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A GENERALIZATION OF MALLIAVIN'S UNIQUENESS THEOREM*†‡

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Abstract

Using Malliavin's uniqueness theorem about Watson's Problem, we obtain a generalization of Malliavin's uniqueness results and a discrete version of a Phragmén-Lindelöf theorem.

1. Introduction

Recently, B. Korenblum and the others give some results about a generalization of Carleman's uniqueness theorem and a discrete Phragmén-Lindelöf theorem. In this paper, we will give a further generalization about these results by using a generalization of Malliavin's uniqueness theorem.

Let $v(x)$ be a function defined on $[0, +\infty)$ and let $H(v)$ be the set of such functions $f(z)$ which are holomorphe in the half-plane $\boldsymbol{C}_+ = \{z = x + iy : x > 0\}$ continuous in the closed half-plane $\mathrm{cl}(\boldsymbol{C}_+) = \{z = x + iy : x \geq 0\}$ such that the following condition

$$(1) \qquad |f(z)| \leq A \exp\{Ax + xv(x)\}$$

hold for $z = x + iy \in \boldsymbol{C}_+$, $r = |z|$. (The symbol A is used for the large enough, positive constant, not necessarily the same at each occurrence.) Let $\Lambda = \{\lambda_n\}$ be an increasing sequence of positive real numbers such that the following separation condition

$$(2) \qquad 8\delta = \inf\{\lambda_{n+1} - \lambda_n : n = 1, 2, \ldots\} > 0$$

and the following Malliavin's uniqueness condition ([11]) for $H(v)$

$$(3) \qquad \int_1^{\infty} S(\Lambda(r) - a) r^{-2}\, dr = +\infty$$

holds for any real number a, where

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$$(4) \qquad \Lambda(r) = 2 \sum_{\lambda \le r, \lambda \in \Lambda} \frac{1}{\lambda} \quad \text{if } r \ge \lambda_1, \quad \text{and} \quad \Lambda(r) = 0, \quad \text{if } r < \lambda_1;$$

$$S(t) = \sup\{xt - xv(x) : x \ge 0\}.$$

Malliavin's uniqueness theorem ([8] and [11]) says that, if $f \in H(v)$ and $f(\lambda) = 0$ for $\lambda \in \Lambda$, (2) and (3) hold, then $f \equiv 0$ on $\boldsymbol{C}_+$. Therefore we shall call the set Λ, which satisfies (2) and (3), Malliavin's uniqueness set for $H(v)$. A well-known Phragmén-Lindelöf Theorem says that if $f \in H(1)$ is bounded in the positive real axis, then f is bounded in $\mathrm{cl}(\boldsymbol{C}_+)$. In this paper, we prove that if the separation condition (2) and Malliavin's uniqueness condition (3) hold, $f \in H(v)$ and

$$(5) \qquad \rho = \overline{\lim_{n \to +\infty}} \lambda_n^{-1} \log|f(\lambda_n)| < \infty$$

then f is of exponential type ρ. So we write our theorem as follows:

Theorem 1. *Suppose that the set $\Lambda = \{\lambda_n\}$ is Malliavin's uniqueness set for $H(v)$. If $f \in H(v)$ and (5) holds, then f is of exponential type ρ and*

$$(6) \qquad |f(z)| \le A \exp(\rho x)$$

holds for $z = x + iy \in \boldsymbol{C}_+$.

Remark 1. If $\rho = -\infty$ then $f \equiv 0$, so our theorem is a generalization of Malliavin's uniqueness theorem.

As a corollary of Theorem 1, we have the following theorem about a signed Borel measure.

Theorem 2. *Suppose that the set $\Lambda = \{\lambda_n\}$ is Malliavin's uniqueness set for $H(v)$. If μ is a signed Borel measure on $(-\infty, +\infty)$ and*

$$(7) \qquad \int_{-\infty}^{+\infty} e^{tx} |d\mu(t)| \le A \exp(xv(x) + xA) \quad \text{for } x > 0,$$

$$(8) \qquad \rho = \overline{\lim_{n \to +\infty}} \lambda_n^{-1} \log\left(\int_{-\infty}^{+\infty} e^{\lambda_n t} |d\mu(t)|\right) < \infty$$

then $d\mu$ is a measure supported on $(-\infty, \rho]$.

2. Proof of Theorems

In order to prove Theorem 1, we need a generalization of Malliavin's uniqueness theorem about Watson's problem ([4], [5], [6] and [11]).

Lemma 1. *Let $\Lambda = \{\lambda_n\}$ be a sequence of positive numbers such that* (2) *holds, let $v(x)$ be a continuous, increasing function on $[0, +\infty)$ and let $\Lambda(r)$ be defined by* (4). *Suppose that the function $g(z)$ is analytic in $\boldsymbol{C}_+$, continuous in* cl($\boldsymbol{C}_+$) *such that*

$$|g(z)| \leq 1 + \exp\{xv(x) - x\Lambda(r) + Ax\}.$$

If (3) *holds, then* $g(z)$ *is bounded in* $\boldsymbol{C}_+$ *and the upper boundedness is not greater than* 2.

