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KODAI MATH. J.
23 (2000), 255–265

ON SPECIAL VALUES OF STANDARD L-FUNCTIONS ATTACHED TO VECTOR VALUED SIEGEL MODULAR FORMS

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1. Introduction

Let V be a vector space of dimension $n \in \boldsymbol{Z}_{>0}$ over $\boldsymbol{C}$ and $\mathrm{sym}^l(V)$ the l-th symmetric tensor product of V with $l \in \boldsymbol{Z}_{\geq 0}$. For $k \in \boldsymbol{Z}_{\geq 0}$, let f be a $\mathrm{sym}^l(V)$-valued Siegel modular form of type $\det^k \otimes \mathrm{sym}^l$ with respect to $Sp(n, \boldsymbol{Z})$ (size $2n$). Suppose f is a cuspform and an eigenform (i.e., a non-zero common eigenfunction of the Hecke algebra). Then we define the standard L-function attached to f by

$$L(s, f, \underline{\mathrm{St}}) := \prod_p \left\{ (1 - p^{-s}) \prod_{j=1}^{n} (1 - \alpha_j(p) p^{-s})(1 - \alpha_j(p)^{-1} p^{-s}) \right\}^{-1}, \tag{1.1}$$

where p runs over all prime numbers and $\alpha_j(p) (j = 1, \dots, n)$ are the Satake p-parameters of f. The right-hand side of (1.1) converges absolutely and locally uniformly for $\mathrm{Re}(s) > n + 1$. We put

$$\Lambda(s, f, \underline{\mathrm{St}}) := \Gamma_{\boldsymbol{R}}(s + \varepsilon) \Gamma_{\boldsymbol{C}}(s + k + l - 1) \prod_{j=2}^{n} \Gamma_{\boldsymbol{C}}(s + k - j) L(s, f, \underline{\mathrm{St}})$$

with

$$\Gamma_{\boldsymbol{R}}(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\boldsymbol{C}}(s) := 2(2\pi)^{-s} \Gamma(s),$$

and

$$\varepsilon := \begin{cases} 0 & \text{for } n \text{ even,} \\ 1 & \text{for } n \text{ odd.} \end{cases}$$

Then by Takayanagi [9, Theorem 2, Theorem 3], we have:

If $k, l \in 2\boldsymbol{Z}$, $k > 0$, $l \geq 0$, then $\Lambda(s, f, \underline{\mathrm{St}})$ has a meromorphic continuation to the whole s-plane and satisfies the functional equation

$$\Lambda(s, f, \underline{\mathrm{St}}) = \Lambda(1 - s, f, \underline{\mathrm{St}}),$$

Received May 14, 1999.

and if $k > n$, *then* $\Lambda(s, f, \underline{\mathrm{St}})$ *is holomorphic except for possible simple poles at* $s = 0$ *and* $s = 1$. *Moreover if* $n \not\equiv 0 \pmod 4$, *then* $\Lambda(s, f, \underline{\mathrm{St}})$ *is entire.*

Therefore the right half of critical points of $L(s, f, \underline{\mathrm{St}})$ is

$$\{m \in \boldsymbol{Z} \mid 1 \leq m \leq k - n \text{ and } m \equiv n \pmod 2\}.$$

For scalar valued cases (i.e., $l = 0$), special values of $L(s, f, \underline{\mathrm{St}})$ were studied by several authors: Sturm [8], Harris [5], Böcherer [2], and Mizumoto [6]. In this paper, we give some algebraic results for the values in the case of vector valued modular forms. The main theorem is follows:

THEOREM (Precise statements are given below). *Let* $k, l \in 2\boldsymbol{Z}_{\geq 0}$ *and* $k \geq 2n + 2$. *Let* f *be a* $\mathrm{sym}^l(V)$*-valued cuspidal eigenform of type* $\det^k \otimes \mathrm{sym}^l$. *Let* $\boldsymbol{Q}(f)$ *be the extension field of* $\boldsymbol{Q}$ *generated by the eigenvalues on* f *of the Hecke algebra over* $\boldsymbol{Q}$. *Suppose the Fourier coefficients of* f *in* $\boldsymbol{Q}(f)$.

