < \theta < 1$, *and* $0 < r < R$; *there must exist a constant* $C_1 > 0$, *depending only on* q, μ, C_s, p, *such that*:

$$\sup_{B_o(\theta r)} f \leq C_1 \cdot (Ar^P + (1-\theta)^{-p})^{\mu\cdot(q+p)/(p\cdot q)} \cdot V(B_o(r))^{-1/q} \cdot \left(\int_{B_o(r)} f^q\right)^{1/q} \tag{2.4}$$

Proof. Setting $0 < r_1 < r_2 \leq r$, $\tilde{q} \geq p$, and let $\eta \in C_0^\infty(B_o(R))$ be the cut-off function

$$\eta(x) = \begin{cases} 1; & x \in B_o(r_1) \\ 0; & x \in B_o(R)\backslash B_o(r_2) \end{cases}$$

$\eta(x) \in [0,1]$, $|\nabla\eta| \leq 2/(r_2 - r_1)$. Using the sobolev inequality (2.3) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
&\left(\int_{B_o(r_1)} f^{\tilde{q}\cdot\mu/(\mu-p)}\right)^{(\mu-p)/p\mu} \\
&\leq \left(\int_{B_o(r_2)} (\eta\cdot f^{\tilde{q}/p})^{p\cdot\mu/(\mu-p)}\right)^{(\mu-p)/p\mu} = \left(\int_{B_o(r)} (\eta\cdot f^{\tilde{q}/p})^{p\cdot\mu/(\mu-p)}\right)^{(\mu-p)/p\mu} \\
&\leq C_s \cdot V(B_o(r))^{-1/\mu} \cdot \left[r\left(\int_{B_o(r)} |\nabla(\eta\cdot f^{\tilde{q}/p})|^p\right)^{1/p} + \left(\int_{B_o(r)} \eta^p \cdot f^{\tilde{q}}\right)^{1/p}\right] \\
&\leq C_s \cdot V(B_o(r))^{-1/\mu} \cdot \left[\left(\int_{B_o(r)} \eta^p \cdot f^{\tilde{q}}\right)^{1/p}\right. \\
&\quad \left. + 2r\left(\int_{B_o(r)} (|\nabla\eta|^p \cdot f^{\tilde{q}} + \left(\frac{\tilde{q}}{p}\right)^p \cdot \eta^p \cdot f^{\tilde{q}-p}\cdot|\nabla f|^p)\right)^{1/p}\right]
\end{aligned}$$

by formula (2.2), we have:

$$
\begin{aligned}
(2.5)\quad & \left(\int_{B_o(r_1)} f^{\tilde{q}\cdot\mu/(\mu-p)}\right)^{(\mu-p)/p\mu} \\
&\le C_s\cdot V(B_o(r))^{-1/\mu}\cdot\left[\left(\left(\int_{B_o(r)}\eta^p\cdot f^{\tilde{q}}\right)^{1/p}\right.\right. \\
&+2r\left(\int_{B_o(r)}|\nabla\eta|^p\cdot f^{\tilde{q}}+\left(\frac{\tilde{q}}{p}\right)^p\cdot\frac{2^p\cdot(p-1)^{p-1}}{(\tilde{q}-p+1)^p}\cdot\int_{B_o(r)} f^{\tilde{q}}\cdot|\nabla\eta|^p\right. \\
&\left.\left.+\left(\frac{\tilde{q}}{p}\right)^p\cdot\frac{2A}{\tilde{q}-p+1}\cdot\int_{B_o(r)}\eta^p\cdot f^{\tilde{q}}\right)^{1/p}\right] \\
&\le C_s\cdot V(B_o(r))^{-1/\mu}\cdot\left[16\tilde{q}\cdot\left(Ar^p+\frac{r^p}{(r_2-r_1)^p}\right)^{1/p}+1\right]\cdot\left(\int_{B_o(r_2)} f^{\tilde{q}}\right)^{1/p} \\
&\le 17\cdot\tilde{q}\cdot C_s\cdot V(B_o(r))^{-1/\mu}\cdot\left(Ar^p+\frac{r^p}{(r_2-r_1)^p}\right)^{1/p}\cdot\left(\int_{B_o(r_2)} f^{\tilde{q}}\right)^{1/p}
\end{aligned}
$$

Let:

$$
\begin{cases}
R_i = r_3+(r_4-r_3)\cdot 2^{-i}, \\
q_i = p\cdot\left(\dfrac{\mu}{\mu-p}\right)^i.
\end{cases}
$$

where $0<r_3<r_4\le r$. Denote $k=\mu/(\mu-p)$, appling (2.5) to $r_1=R_{i+1}$, $r_2=R_i$, $\tilde{q}=q_i$, we have:

$$
\begin{aligned}
(2.6)\quad & \left\{\int_{B_o(R_{i+1})} f^{p\cdot k^{i+1}}\right\}^{k^{-(i+1)}} \\
&\le (17\cdot C_s\cdot V(B_o(r))^{-1/\mu})^{p/k^i}\cdot(p\cdot k^i)^{p/k^i} \\
&\left(Ar^p+\frac{r^p}{(r_4-r_3)^p}\right)^{1/k^i}\cdot 2^{(i+1)p/k^i}\cdot\left\{\int_{B_o(R_i)} f^{p\cdot k^i}\right\}^{k^{-i}}
\end{aligned}
$$

Observe that $\lim_{i\to\infty} R_i = r_3$, and iterating the inequality (2.6), we conclude that:

$$
(2.7)\quad \sup_{B_o(r_3)} f^p \le C_2\cdot\left(Ar^p+\frac{r^p}{(r_4-r_3)^p}\right)^{\mu/p}\cdot V(B_o(r))^{-1}\cdot\int_{B_o(r_4)} f^p
$$

where we have used $\sum_{i=0}^{\infty} 1/k^i=\mu/p$, $\sum_{i=0}^{\infty}(i+1)/k^i=\mu^2/p^2$, and denote $C_2=(17\cdot p\cdot C_s/k)^{\mu}\cdot(2k)^{\mu^2/p}$.