The proof of Lemma 1 is similar to that given in [5], [6], [8], [11] and is here omitted.

Proof of Theorem 1. W. H. J. Fuchs ([8]) has proved that the function v in (1) can be replaced by a continuous, increasing function and the uniqueness condition (3) also satisfy. So we suppose that such conditions also hold, we suppose also that $\lambda_1 \geq 8\delta$ hold. First the function

$$f_1(z) = \frac{f(z)}{G(z)(1+z)^2}$$

is analytic in $\boldsymbol{C}_+ - \Lambda$, where $G(z)$ is Fuch's function ([7] and [12]) defined by

$$G(z) = \prod_{n=1}^{\infty}\left(\frac{z-\lambda_n}{z+\lambda_n}\right)\exp\left(\frac{2z}{\lambda_n}\right).$$

W. H. J. Fuchs ([7] and [12]) has proved that the function $G(z)$ is analytic in the half plane $\{z = x + iy : x > -\lambda_1\}$, and that

$$|G(z)| \leq \exp\{x\Lambda(r) + Ax\}, \quad z \in \boldsymbol{C}_+, \quad r = |z|;$$

$$|G(z)| \geq \exp\{x\Lambda(r) - Ax\}, \quad z \in C(\Lambda, \delta)$$

$$|G'(\lambda_n)| \geq \exp\{\lambda_n\Lambda(\lambda_n) - A\lambda_n\}, \quad n = 1, 2, \ldots,$$

where $C(\Lambda,\delta) = \boldsymbol{C}_+ - \bigcup_{n=1}^{+\infty} D(\lambda_n,\delta)$, $D(\lambda_n,\delta) = \{z : |z-\lambda_n| \leq \delta\}$. Therefore we obtain that

$$|f_1(z)| \leq \frac{A}{1+|z|^2}\exp\{xv(x) - x\Lambda(r) + Ax\},$$

holds for $z \in C(\Lambda,\delta)$, $r = |z|$. The function $h_2(t)$ defined by

$$h_2(t) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} f_1(\zeta)e^{-\zeta t}\,d\zeta$$

is continuous on $(-\infty, +\infty)$ and that

$$h_2(t) = \sum_{\lambda \in \Lambda, \lambda < \xi} a(\lambda)e^{-\lambda t} - \frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} f_1(\zeta)e^{-\zeta t}\,d\zeta$$

holds for $\xi > 0$, $\xi \notin \Lambda$, where the coefficients $a(\lambda)$ are the residues of $f_1(z)$ at the points $\lambda \in \Lambda$. Since

$$a(\lambda) = \frac{f(\lambda)}{G'(\lambda)(1+\lambda)^2}; \quad \lambda \in \Lambda, \quad \lim_{n\to+\infty} \frac{\log|a(\lambda_n)|}{\lambda_n} = -\infty,$$

(Malliavin's Uniqueness condition (3) implies that $\Lambda(r)$ is unbounded in $[0, \infty)$.) the function

$$h_3(t) = \sum_{\lambda\in\Lambda} a(\lambda)e^{-\lambda t}$$

is an entire function of $t = \sigma + i\tau$. The function $g_\sigma(z)$ defined by

$$g_\sigma(z) = \int_\sigma^{+\infty} (h_3(t) - h_2(t)) \exp\{z(t-\sigma)\}\, dt$$

is an entire function of $z = x + iy$, and for $z \notin \Lambda$, $x < \xi$, $\xi \notin \Lambda$

$$g_\sigma(z) = -\frac{1}{2\pi i}\int_{\xi-i\infty}^{\xi+i\infty} \frac{f_1(\zeta)}{\zeta - z} e^{-\zeta\sigma}\, d\zeta - \sum_{\lambda<\xi,\, \lambda\in\Lambda} \frac{a(\lambda)}{\lambda - z} e^{-\lambda\sigma}$$

and for $x > 0$,

$$g_\sigma(z) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{f_1(\zeta)}{\zeta - z} e^{-\zeta\sigma}\, d\zeta - \sum_{\lambda\in\Lambda} \frac{a(\lambda)}{\lambda - z} e^{-\lambda\sigma} - f_1(z)e^{-z\sigma},$$

since, for any $\xi > 0$, $\xi \notin \Lambda$ and any σ, there exists a constant $A(\xi, \sigma)$, $\lambda_\xi = \inf\{\lambda : \lambda \in \Lambda, \lambda > \xi\}$ such that

$$|h_3(t) - h_2(t)| \le A(\xi, \sigma)[\exp(-\xi t) + \exp\{-\lambda_\xi(t - \sigma)\}].$$