Let $m \in \boldsymbol{Z}$ *be in the right half of critical points of* $L(s, f, \underline{\mathrm{St}})$. *If* $m = 1$, *then we assume* $n \equiv 3 \pmod 4$. *Let*

$$A(f) := \frac{L(m, f, \underline{\mathrm{St}})}{\pi^{nk+l+m(n+1)-n(n+1)/2}(f, f)}.$$

Then we have

$$A(f)^\sigma = A(f^\sigma) \quad \textit{for all } \sigma \in \mathrm{Aut}(\boldsymbol{C}).$$

In particular,

$$A(f) \in \boldsymbol{Q}(f).$$

2. Preliminaries

In this section, we describe notations and basic notions (see [3], [6], [9] and [10]).

Let $n \in \boldsymbol{Z}_{>0}, k, l \in \boldsymbol{Z}_{\geq 0}$. Let $x = (x_1, \ldots, x_n)$ be a row vector consisting of n indeterminates. We put

$$V := \boldsymbol{C}x_1 \oplus \cdots \oplus \boldsymbol{C}x_n,$$

and define a Hermitian inner product on V by

$$\left\langle \sum_{j=1}^n a_j x_j, \sum_{j=1}^n b_j x_j \right\rangle := \sum_{j=1}^n a_j \overline{b_j},$$

where $a_j, b_j \in \boldsymbol{C}$ $(j = 1, \ldots, n)$.

We identify $\mathrm{sym}^l(V)$ with the $\boldsymbol{C}$-vector space of homogeneous polynomials in x of degree l. The inner product on V induces an inner product on $\mathrm{sym}^l(V)$ defined by

$$\langle v_1 \cdots v_l, w_1 \cdots w_l \rangle := \frac{1}{l!} \sum_{\tau \in \mathfrak{S}_l} \prod_{j=1}^{l} \langle v_{\tau(j)}, w_j \rangle,$$

where $v_j, w_j \in V$ $(j = 1, \ldots, l)$.

Let ρ be the representation of $GL(n, \boldsymbol{C})$ on $\mathrm{sym}^l(V)$ defined by

$$\rho(g)(v(x)) = (\det g)^k v(xg), \quad v(x) \in \mathrm{sym}^l(V).$$

Let $\Gamma^n := Sp(n, \boldsymbol{Z})$ be the Siegel modular group of degree n, and $\mathfrak{H}_n$ be the Siegel upper half space of degree n. For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^n$ and $Z \in \mathfrak{H}_n$, we put

$$M\langle Z \rangle := (AZ + B)(CZ + D)^{-1}, \quad j(M, Z) := \det(CZ + D),$$

and for $f : \mathfrak{H}_n \to \mathrm{sym}^l(V)$,

$$(f|M)(Z) := \rho((CZ + D)^{-1}) f(M\langle Z \rangle).$$

A C^∞-function $f : \mathfrak{H}_n \to \mathrm{sym}^l(V)$ is called a $\mathrm{sym}^l(V)$-valued C^∞-modular form of type ρ if it satisfies $f|M = f$ for all $M \in \Gamma^n$. The space of all such functions is denoted by $M_{k,l}^n(\mathrm{sym}^l(V))^\infty$. The space of $\mathrm{sym}^l(V)$-valued Siegel modular forms of type ρ is defined by

$$M_{k,l}^n(\mathrm{sym}^l(V))$$
$$:= \{ f \in M_{k,l}^n(\mathrm{sym}^l(V))^\infty \mid f \text{ is holomorphic on } \mathfrak{H}_n \text{ (and its cusps)} \},$$

and the space of cuspforms by

$$S_{k,l}^n(\mathrm{sym}^l(V))$$
$$:= \left\{ f \in M_{k,l}^n(\mathrm{sym}^l(V)) \,\middle|\, \lim_{\lambda \to \infty} f \begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix} = 0 \text{ for all } Z \in \mathfrak{H}_{n-1} \right\}.$$