(a) When $q \geq p$, appling (2.7) to $r_3 = \theta \cdot R$, and $r_4 = R$, by Hölder inequality, we have:

$$\sup_{B_o(\theta r)} f \leq C_2^{1/p} \cdot (Ar^p + (1-\theta)^{-p})^{\mu/p^2} \cdot \left(\frac{\int_{B_o(r)} f^p}{V(B_o(r))}\right)^{1/p} \tag{2.8}$$

$$\leq C_2^{1/p} \cdot (Ar^p + (1-\theta)^{-p})^{\mu/p^2} \cdot \left(\frac{\int_{B_o(r)} f^q}{V(B_o(r))}\right)^{1/q}$$

(b) When $0 < q < p$. Let $h_0 = \theta r$, $h_1 = \theta r + 2^{-1} \cdot (1-\theta) \cdot r_1, \ldots, h_i = h_{i-1} + 2^{-i} \cdot (1-\theta) \cdot r$, for each $i = 1, 2, 3, \ldots$; applying (2.7) to $r_3 = h_i$, $r_4 = h_{i+1}$, we have:

$$\sup_{B_o(h_i)} f^p \leq C_2 \cdot \left(Ar^p + \frac{r^p}{(h_{i+1} - h_i)^p}\right)^{\mu/p} \cdot V(B_o(r))^{-1} \cdot \int_{B_o(h_{i+1})} f^p \tag{2.9}$$

$$\leq C_2 \cdot (Ar^p + (1-\theta)^{-p})^{\mu/p} \cdot 2^{(i+1)\mu} \cdot V(B_o(r))^{-1}$$

$$\cdot \int_{B_o(h_{i+1})} f^q \cdot \sup_{B_o(h_{i+1})} f^{p-q}$$

denote $M(i) = \sup_{B_o(h_i)} f^p$, (2.9) becomes:

$$M(i) \leq C_2 \cdot (Ar^p + (1-\theta)^{-p})^{\mu/p} \cdot 2^{(i+1)\mu} \cdot V(B_o(r))^{-1} \tag{2.10}$$

$$\cdot \int_{B_o(r)} f^q \cdot M(1+i)^{1-(q/p)}$$

Let $\lambda = 1 - (q/p)$, interating the inequality, we conclude that:

$$M(0) \leq \prod_{i=0}^{j-1} \left\{ C_2 \cdot (Ar^p + (1-\theta)^{-p})^{\mu/p} \cdot V(B_o(r))^{-1} \cdot \int_{B_o(r)} f^q \right\}^{\lambda^i} \tag{2.11}$$

$$\cdot 2^{(i+1)\mu\lambda^i} \cdot M(j)^{\lambda^j}$$

let $j \to +\infty$, we have

$$\sup_{B_o(\theta r)} f \leq \{C_2 \cdot 2^{\mu \cdot p^2 \; q^{-1}} (Ar^p + (1-\theta)^{-p})^{\mu/p} \tag{2.12}$$

$$\cdot V(B_o(r))^{-1}\} q^{-1} \cdot \left(\int_{B_o(r)} f^q\right)^{1/q}$$

In any event, (2.8), (2.12) imply that, for any $q > 0$, we have the inequality

$$\sup_{B_o(\theta r)} f \leq C_1 \cdot (Ar^P + (1-\theta)^{-p})^{\mu \cdot (q+p)/(p \cdot q)} \cdot V(B_o(r))^{-1/q} \cdot \left(\int_{B_o(r)} f^q\right)^{1/q}$$

for some appropriate constant $C_1 > 0$ depending only on μ, p, q, C_s. □

3. Harnack inequality

LEMMA 3.1. *Let M be a complete Riemannian manifold, and a geodesic ball $B_o(R)$ satisfies: $B_o(R) \cap \partial M = \emptyset$. If it satisfies the following conditions:*

(1) *For any $0 < r < R$, there exist a constant $\eta > 0$, such that*

$$V(B_o(r)) \leq 2^\eta \cdot V\left(B_o\left(\frac{r}{2}\right)\right) \tag{3.1}$$

(2) *Poincaré inequality, i.e there exist a constant $C_p > 0$ such that*

$$\int_{B_o(r)} |f - f_B|^p \leq C_p \cdot r^p \cdot \int_{B_o(r)} |\nabla f|^p \tag{3.2}$$

for any $0 < r < R$, $f \in H_{1,p}(B_o(r))$. Where $f_B = \int_{B_o(r)} f / V(B_o(r))$.

(3) *Sobolev inequality, i.e there exist a constant $C_s > 0$ such that:*

$$\left(\int_{B_o(r)} \phi^{p\mu/(\mu-p)}\right)^{(\mu-p)/p\mu} \leq C_s \cdot V(B_o(r))^{-1/\mu} \cdot r \cdot \left\{\left(\int_{B_o(r)} |\nabla\phi|^p\right)^{1/p} + r^{-1} \cdot \left(\int_{B_o(r)} |\phi|^p\right)^{1/p}\right\} \tag{3.3}$$

for any $\phi \in H^c_{1,p}(B_o(r))$, $0 < r < R$. Where constants $\mu > p > 1$, $C_s > 0$, and $V(B_o(r))$ denotes the volume of geodesic ball $B_o(r)$.