So there exists a constant $B(\sigma)$ depending only on σ, and δ such that, for $x \le 4\delta$, $z = x + iy$, we have

$$|g_\sigma(z)| \le \frac{B(\sigma)}{6\delta - x}.$$

and for $x \ge 0$, we have

$$|g_\sigma(z)| \le A(\sigma) + \exp\{A + A\sigma - x\sigma + xv(x) - x\Lambda(|z|)\},$$

where $A(\sigma)$ is a constant depending only on σ. Lemma implies that the function $g_\sigma(z)$ is bounded in the entire complex plane, so it follows from the Liouville theorem that the entire function $g_\sigma(z)$ is identically equal to a constant, thus the entire function $g_\sigma(z)$ is identically equal to zero. Therefore the following equality

$$(9) \qquad f_1(z)\exp(-z\sigma) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{f_1(\zeta)}{\zeta - z} \exp(-\zeta\sigma)\, d\zeta - \sum_{\lambda\in\Lambda} \frac{a(\lambda)}{\lambda - z} \exp(-\lambda\sigma)$$

holds for $z = x + iy$, $x > 0$ and for any σ. By taking $\sigma = -\Lambda(r) + A$ in (9), we obtain, from (5) and (2), that

$$|f(z)| \le A + A\sum_{\lambda_n\le r} \exp(\lambda_n[\Lambda(r) - \Lambda(\lambda_n)]) + A\sum_{\lambda_n\ge r} \exp(-\lambda_n) \le A\exp(Ar)$$

(Using $|\Lambda(x)-\Lambda(y)|\leq A|\log x-\log y|$, and $\sup\{t(\log r-\log t : t>0\}=re^{-1})$ where $r=|z|$. Therefore the function $f(z)$ is of exponential type in the half plane, bounded on the imaginary axis by (1), so a well-known discrete Phramén-Lindelöf theorem ([1] p. 200 and [2]) implies that (6) holds.

Proof of Theorem 2. The condition (7) implies that the function $f(z)$ defined by

$$f(z)=\int_{-\infty}^{+\infty} e^{tz}\,d\mu(t)$$

is analytic in $\boldsymbol{C}_+=\{z=x+iy : x>0\}$ continuous in the closed half-plane $\mathrm{cl}(\boldsymbol{C}_+))=\{z=x+iy : x\geq 0\}$ and the conditions (1) and (5) are satisfied. Theorem 1 implies that (6) holds.

Define the function $F(z)$ by setting

$$F(z)=\frac{f(z)}{1+z}e^{-\rho z}$$

Clearly, F is analytic on $\boldsymbol{C}_+=\{z=x+iy : x>0\}$ continuous in the closed half-plane $\mathrm{cl}(\boldsymbol{C}_+))=\{z=x+iy : x\geq 0\}$ and is square summable on the imaginary axis. Thus we can apply the Paley-Wiener theorem ([13], p. 8, Theorem V) to concludde that

$$F(z)=\int_{-\infty}^{0}\psi(t)e^{tz}\,dt$$

for some $\psi\in L^2((-\infty,0))$. We shall assume that ψ is defined for all real numbers (by setting $\psi(t)=0$ for all $t>0$).

On the imaginary axis, we have two representation for F

$$\int_{-\infty}^{+\infty}\psi(t)e^{iyt}\,dt=F(iy)=\frac{e^{-\rho iy}}{1+iy}\int_{-\infty}^{+\infty}e^{ity}\,d\mu(t).$$

Using notation $\tilde{f}$ for the Fourier transform of f (i.e., $\tilde{f}(x)=\int_{-\infty}^{+\infty}f(y)e^{iyx}\,dy$), and letting, $\psi_\rho(t)=\psi(t-\rho)$, we arrive at

$$\widetilde{\psi_\rho}(y)=\tilde{\gamma}(y)\,\widetilde{d\mu}(y)$$

where γ is the function

$$\gamma(t)=e^t \quad \text{if } t\leq 0; \quad \text{and} \quad \gamma(t)=0 \quad \text{if } t>0$$

Hence

$$\psi_\rho(y)=\psi(y-\rho)=(\gamma * d\mu)(y)=\int_{-\infty}^{+\infty}\gamma(y-x)\,d\mu(x)=\int_{y}^{+\infty}e^{y-x}\,d\mu(x)$$

Thus, for all $t>\rho$, we have $\int_t^{+\infty}e^{-x}\,d\mu(x)=0$, which implies that the total variation of $d\mu$ on $(\rho,+\infty)$ is 0; i.e., the measure $d\mu(t)$ is supported in $(-\infty,\rho]$. This complete the proof of Theorem 2.

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