For $f, g \in M_{k,l}^n(\mathrm{sym}^l(V))^\infty$, the Petersson inner product of f and g is defined by

$$(f, g) := \int_{\Gamma^n \backslash \mathfrak{H}_n} \langle \rho(\sqrt{\mathrm{Im}\, Z}) f(Z), \rho(\sqrt{\mathrm{Im}\, Z}) g(Z) \rangle \det(\mathrm{Im}\, Z)^{-n-1} \, dZ$$

if the right-hand side is convergent.

For $f \in M_{k,l}^n(\mathrm{sym}^l(V))$, f has a Fourier expansion of the following type:

$$f(Z) = \sum_{R \geq 0} a_R(f) e^{2\pi i \,\mathrm{trace}(RZ)}, \quad (a_R(f) \in \mathrm{sym}^l(V), Z \in \mathfrak{H}_n)$$

where R runs through symmetric, semi-integral, semi-positive matrices of size n, we denote such R by "$R \geq 0$".

Let K be any subfield of $\boldsymbol{C}$. We put

$$V_K := Kx_1 \oplus \cdots \oplus Kx_n,$$
$$M_{k,l}^n(\mathrm{sym}^l(V))_K := \{f \in M_{k,l}^n(\mathrm{sym}^l(V)) \mid a_R(f) \in \mathrm{sym}^l(V_K) \text{ for all } R \geq 0\},$$

and for any subset X of $M_{k,l}^n(\mathrm{sym}^l(V))$,

$$X_K := X \cap M_{k,l}^n(\mathrm{sym}^l(V))_K.$$

For $\sigma \in \mathrm{Aut}(\boldsymbol{C})$, we put

$$f^\sigma(Z) := \sum_{R \geq 0} a_R(f)^\sigma e^{2\pi i\, \mathrm{trace}(RZ)}.$$

Then by Takei [10], if $k \geq 2n+2$ then $S_{k,l}^n(\mathrm{sym}^l(V)) = S_{k,l}^n(\mathrm{sym}^l(V))_{\boldsymbol{Q}} \otimes_{\boldsymbol{Q}} \boldsymbol{C}$. Therefore $\mathrm{Aut}(\boldsymbol{C})$ acts on $S_{k,l}^n(\mathrm{sym}^l(V))$ by $f \mapsto f^\sigma$.

Let $L_{\boldsymbol{C}}^{(n)}$ (resp. $L_{\boldsymbol{Q}}^{(n)}$) be the abstract Hecke algebra of degree n over $\boldsymbol{C}$ (resp. $\boldsymbol{Q}$), and let

$$t : L_{\boldsymbol{C}}^{(n)} \to \mathrm{End}_{\boldsymbol{C}}(S_{k,l}^n(\mathrm{sym}^l(V)))$$

be the $\boldsymbol{C}$-algebra homomorphism defined as in [1]. We put $\mathbf{T}_{\boldsymbol{C}} := t(L_{\boldsymbol{C}}^{(n)})$ and $\mathbf{T}_{\boldsymbol{Q}} := t(L_{\boldsymbol{Q}}^{(n)})$.

Let $f \in S_{k,l}^n(\mathrm{sym}^l(V))$ be an eigenform, and for $T \in \mathbf{T}_{\boldsymbol{C}}$, let $\lambda(T) \in \boldsymbol{C}$ be an eigenvalue on f:

$$Tf = \lambda(T)f \quad \text{for all } T \in \mathbf{T}_{\boldsymbol{C}}.$$

Then λ defines an element of

$$\widehat{\mathbf{T}_{\boldsymbol{C}}} := \{\mathbf{T}_{\boldsymbol{C}} \to \boldsymbol{C} : \boldsymbol{C}\text{-algebra homomorphisms}\},$$

and each element of $\widehat{\mathbf{T}_{\boldsymbol{C}}}$ is obtained in this way.