Assuming $f \in H_{1,p}(B_o(R))$ is a nonnegative function, and satisfies the following inequality in the weak sence

$$\operatorname{div}(|\nabla f|^{p-2}\nabla f) \leq A \cdot f^{p-1} \tag{3.4}$$

for some constant $A \geq 0$; then for $q > 0$ sufficiently small, there must be exist a constant $C_5 > 0$, depending only on q, μ, C_s, $p\eta$, C_p, $(AR^p + 1)$, such that:

$$\left\{\frac{\int_{B_o(R/8)} f^q \, dv_g}{V(B_o(R/8))}\right\}^{1/q} \leq C_5 \cdot \inf_{B_o(R/16)} f \tag{3.5}$$

Proof. For any $\varepsilon > 0$, setting $f_\varepsilon = f + \varepsilon$, f_ε satisfies the inequality (3.4). Letting $\varepsilon \to 0$, it is sufficient to prove that f_ε satisfies the inequality (3.5). So we can assume that $f \geq \varepsilon > 0$, then the function f^{-1} is in $H_{1,p}(B_o(R))$ and satisfies:

$$\begin{aligned}
\operatorname{div}(|\nabla(f^{-1})|^{p-2}\nabla(f^{-1})) &= \operatorname{div}(-f^{-2(p-1)} \cdot |\nabla f|^{p-2} \cdot \nabla f) \\
&= -f^{-2(p-1)} \cdot \operatorname{div}(|\nabla f|^{p-2} \cdot \nabla f) \\
&\quad + 2(p-1) \cdot f^{-2p+1} \cdot |\nabla f|^p \\
&\geq -A \cdot f^{-(p-1)}
\end{aligned}$$

Applying proposition 2.2, there exist a constant $C_2' > 0$, depending only on q, p, μ, C_s such that:

$$(3.6) \qquad \left(\inf_{B_o(R/16)} f\right)^{-1} = \sup_{B_o(R/16)} f^{-1}$$
$$\leq C_2' \cdot (AR^p + 1)^{\mu(p+q)/(p\cdot q)} \cdot \left\{\frac{\int_{B_o(R/8)} f^{-q}\, dv_g}{V(B_o(R/8))}\right\}^{1/q}$$

Clearly, the lemma follows if we can estimate the product

$$\left\{\frac{\int_{B_o(R/8)} f^{q}\, dv_g}{V(B_o(R/8))}\right\}^{1/q} \cdot \left\{\frac{\int_{B_o(R/8)} f^{-q}\, dv_g}{V(B_o(R/8))}\right\}^{1/q}$$

from above for some value of $q > 0$.

To achieve this, let us consider the function $u = \beta + \log f$, where $\beta = -\int_{B_o(R/2)} \log f\, dv_g$; then u satisfies:

$$(3.7) \qquad \begin{aligned} \operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \operatorname{div}(f^{-(p-1)} \cdot |\nabla f|^{p-2} \cdot \nabla f) \\ &= f^{-(p-1)} \cdot \operatorname{div}(|\nabla f|^{p-2} \cdot \nabla f) - (p-1) f^{-p} \cdot |\nabla f|^p \\ &\leq A - (p-1) \cdot |\nabla u|^p \end{aligned}$$

Let ψ the cut-off function defined by:

$$\psi(x) = \begin{cases} 0, & \text{for } x \in M \backslash B_o(R) \\ \dfrac{2(R - r(x))}{R}, & \text{for } x \in B_o(R) \backslash B_o\left(\dfrac{R}{2}\right) \\ 1, & \text{for } x \in B_o\left(\dfrac{R}{2}\right) \end{cases}$$

where $r(x)$ is the distance from o to x.

Multiplying (3.7) by ψ^p and integrating, we have:

$$(3.8) \qquad \begin{aligned} (p-1) \cdot \int |\nabla u|^p \cdot \psi^p &\leq A \int \psi^p - \int \psi^p \cdot \operatorname{div}(|\nabla u|^{p-2}\nabla u) \\ &\leq A \int \psi^p + p \int |\nabla u|^{p-1} \cdot \psi^{p-1} \cdot |\nabla \psi| \\ &\leq A \int \psi^p + \frac{p-1}{2} \int \psi^p \cdot |\nabla u|^p + 2^{p-1} \cdot \int |\nabla \psi|^p \end{aligned}$$

where we have used Green's formula, Schwartz inequality, and Young inequality. by the above ineguality, we have

$$\int_{B_o(R/2)} |\nabla u|^p \le \int \psi^p \cdot |\nabla u|^p \le \frac{2A}{p-1}\int \psi^p + \frac{2^p}{p-1}\int |\nabla \psi|^p \le \frac{4^p}{p-1}\cdot (AR^p+1)\cdot \frac{V(B_o(R))}{R^p} \tag{3.9}$$

the Poincare inequality (3.2) and (3.9) implies that:

$$\int_{B_o(R/2)} |u|^p \le \frac{C_p \cdot R^p}{2^p}\int_{B_o(R/2)} |\nabla u|^p \le C_6 \cdot V(B_o(R)) \tag{3.10}$$

denoted $C_6 = 2^p \cdot C_p/(p-1)\cdot(AR^p+1)$.

For $\forall q \le p$, using Hölder inequality, we have

$$\int_{B_o(R/2)} |u|^q \le \left(\int_{B_o(R/2)} |u|^p\right)^{q/p} \cdot \left(\int_{B_o(R/2)} 1\right)^{1-(q/p)} \le C_6^{q/p}\cdot V(B_o(R)) \tag{3.11}$$

On the other hand, let ϕ be a Lipschitz cut-off function, given by

$$\phi(x) = \begin{cases} 0, & \text{for } x \in M\backslash B_o(\rho+\sigma) \\ \dfrac{\rho+\sigma-r(x)}{\sigma}, & \text{for } x\in B_o(\rho+\sigma)\backslash B_o(\rho) \\ 1, & \text{for } x \in B_o(\rho). \end{cases}$$

where $\rho, \sigma > 0$, $\rho+\sigma \le R$.