For $\lambda \in \widehat{\mathbf{T}_{\boldsymbol{C}}}$, we put

$$S_{k,l}^n(\lambda) := \{f \in S_{k,l}^n(\mathrm{sym}^l(V)) \mid Tf = \lambda(T)f \text{ for all } T \in \mathbf{T}_{\boldsymbol{C}}\}.$$

Then the space of cuspforms decomposes into eigenspaces:

$$S_{k,l}^n(\mathrm{sym}^l(V)) = \bigoplus_{\lambda \in \widehat{\mathbf{T}_{\boldsymbol{C}}}} S_{k,l}^n(\lambda).$$

We note that for any $f_j \in S_{k,l}^n(\lambda_j)$ $(j = 1, 2)$, $(f_1, f_2) = 0$ if $\lambda_1 \neq \lambda_2$.

For $\lambda \in \widehat{\mathbf{T}_{\boldsymbol{C}}}$, we define an extension field of $\boldsymbol{Q}$ by

$$\boldsymbol{Q}(\lambda) := \boldsymbol{Q}(\lambda(T) \mid T \in \mathbf{T}_{\boldsymbol{Q}}),$$

and for $f \in S_{k,l}^n(\lambda)$, we put $\boldsymbol{Q}(f) := \boldsymbol{Q}(\lambda)$. Then by Takei [10], $\boldsymbol{Q}(\lambda)$ is a totally real finite extension of $\boldsymbol{Q}$.

3. Differential operator and the pullback formula

We put

$$V_1 := \boldsymbol{C}x_1 \oplus \cdots \oplus \boldsymbol{C}x_n, \quad e_1 := (x_1, \ldots, x_n),$$
$$V_2 := \boldsymbol{C}x_{n+1} \oplus \cdots \oplus \boldsymbol{C}x_{2n}, \quad e_2 := (x_{n+1}, \ldots, x_{2n}).$$

Let ι be an isomorphism from V_1 to V_2 defined by $\iota(x_j) = x_{n+j}$ $(j = 1, \ldots, n)$. It induces an isomorphism (also denoted by ι) from $\mathrm{sym}^l(V_1)$ to $\mathrm{sym}^l(V_2)$. For $j = 1, 2$, let ρ_j be the representation $\det^k \otimes \mathrm{sym}^l$ of $GL(n, \boldsymbol{C})$ on $\mathrm{sym}^l(V_j)$ as in Sect. 2.

For $s \in \boldsymbol{C}$ and $\lambda \in \boldsymbol{Z}_{\geq 0}$, we put

$$(s)_\lambda := \begin{cases} s(s+1)\cdots(s+\lambda-1), & \text{if } \lambda > 0, \\ 1, & \text{if } \lambda = 0. \end{cases}$$

For $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in Sp(n, \boldsymbol{R})$, we put

$$M^\uparrow := \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1_n & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1_n \end{pmatrix}, \quad M^\downarrow := \begin{pmatrix} 1_n & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1_n & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

For $k \in 2\boldsymbol{Z}_{>0}$, $s \in \boldsymbol{C}$ and $Z \in \mathfrak{H}_n$, we define the Eisenstein series by

$$G_k^{(n)}(Z, s) := \sum_{M \in P_{n,0} \backslash \Gamma^n} j(M, Z)^{-k} |j(M, Z)|^{-2s}, \tag{3.1}$$

where

$$P_{n,r} := \left\{ \begin{pmatrix} * & * \\ 0^{(n+r, n-r)} & * \end{pmatrix} \in \Gamma^n \right\}.$$