Then multiplying $\phi^p \cdot |u|^{pa-p}$ to (3.7) for $a \ge 2$, and integrating by parts yields

$$\begin{aligned}(p-1)&\int \phi^p\cdot |u|^{pa-p}\cdot|\nabla u|^p \\ &\le A\int \phi^p \cdot |u|^{pa-p} - \int \operatorname{div}(|\nabla u|^{p-2}\nabla u)\cdot \phi^p\cdot |u|^{pa-p} \\ &\le A\int \phi^p |u|^{pa-p} + (pa-p)\cdot \int \phi^p \cdot |u|^{pa-p-1}\cdot |\nabla u|^p \\ &\quad + p\cdot \int |\nabla u|^{p-2}\cdot \phi^{p-1}\cdot |u|^{pa-p}\cdot \langle \nabla u, \nabla\phi\rangle \end{aligned} \tag{3.12}$$

by Young inequality we have:

$$\begin{aligned}(pa-p)&\cdot \phi^p\cdot |u|^{pa-p-1}\cdot|\nabla u|^p \\ &\le \frac{p-1}{4}\cdot |u|^{pa-p}\cdot \phi^p \cdot |\nabla u|^p + \left(\frac{4(pa-p-1)}{p-1}\right)^{pa-p-1}\cdot \phi^p\cdot |\nabla u|^p\end{aligned} \tag{3.13}$$

and

$$p \cdot |\nabla u|^{p-2} \cdot \phi^{p-1} \cdot |u|^{pa-p} \cdot \langle \nabla u, \nabla \phi \rangle \leq \frac{p-1}{4} |u|^{pa-p} \cdot \phi^p \cdot |\nabla u|^p + 4^{p-1} \cdot |\nabla u|^{pa-p} \cdot |\nabla \phi|^p \tag{3.14}$$

Using (3.12), (3.13), (3.14); then

$$\begin{aligned} \int \phi^p \cdot |u|^{pa-p} \cdot |\nabla u|^p &\leq \frac{2A}{p-1} \int \phi^p \cdot |u|^{pa-p} + \frac{2}{p-1} \cdot \left(\frac{4(pa-p-1)}{p-1} \right)^{pa-p-1} \cdot \int \phi^p \cdot |\nabla u|^p \\ &+ \frac{2 \cdot 4^{p-1}}{p-1} \cdot \int |u|^{pa-p} \cdot |\nabla \phi|^p \\ &\leq \left(\frac{2A}{p-1} + \frac{2 \cdot 4^{p-1}}{(p-1) \cdot \sigma^p} \right) \cdot \int_{B_o(\rho+\sigma)} |u|^{pa-p} \\ &+ \frac{2}{p-1} \cdot \left(\frac{4(pa-p-1)}{p-1} \right)^{pa-p-1} \cdot \int_{B_o(\rho+\sigma)} |\nabla u|^p \end{aligned} \tag{3.15}$$

By setting $a = 2$, $\rho = R/4$, $\sigma = R/4$; (3.15) becomes:

$$\int_{B_o(R/4)} |u|^p \cdot |\nabla u|^p \leq \left(\frac{2A}{p-1} + \frac{2 \cdot 4^{2p-1}}{(p-1) \cdot R^p} \right) \cdot \int_{B_o(R/2)} |u|^p + \frac{2 \cdot 4^{p-1}}{p-1} \cdot \int_{B_o(R/2)} |\nabla u|^p$$

Using (3.9), (3.10), and the last inequality; then:

$$\int_{B_o(R/4)} |u|^p \cdot |\nabla u|^p \leq C_7 \cdot \frac{V(B_o(R))}{R^p} \tag{3.16}$$

where we denoted $C_7 = 2 \cdot 4^{2p-1}/(p-1) \cdot (C_6 + 1/(p-1)) \cdot (AR^p + 1)$.

Then, we want to estimate $\int_{B_o(R/4)} |u|^2$ from above.

(1) When $1 < p < 2$, for any $\tilde{q} \leq p$, by Hölder inequality, (3.9), and (3.16), we have:

$$\begin{aligned} \int_{B_o(R/4)} |u|^{\tilde{q}} \cdot |\nabla u|^p &\leq \left(\int_{B_o(R/4)} |u|^p \cdot |\nabla u|^p \right)^{\tilde{q}/p} \cdot \left(\int_{B_o(R/4)} |\nabla u|^p \right)^{1-(\tilde{q}/p)} \\ &\leq C_7^{\tilde{q}/p} \cdot \left(\frac{4^p}{p-1} \cdot (AR^p + 1) \right)^{1-(\tilde{q}/p)} \cdot \frac{V(B_o(R))}{R^p} \end{aligned} \tag{3.17}$$

Let $l \in Z^+$, such that $p^{l-1} < 2 \leq p^l$; and let $1 \leq i \leq l - 1$, by Minkowski inequality and Poincare inequality (3.2), then

$$\int_{B_o(R/4)} p'^p \cdot |u|^{(p'-1)\cdot p} \cdot |\nabla u|^p \geq \int_{B_o(R/4)} |\nabla(|u|^{p'})|^p$$

$$\geq \frac{4^p}{C_p \cdot R^p} \cdot \int_{B_o(R/4)} \left| \left(|u|^{p'} - V\left(B_o\left(\frac{R}{4}\right)\right)^{-1} \cdot \int_{B_o(R/4)} |u|^{p'} \right) \right|^p$$

$$\geq \frac{4^p}{C_p \cdot R^p} \left[\left(\int_{B_o(R/4)} |u|^{p'^{+1}} \right)^{1/p} - V\left(B_o\left(\frac{R}{4}\right)\right)^{-1+(1/p)} \cdot \int_{B_o(R/4)} |u|^{p'} \right]^p$$

By Hölder inequality, it is easy to show that $(\int_{B_o(R/4)} |u|^{p'^{+1}})^{1/p} \geq V(B_o(R/4))^{-1+(1/p)} \cdot \int_{B_o(R/4)} |u|^{p'}$. then the last inequality becomes:

$$\left(\int_{B_o(R/4)} |u|^{p'^{+1}} \right)^{1/p} \leq \left\{ \frac{p'^p \cdot C_p \cdot R^p}{4^p} \cdot \int_{B_o(R/4)} |u|^{(p'-1)\cdot p} \cdot |\nabla u|^p \right\}^{1/p} + V\left(B_o\left(\frac{R}{4}\right)\right)^{-1+(1/p)} \cdot \int_{B_o(R/4)} |u|^{p'}$$

Using (3.17) and the last inequality, we have:

$$(3.18) \quad \left(\int_{B_o(R/4)} |u|^{p'^{+1}} \right)^{1/p} \leq \left\{ \frac{p'^p \cdot C_p}{4^p} \cdot C_7^{p'-1} \cdot \left[\frac{4^p}{p-1} \cdot (AR^p + 1) \right]^{p'} \cdot V(B_o(R)) \right\}^{1/p} + 4^{\eta\cdot(1-(1/p))} \cdot V(B_o(R))^{-1+(1/p)} \cdot \int_{B_o(R/4)} |u|^{p'}$$