The right-hand side of (3.1) converges absolutely and locally uniformly for $k + 2\,\mathrm{Re}(s) > n + 1$. We consider also

$$E_k^{(n)}(Z, s) := \det(\mathrm{Im}(Z))^s G_k^{(n)}(Z, s).$$

As is well known from the Langlands theory, the Eisenstein series $E_k^{(n)}(Z, s)$ has a meromorphic continuation to the whole s-plane. Moreover by [11], $E_k^{(n)}(Z, s)$ is holomorphic in s at $s = 0$ for each $Z \in \mathfrak{H}_n$. So we define $E_k^{(n)}(Z) := E_k^{(n)}(Z, 0)$.

For $\nu \in \boldsymbol{Z}_{\geq 0}$ and $\lambda \in \boldsymbol{C} - \{j/2 \mid j \in \boldsymbol{Z}, n - 2\nu + 2 \leq j \leq 2n - 1\}$, let $\tilde{\mathscr{D}}_\lambda^\nu$ be the differential operator defined in [2], which acts on $C^\infty(\mathfrak{H}_{2n}, \boldsymbol{C})$. For $k, l \in \boldsymbol{Z}_{\geq 0}$, let $L^{k,l}$ be the differential operator defined in [3], which maps each element of $C^\infty(\mathfrak{H}_{2n}, \boldsymbol{C})$ to in $C^\infty(\mathfrak{H}_n \times \mathfrak{H}_n, \mathrm{sym}^{2l}(V_1 \oplus V_2))$.

Let $k, l, \nu \in 2\boldsymbol{Z}_{\geq 0}$, $k - \nu > 0$, $s \in \boldsymbol{C}$ and $k - \nu + 2\,\mathrm{Re}(s) > 2n + 1$. For $Z, W \in \mathfrak{H}_n$, we put

$$F^{(n)}_{k,\nu,l}(Z,W,s) := (\mathscr{D}_{k,\nu,l,s}G^{(2n)}_{k-\nu})\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}, s\right),$$

where a differential operator $\mathscr{D}_{k,\nu,l,s}$ is defined by

$$\mathscr{D}_{k,\nu,l,s} := L^{k,l}\det(\mathrm{Im}(\mathfrak{Z}))^s\tilde{\mathscr{D}}^{\nu}_{k-\nu+s}.$$

Then $F^{(n)}_{k,\nu,l}(Z,W,s) \in M^n_{k,l}(\mathrm{sym}^l(V_1))^{\infty} \otimes M^n_{k,l}(\mathrm{sym}^l(V_2))^{\infty}$ and we have the following:

PROPOSITION. *We assume $\nu \neq 0$. Then we get*

$$F^{(n)}_{k,\nu,l}(Z,W,s)$$

$$(3.2)\qquad = \frac{\varrho^{(n)}_{k,\nu}(s)}{(2\pi i)^l}\sum_{\mu=0}^{l/2}\left(-\frac{1}{4}\right)^{\mu} a(l,\mu,k,s)\sum_{T\in \boldsymbol{T}^{(n)}}\mathscr{P}_{\mu}(Z,W,T,s)\det(T)^{\nu},$$

where

$$\varrho^{(n)}_{k,\nu}(s) := \prod_{\lambda=0}^{\nu-1}\prod_{j=1}^{n}\left(-(k-\nu+s)+\frac{j-1}{2}-\lambda\right),$$

$$a(l,\mu,k,s) := \frac{1}{(k)_l}\sum_{J=\mu}^{l/2}(-1)^{J-\mu}\binom{J}{\mu}\frac{(2k-2+2J)_{l-2J}(-s)_J(k+s)_{l-J}}{J!(l-2J)!(k-1+J)_{l-J}},$$

$$\boldsymbol{T}^{(n)} := \left\{\begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \,\middle|\, t_j \in \boldsymbol{Z}_{>0}(j=1,\dots,n),\ t_1|\cdots|t_n\right\},$$