Where we have used the condition (1) $V(B_o(r)) \leq 2^\eta \cdot V(B_o(r/2))$, $0 < r \leq R$. By formula (3.10), one can conclude that: $\int_{B_o(R/4)} |u|^p \leq \int_{B_o(R/2)} |u|^p \leq C_6 \cdot V(B_o(R))$. Iterating the inequality (3.18) by finite times, one can conclude that there must be exist a constant $C_8 > 0$, depending only on $p, \eta, C_p, (AR^p + 1)$, such that:

$$\int_{B_o(R/4)} |u|^{p'} \leq C_8 \cdot V(B_o(R))$$

By Hölder inequality, we have:

$$(3.19) \quad \int_{B_o(R/4)} |u|^2 \leq \left(\int_{B_o(R/4)} |u|^{p'} \right)^{2/p'} \cdot \left(\int_{B_o(R/4)} 1 \right)^{1-(2/p')} \leq C_8^{2/p'} \cdot V(B_o(R))$$

(2) When $p \geq 2$, we have:

$$\int_{B_o(R/4)} |u|^2 \leq \left(\int_{B_o(R/4)} |u|^p\right)^{2/p} \cdot \left(\int_{B_o(R/4)} 1\right)^{1-(2/p)} \leq C_6^{2/p} \cdot V(B_o(R)) \tag{3.20}$$

In any event, (3.19), (3.20) imply that, for any $p > 1$, we have the inequality

$$\int_{B_o(R/4)} |u|^2 \leq C_9 \cdot V(B_o(R)) \tag{3.21}$$

for some appropriate constant $C_9 > 0$, depending only on $p, \eta, C_p, (AR^p + 1)$.

On the other hand, using the Minkowski inequality and the poincare inequality (3.2), we have:

$$\begin{aligned} \int_{B_o(R/4)} |u|^p \cdot |\nabla u|^p &= \frac{1}{2^p} \int_{B_o(R/4)} |\nabla(u^2)|^p \\ &\geq \frac{2^p}{C_p \cdot R^p} \cdot \int_{B_o(R/4)} \left| \left(u^2 - V\left(B_o\left(\frac{R}{4}\right)\right)^{-1} \cdot \int_{B_o(R/4)} u^2 \right) \right|^p \\ &\geq \frac{2^p}{C_p \cdot R^p} \left[\left(\int_{B_o(R/4)} u^{2p} \right)^{1/p} - V\left(B_o\left(\frac{R}{4}\right)\right)^{-1+(1/p)} \cdot \int_{B_o(R/4)} u^2 \right]^p \end{aligned} \tag{3.22}$$

by (3.16), (3.21), then (3.22) becomes:

$$\int_{B_o(R/4)} u^{2p} \leq C_{10} \cdot V(B_o(R)) \tag{3.23}$$

where we denoted: $C_{10} = ((C_7 \cdot C_p/2^p)^{1/p} + 4^{\eta(1-(1/p))} \cdot C_9)^p$.

For any $\tilde{q} \leq 2p$, using Hölder inequality one can conclude:

$$\int_{B_o(R/4)} |u|^{\tilde{q}} \leq \left(\int_{B_o(R/4)} |u|^{2p}\right)^{\tilde{q}/2p} \cdot \left(\int_{B_o(R/4)} 1\right)^{1-(\tilde{q}/2p)} \leq C_{10}^{\tilde{q}/2p} \cdot V(B_o(R)) \tag{3.24}$$

Let $a \geq 2$, by Cauchy-Schwartz inequality, we have

$$|\nabla(\phi|u|^a)|^p \leq 2^p[|\nabla\phi|^p \cdot |u|^{ap} + a^p|u|^{pa-p}\phi^p|\nabla u|^p] \tag{3.25}$$

By the Sobolev inequality, one can conclude:

$$\left(\int_{B_o(\rho)} |u|^{a\cdot p\cdot\mu/(\mu-p)}\right)^{(\mu-p)/\mu} \left(\int_{B_o(\rho)} (\phi|u|^a)^{p\cdot\mu/(\mu-p)}\right)^{(\mu-p)/\mu}$$

$$\leq \left\{ C_s \cdot V(B_o(R))^{-1/\mu}\left[R\left(\int_{B_o(R)} |\nabla(\phi|u|^a)|^p\right)^{1/p} + \left(\int_{B_o(R)} \phi^p|u|^{ap}\right)^{1/p}\right]\right\}^p$$

$$\leq 2^p \cdot C_s^p \cdot V(B_o(R))^{-p/\mu}\left[R^p\int_{B_o(R)} |\nabla(\phi|u|^a)|^p + \int_{B_o(R)} \phi^p|u|^{ap}\right]$$

$$\leq 2^p \cdot C_s^p \cdot V(B_o(R))^{-p/\mu}\left[R^p \cdot 2^p \cdot a^p \int_{B_o(R)} \phi^p \cdot |u|^{pa-p}|\nabla u|^p\right.$$

$$\left. + R^p \cdot 2^p \int_{B_o(R)} |u|^{ap} \cdot |\nabla\phi|^p + \int_{B_o(R)} \phi^p|u|^{ap}\right]$$

Using (3.9), (3.15), and the last inequality, we have:

$$(3.26) \quad \left(\int_{B_o(\rho)} |u|^{a\cdot p\cdot k}\right)^{1/k}$$

$$\leq 2 \cdot C_{11} \cdot V(B_o(R))^{(1-k)/k}\left[a^p \cdot \left(AR^p + \frac{R^p}{\sigma^p}\right)\int_{B_o(\rho+\sigma)} |u|^{pa-p}\right.$$

$$+ a^p\left(\frac{4(pa-p-1)}{p-1}\right)^{pa-p-1} \cdot (AR^p+1) \cdot V(B_o(R))$$

$$\left. + \frac{R^p}{\sigma^p} \cdot V(B_o(R))^{-1} \cdot \int_{B_o(\rho+\sigma)} |u|^{pa}\right]$$

where we denoted: $C_{11} = 2^P \cdot C_s^p \cdot \max\{2^{3p-1}/(p-1), 2^{3p+1}/(p-1)^2, 2^{p+1}\}$, $k = \mu/(\mu-p)$.