$$\mathscr{P}_{\mu}(Z,W,T,s) := \sum_{\tilde{g}_1\in\Gamma^n}\sum_{\tilde{g}'_1\in\Gamma^n(T)\backslash\Gamma^n}\{\det(\mathrm{Im}(Z))^s\det(\mathrm{Im}(W))^s$$
$$\times|\det(1_n-TWTZ)|^{-2s}\rho_1((1_n-TWTZ)^{-1})(e_1T{}^te_2)^{l-2\mu}$$
$$\times(e_1(1_n-TWT\bar{Z})\,\mathrm{Im}(Z)^{-1}{}^t(1_n-TWTZ){}^te_1)^{\mu}$$
$$\times(e_2(1_n-TZTW)^{-1}(1_n-TZT\overline{W})\,\mathrm{Im}(W)^{-1}{}^te_2)^{\mu}\}|(\tilde{g}'_1)_W|(\tilde{g}_1)_Z,$$

where $(\ )_Z$ (resp. $(\ )_W$) denotes the action on Z (resp. W) and

$$\Gamma^n(T) := \left\{g\in\Gamma^n \,\middle|\, \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} g \begin{pmatrix} 0 & T^{-1} \\ T & 0 \end{pmatrix} \in \Gamma^n\right\}.$$

Proof. For $\mathfrak{Z} \in \mathfrak{H}_{2n}$,

$$G^{(2n)}_{k-\nu}(\mathfrak{Z},s) = \sum_{\mathfrak{M}\in P_{2n,0}\backslash\Gamma^{2n}} j(\mathfrak{M},\mathfrak{Z})^{-k+\nu}|j(\mathfrak{M},\mathfrak{Z})|^{-2s}.$$

By Garrett [4], the left coset $P_{2n,0}\backslash\Gamma^{2n}$ has a complete system of representatives $g_{\tilde{T}}\tilde{g}_1^{\uparrow}g_2^{\uparrow}\tilde{g}_1'^{\downarrow}g_2'^{\downarrow}$ with

$$g_{\tilde{T}}=\begin{pmatrix}1_n&0&0&0\\0&1_n&0&0\\0&\tilde{T}&1_n&0\\\tilde{T}&0&0&1_n\end{pmatrix},\quad \tilde{T}=\begin{pmatrix}0&0\\0&T^{(r)}\end{pmatrix},\quad T\in\boldsymbol{T}^{(r)}\quad(r=0,\dots,n),$$

$$\tilde{g}_1\in G_{n,r},\quad g_2\in P_{n,r}\backslash\Gamma^n,\quad \tilde{g}_1'\in\Gamma^r(T)\backslash G_{n,r},\quad g_2'\in P_{n,r}\backslash\Gamma^n,$$

where

$$G_{n,r}:=\left\{\begin{pmatrix}1_{n-r}&0&0&0\\0&A^{(r)}&0&B^{(r)}\\0&0&1_{n-r}&0\\0&C^{(r)}&0&D^{(r)}\end{pmatrix}\in\Gamma^n\right\},\quad \Gamma^r(T)\subset\Gamma^r\simeq G_{n,r}.$$

Hence we put $\mathfrak{M}=g_{\tilde{T}}\tilde{g}_1^{\uparrow}g_2^{\uparrow}\tilde{g}_1'^{\downarrow}g_2'^{\downarrow}$.