It is easy to show that: $|u|^{pa-p} \leq |u|^{pa}+1$, and let $\rho \geq R/8$, then (3.26) becomes:

$$(3.27) \quad \left(V(B_o(\rho))^{-1}\int_{B_o(\rho)} |u|^{a\cdot p\cdot k}\right)^{1/(k\cdot p\cdot a)}$$

$$\leq \left(8^\eta \cdot V(B_o(R))^{-1}\int_{B_o(\rho)} |u|^{a\cdot p\cdot k}\right)^{1/(k\cdot p\cdot a)}$$

$$\leq (2\cdot 8^{\eta/k}\cdot C_{11})^{1/pa} \cdot \left[a^p \cdot \left(AR^p + \frac{R^p}{\sigma^p}\right) V(B_o(R))^{-1} \cdot \int_{B_o(\rho+\sigma)} |u|^{pa}\right.$$

$$\left. + a^{p+pa}\left(\frac{4p}{p-1}\right)^{pa} \cdot (AR^p+1)\right]^{1/pa}$$

$$\leq C_{12}^{1/pa}\cdot a^{1/a}\cdot\left(AR^p+\frac{R^p}{\sigma^p}\right)^{1/pa}\cdot\left(V(B_o(R))^{-1}\cdot\int_{B_o(\rho+\sigma)}|u|^{pa}\right)^{1/pa}$$

$$+\,C_{12}^{1/pa}a^{1+(1/a)}\left(\frac{4p}{p-1}\right)\cdot(AR^p+1)^{1/pa}$$

where $C_{12}=2\cdot 8^{\eta/k}\cdot C_{11}$.

Let: $a_i=2k^\imath$, $\sigma_\imath=2^{-4-\imath}$, $\rho_i=R/4-\sum_{\jmath=0}^{\imath}\sigma_\jmath$, for $i=0,1,2,\ldots$; $\rho_{-1}=R/4$. applying (3.27) to $a=a_i$, $\rho=\rho_i$, $\sigma=\sigma_\imath$; then

$$\text{(3.28)}\qquad \left(V(B_o(\rho_i))^{-1}\cdot\int_{B_o(\rho_\imath)}|u|^{2pk^{\imath+1}}\right)^{1/(2pk^{\imath+1})}$$

$$\leq C_{13}^{k^{-\imath}}\cdot D^{ik^{-\imath}}\left(V(B_o(\rho_{i-1}))^{-1}\cdot\int_{B_o(\rho_{\imath-1})}|u|^{2pk^\imath}\right)^{1/2pk^\imath}$$

$$+\,C_{13}^{k^{-\imath}}\cdot D^{ik^{-\imath}}\cdot k^\imath\cdot\left(\frac{8p}{p-1}\right)$$

where we denoted: $C_{13}=(C_{12}\cdot(AR^p+1)\cdot 2^{4+p})^{1/2p}$, $D=2^{1/2p}\cdot k^{1/2}$.

Iterating the inequality (3.28), we have:

$$\text{(3.29)}\qquad \left(V(B_o(\rho_l))^{-1}\cdot\int_{B_o(\rho_l)}|u|^{2pk^{l+1}}\right)^{1/(2pk^{l+1})}$$

$$\leq\prod_{\imath=0}^{l}C_{13}^{k^{-\imath}}\cdot D^{ik^{-\imath}}\cdot\left(V\left(B_o\left(\frac{R}{4}\right)\right)^{-1}\cdot\int_{B_o(R/4)}|u|^{2p}\right)^{1/2p}$$

$$+\left(\frac{8p}{p-1}\right)\sum_{\imath=0}^{l-1}C_{13}^{k^{-\imath}}\cdot D^{ik^{-\imath}}\cdot k^\imath\cdot\prod_{\jmath=\imath+1}^{l}(C_{13}^{k^{-\jmath}}\cdot D^{ik^{-\imath}})$$

$$+\,C_{13}^{k^{-l}}\cdot K^l\cdot D^{lk^{-l}}\cdot\left(\frac{8p}{p-1}\right)$$

$$\leq C_{14}\left(\left(V\left(B_o\left(\frac{R}{4}\right)\right)^{-1}\cdot\int_{B_o(R/4)}|u|^{2p}\right)^{1/2p}+\sum_{\imath=0}^{l}k^\imath\right)$$

$$\leq C_{14}\left(\left(V\left(B_o\left(\frac{R}{4}\right)\right)^{-1}\cdot\int_{B_o(R/4)}|u|^{2p}\right)^{1/2p}+\frac{\mu}{p}\cdot k^l\right)$$

where $C_{14}=(8p/(p-1))\prod_{\imath=0}^{\infty}(C_{13}+1)k^{-\imath}\cdot D^{ik^{-\imath}}$.

For any $j > 2p$; let $l \in N$ such that: $2pk^l < j \le 2pk^{l+1}$, then

$$V\left(B_o\left(\frac{R}{8}\right)\right)^{-1} \cdot \int_{B_o(R/8)} |u|^j \tag{3.30}$$
$$\le \left\{ V\left(B_o\left(\frac{R}{8}\right)\right)^{-1} \cdot \int_{B_o(R/8)} |u|^{2pk^{l+1}} \right\}^{j/(2pk^{l+1})}$$
$$\le \left\{ 2^\eta \cdot V(B_o(\rho_l))^{-1} \cdot \int_{B_o(\rho_l)} |u|^{2pk^{l+1}} \right\}^{j/(2pk^{l+1})}$$
$$\le \left\{ 2^\eta \cdot C_{14} \left(\left(V\left(B_o\left(\frac{R}{4}\right)\right)^{-1} \cdot \int_{B_o(R/4)} |u|^{2p} \right)^{1/2p} + \frac{\mu}{p} \cdot k^l \right) \right\}^j$$
$$\le C_{15}^j \cdot \left(\left(V\left(B_o\left(\frac{R}{4}\right)\right)^{-1} \cdot \int_{B_o(R/4)} |u|^{2p} \right)^{1/2p} + j \right)^j$$

where $C_{15} = 2^\eta \cdot C_{14} \cdot (\mu/(2p^2) + 1)$.

By (3.24), (3.30) we have:

$$V\left(B_o\left(\frac{R}{8}\right)\right)^{-1} \cdot \int_{B_o(R/8)} e^{q\cdot|u|} = \sum_{j=0}^{\infty} (j!)^{-1} \cdot q^j \cdot V\left(B_o\left(\frac{R}{8}\right)\right)^{-1} \int_{B_o(R/8)} |u|^j \tag{3.31}$$
$$\le C_{16} + \sum_{j>2p}^{\infty} (j!)^{-1} \cdot (C_{17} q \cdot j)^j$$

where C_{16}, C_{17} is appropriate positive constantes depending only on C_p, C_s, η, p, μ, $AR^p + 1$. By the Stirling inequality, we have:

$$j^j \le (j!) \cdot e^j$$

then, (3.31) becomes:

$$V\left(B_o\left(\frac{R}{8}\right)\right)^{-1} \cdot \int_{B_o(R/8)} e^{q\cdot|u|} \le C_{16} + \sum_{j>2p}^{\infty} (C_{17} \cdot q \cdot e)^j \tag{3.32}$$

Let $q \le (1/2) \cdot (C_{17} \cdot e)^{-1}$, we have:

$$V\left(B_o\left(\frac{R}{8}\right)\right)^{-1} \cdot \int_{B_o(R/8)} e^{q|u|} \le C_{18} \tag{3.33}$$

where C_{18} is a appropriate positive constant depending only on C_p, C_s, η, p, μ, $AR^p + 1$. Applying inequalities: $e^{q\beta} \cdot f^q = e^{qu} \le e^{q\cdot|u|}$, $e^{-q\beta} \cdot f^{-q} = e^{-qu} \le e^{q\cdot|u|}$; we have:

$$\left\{V\left(B_o\left(\frac{R}{8}\right)\right)^{-1}\int_{B_o(R/8)} f^{-q}\right\}^{1/q}\cdot\left\{V\left(B_o\left(\frac{R}{8}\right)\right)^{-1}\int_{B_o(R/8)} f^{q}\right\}^{1/q} \tag{3.34}$$
$$\leq\left\{V\left(B_o\left(\frac{R}{8}\right)\right)^{-1}\int_{B_o(R/8)} e^{q\cdot|u|}\right\}^{2/q}\leq C_{18}^{2/q}$$

When $q \leq (1/2)(C_{17}\cdot e)^{-1}$, by (3.6), (3.34), there exist a positive constant depending only on C_p, C_s, η, p, q, μ, AR^p+1; such that:

$$\left\{V\left(B_o\left(\frac{R}{8}\right)\right)^{-1}\int_{B_o(R/8)} f^{q}\right\}^{1/q}\leq C_5\cdot\inf_{B_o(R/16)} f \tag{3.35}$$ □

Combining Proposition 2.2 and Lemma 3.1, we have the following locally Harnack inequality.

THEOREM 3.2. *Let M be a complete Riemannian manifold, and geodesic ball $B_o(R)$ satisfies: $B_o(R)\cap\partial M=\emptyset$. If it satisfies the conditions* (1), (2), (3) *in Lemma* 3.1. *Fix $p>1$, assuming $f\in H_{1,p}(B_o(R))$ is a nonnegative function, and satisfies the following inequality in the distribiution sence,*

$$|\operatorname{div}(|\nabla f|^{p-2}\nabla f)|\leq A\cdot f^{p-1} \tag{3.36}$$

for some constant $A\geq 0$; then, there must be exist a constant $C_{19}>0$, depending only on $p,\mu,C_s,p\ \eta,C_p$, (AR^p+1), such that:

$$\sup_{B_o(R/16)} f\leq C_{19}\cdot\inf_{B_o(R/16)} f \tag{3.37}$$

Remark. When $p=2$, Theorem 3.2 is just the result due to P. Li in [L]. In the special case $A=0$, by Theorem 3.2, we can conclude a globally Harnack inequality which is similar to a result of M. Rigoli, M. Salvatori, and M. Vignati in [2], then Theorem 3.2 can be seen as a generalization of the result in [2].

PROPOSITION 3.3. *Let M be a complete noncompact Riemannian manifold* (*without boudary*), *and o be a fixed point in M. Assuming for any $R>0$ geodesic ball $B_o(R)$ satisfies the conditions* (1), (2), (3) *in Lemma* 3.1. *Fix $p>1$, let $f\in H_{1,p}(M)$ is a nonnegative function, and satisfies the following quality in the distribiution sence,*

$$\operatorname{div}(|\nabla f|^{p-2}\nabla f)=0$$

then, for any $R>0$, there must be exist a constant $C_{20}>0$, depending only on p, $\mu,C_s,p\ \eta,C_p$, such that:

$$\sup_{B_o(R)} f\leq C_{20}\cdot\inf_{B_o(R)} f$$

By the above grobally Harnack inequality, one can conclude a Liouville theorem for weakly p-harmonic function.

COROLLARY 3.4. *Let M be a complete noncompact Riemannian manifold (without boundary), and o be a fixed point in M. Assuming for any $R > 0$ geodesic ball $B_o(R)$ satisfies the conditions* (1), (2), (3) *in Lemma* 3.1. *Fix $p > 1$, let f is a nonnegative weakly p-harmonic function $(p > 1)$, then f must be constantly.*

By the Gromove-Bishop volume comparision theorem and the results due to Saloff-Coste in [5], the conditions (1), (2), (3) in Lemma 3.1 is guaranteed, in the assumption $\mathrm{Ric}_M \geq 0$ on M. Then, we have the following Corollary.