By Böcherer [2, Lemma 10],

$$\tilde{\mathscr{D}}_{k-v+s}^{v}j(\mathfrak{M},\mathfrak{Z})^{-k+v-s}=\varrho_{k,v}^{(n)}(s)\det(\tilde{T})^{v}j(\mathfrak{M},\mathfrak{Z})^{-k-s}.$$

Therefore

$$\begin{aligned}(3.3)\qquad &\det(\mathrm{Im}(\mathfrak{Z}))^{s}\tilde{\mathscr{D}}_{k-v+s}^{v}(j(\mathfrak{M},\mathfrak{Z})^{-k+v}|j(\mathfrak{M},\mathfrak{Z})|^{-2s})\\ &=\varrho_{k,v}^{(n)}(s)\det(\tilde{T})^{v}j(\mathfrak{M},\mathfrak{Z})^{-k}|j(\mathfrak{M},\mathfrak{Z})|^{-2s}\det(\mathrm{Im}(\mathfrak{Z}))^{s}.\end{aligned}$$

If $\operatorname{rank}\tilde{T}\neq n$, then (3.3) is equal to 0. So we suppose $\operatorname{rank}\tilde{T}=n$, i.e. $\tilde{T}=T\in\boldsymbol{T}^{(n)}$. Then we can put $g_2=1_{2n}$ and $g_2'=1_{2n}$. By Takayanagi [9, Lemma 1, Proposition 2], we get

$$\begin{aligned}&\mathscr{D}_{k,v,l,s}(j(\mathfrak{M},\mathfrak{Z})^{-k+v}|j(\mathfrak{M},\mathfrak{Z})|^{-2s})|_{\mathfrak{Z}=\mathfrak{Z}_0}\\ &=\frac{\varrho_{k,v}^{(n)}(s)}{(2\pi i)^{l}}\det(T)^{v}\sum_{\mu=0}^{l/2}\left(-\frac{1}{4}\right)^{\mu}a(l,\mu,k,s)\{\det(\mathrm{Im}(Z))^{s}\det(\mathrm{Im}(W))^{s}\\ &\times|\det(1_n-TWTZ)|^{-2s}\rho_1((1_n-TWTZ)^{-1})(e_1T{}^te_2)^{l-2\mu}\\ &\times(e_1(1_n-TWT\overline{Z})\,\mathrm{Im}(Z)^{-1}{}^t(1_n-TWTZ){}^te_1)^{\mu}\\ &\times(e_2(1_n-TZTW)^{-1}(1_n-TZT\overline{W})\,\mathrm{Im}(W)^{-1}{}^te_2)^{\mu}\}|(\tilde{g}_1')_W|(\tilde{g}_1)_Z,\end{aligned}$$

where $\mathfrak{Z}_0=\begin{pmatrix}Z&0\\0&Q\end{pmatrix}$.

Thus, we obtain (3.2). □

4. Special values of $L(s, f, \underline{\mathrm{St}})$

THEOREM. *Let* $k, l \in 2\boldsymbol{Z}_{\geq 0}$ *and* $k \geq 2n+2$. *Let* $f \in S_{k,l}^n(\mathrm{sym}^l(V_2))$ *be an eigenform with the Fourier coefficients in* $\mathrm{sym}^l((V_2)_{\boldsymbol{Q}(f)})$. *Let* $m \in \boldsymbol{Z}$ *be such that*

$$1 \leq m \leq k-n \quad \textit{and} \quad m \equiv n \pmod 2.$$

We assume

$$n \equiv 3 \pmod 4 \quad \textit{if} \quad m = 1.$$

Let

$$A(f) := \frac{L(m, f, \underline{\mathrm{St}})}{\pi^{nk+l+m(n+1)-n(n+1)/2}(f, f)}.$$

Then we have

$$A(f)^\sigma = A(f^\sigma) \quad \textit{for all} \ \sigma \in \mathrm{Aut}(\boldsymbol{C}).$$