COROLLARY 3.5. *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature, then there is no non-constantly nonnegative weakly p-harmonic function.* $(p > 1)$

4. Hölder estimate

THEOREM 4.1. *Let M be a complete Riemannian manifold, and geodesic ball $B_o(R_0)$ satisfies: $B_o(R_0) \cap \partial M = \emptyset$. If it satisfies the conditions* (1), (2), (3) *in Lemma* (3.1). *Fix $p > 1$, assuming the $u \in H_{1,p}(B_o(R_0)) \cap L^\infty(B_o(R_0))$ and that satisfies the following inequality in the distribiution sence,*

$$|\mathrm{div}(|\nabla u|^{p-2}\nabla u)| \leq A \tag{4.1}$$

for some constant $A \geq 0$; then, u must be α-Hölder contiuous at o. and Hölder exponent α depending only on $p, \mu, C_s, C_p, \eta, C_p$.

Proof. Denote: $S(R) = \sup_{B_o(R)} u$, $i(R) = \inf_{B_o(R)} u$; let $f = S(R) - u + A^{1/(p-1)} \cdot R^{p/(p-1)}$, $g = u - i(R) + A^{1/(p-1)} \cdot R^{p/(p-1)}$, applying Theorem 3.2 to f and g, we have:

$$S(R) - i\left(\frac{R}{16}\right) + A^{1/(p-1)} \cdot R^{p/(p-1)} \leq C_{22}\left(S(R) - S\left(\frac{R}{16}\right) + A^{1/(p-1)} \cdot R^{p/(p-1)}\right)$$

$$S\left(\frac{R}{16}\right) - i(R) + A^{1/(p-1)} \cdot R^{p/(p-1)} \leq C_{22}\left(i\left(\frac{R}{16}\right) - i(R) + A^{1/(p-1)} \cdot R^{p/(p-1)}\right)$$

where C_{22} is a positive constant depending only on C_p, C_s, η, μ, p. Denote: $a = (C_{22} - 1)/(C_{22} + 1) < 1$, $\omega = S(R) - i(R)$, by the above inequalities, we have

$$\omega\left(\frac{R}{16}\right) \leq a(\omega(R) + 2A^{1/(p-1)} \cdot R^{p/(p-1)}) \tag{4.2}$$

Iterating (4.2), we have:

$$\begin{aligned} \omega(16^{-m} \cdot R) &\leq a^m \cdot \omega(R) + 2A^{1/(p-1)} \cdot R^{p/(p-1)} \sum_{i=1}^{m} a^i \\ &\leq a^m \cdot \omega(R) + 2A^{1/(p-1)} \cdot R^{p/(p-1)} \frac{a}{1-a} \end{aligned} \tag{4.3}$$

For $\forall 0 < R < R_1 \leq R_0$, let: $(1/16)^l R_1 < R \leq (1/16)^{l-1} R_1$, by (4.3), we have:

$$\begin{aligned}(4.4)\qquad \omega(R) &\leq \omega\left(\left(\frac{1}{16}\right)^{l-1} \cdot R_1\right) \\ &\leq a^{l-1} \cdot \omega(R_1) + 2 \cdot A^{1/(p-1)} \cdot R_1^{p/(p-1)} \cdot \frac{a}{1-a} \\ &\leq a^{-1} \cdot \left(\frac{R}{R_1}\right)^{-\log a/\log 16} \cdot \omega(R_1) + 2 \cdot A^{1/(p-1)} \cdot R_1^{p/(p-1)} \cdot \frac{a}{1-a}\end{aligned}$$

let $R_1 = R_0^{1-t} \cdot R^t$, $0 < t < 1$, then

$$\begin{aligned}\omega(R) \leq a^{-1} \cdot \left(\frac{R}{R_0}\right)^{-(1-t)\cdot(\log a/\log 16)} \cdot \omega(R_0) \\ + 2 \cdot A^{1/(p-1)} \cdot R_0^{p/(p-1)\cdot(1-t)} \cdot R^{p/(p-1)\cdot t} \frac{a}{1-a}\end{aligned}$$

let $t = (-\log a/\log 16) \cdot (p/(p-1) - \log a/\log 16)^{-1}$, and denote $\alpha = p/(p-1) \cdot (-\log a/\log 16) \cdot (p/(p-1) - \log a/\log 16)^{-1}$, by the last inequality, we have:

$$(4.5)\qquad \omega(R) \leq R^{\alpha} \cdot \left(\frac{\omega(R_0)}{a \cdot R_0^{\alpha}} + \frac{2aA^{1/(p-1)} \cdot R_0^{p/(p-1)-\alpha}}{1-a}\right)$$

for any $0 < R < R_0$. □

When $A = 0$, by inequality (4.5), for any $0 < R < R_0$, we have:

$$(4.6)\qquad \omega(R) \leq R^{\alpha} \cdot \left(\frac{\omega(R_0)}{a \cdot R_0^{\alpha}}\right)$$

if $|f(x)| = o(\rho^{\alpha}(x))$, as $x \to 0$, where $\rho(x)$ denotes the geodesic distance from o to x; letting $R_0 \to 0$, then $f \equiv$ *constant.* This is the proof of the following theorem.

THEOREM 4.2. *Let M be a complete noncompact Riemannian manifold satisfies the conditions* (1), (2), (3) *in Lemma* 3.1. *Then there exists a constant* $0 < \alpha \leq 1$ *such that any p-harmonic function f defined on M satisfying the growth condition*

$$|f(x)| = o(\rho^{\alpha}(x))$$

as $x \to 0$, *where* $\rho(x)$ *denotes the geodesic distance from o to x; must be identically constant.*

COROLLARY 4.3. *Let M be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Then there exists a constant* $0 < \alpha \leq 1$ *such that any p-harmonic function f defined on M satisfying the growth condition*

$$|f(x)| = o(\rho^{\alpha}(x)),$$

as $x \to 0$, *where* $\rho(x)$ *denotes the geodesic distance from o to x; Must be identically constant.*

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