In particular,

$$A(f) \in \boldsymbol{Q}(f).$$

Proof. Let $f \in S_{k,l}^n(\mathrm{sym}^l(V_2))$ is an eigenform. Taking the inner product of f and (3.2) in the variable W, we obtain the following in the same way as in [9]:

$$\begin{aligned}
&(f, F_{k,\nu,l}^{(n)}(-\bar{Z}, *, \bar{s})) \\
&= \overline{\varrho_{k,\nu}^{(n)}(\bar{s})} \frac{1}{(k)_l l!} 2^{n(n+1-k-2s)-l+l} i^{nk} \pi^{n(n+1)/2-l} \prod_{j=1}^{n-1} \frac{\Gamma(2k+2s+2j-2n-1)}{\Gamma(2k+2s+j-n-2)} \\
&\quad \times \frac{\Gamma(k+s+l/2-1)\Gamma(k+s+l/2-1/2)\Gamma(k+s-n)\Gamma(2k+2s+l-n-1)}{\Gamma(k+s)\Gamma(k+s-1/2)\Gamma(k+s-1)\Gamma(2k+2s+l-2)} \\
&\quad \times \zeta(2s+k-\nu)^{-1} \prod_{j=1}^{n} \zeta(4s+2k-2\nu-2j)^{-1} L(2s+k-\nu-n, f, \underline{\mathrm{St}}) \\
&\quad \times (\iota^{-1}(f))(Z).
\end{aligned}$$

Here the convergence of the left-hand side follows from [7, Theorem 5.4]. Moreover the expression holds if $\nu = 0$ in [9]. We note that $L(s, f, \underline{\mathrm{St}})$ is holomorphic at m. We put $s = 0$ and $\nu = k-n-m$, which satisfies the required condition: $\nu \in 2\boldsymbol{Z}_{\geq 0}$ and $k - \nu > 0$. Then

$$(f, g(-\bar{Z}, *)) = c(f)(\iota^{-1}(f))(Z), \tag{4.1}$$

where

$$\begin{aligned}
g(Z, W) &:= \pi^{-n(k-n-m)} F_{k,k-n-m,l}^{(n)}(Z, W, 0) \\
&= (\pi^{-n(k-n-m)} L^{k,l} \tilde{\mathscr{D}}_{m+n}^{k-n-m} E_{m+n}^{(2n)}) \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix},
\end{aligned}$$

$$c(f) := \varrho_{k,\nu}^{(n)}(0)\frac{1}{(k)_l l!}2^{n(n+1-k)-l+1}i^{nk}\pi^{n(n+1)/2-l-n(k-n-m)}$$
$$\times \prod_{j=1}^{n-1}\frac{\Gamma(2k+2j-2n-1)}{\Gamma(2k+j-n-2)}$$
$$\times \frac{\Gamma(k+l/2-1)\Gamma(k+l/2-1/2)\Gamma(k-n)\Gamma(2k+l-n-1)}{\Gamma(k)\Gamma(k-1/2)\Gamma(k-1)\Gamma(2k+l-2)}$$
$$\times \zeta(m+n)^{-1}\prod_{j=1}^{n}\zeta(2m+2n-2j)^{-1}L(m,f,\underline{\mathrm{St}}).$$

Since

$$\zeta(m+n)^{-1}\prod_{j=1}^{n}\zeta(2m+2n-2j)^{-1} \in \pi^{-(m+2mn+n^2)}\boldsymbol{Q}^{\times},$$

$$c(f) = \frac{L(m,f,\underline{\mathrm{St}})}{\pi^{nk+l+m(n+1)-n(n+1)/2}} \times (\text{rational}).$$

Therefore it suffices to show that

$$\left(\frac{c(f)}{(f,f)}\right)^{\sigma} = \frac{c(f^{\sigma})}{(f^{\sigma},f^{\sigma})} \quad \text{for all } \sigma \in \mathrm{Aut}(\boldsymbol{C}).$$

We have a partial Fourier expansion of $g(Z,W)$:

$$g(Z,W) = \sum_{R\geq 0}\sum_{\xi \in X_n} g_{R,\xi}(W)\xi e^{2\pi i\,\mathrm{trace}(RZ)},$$

where $X_n := \{\prod_{j=1}^{n} x_j^{\alpha_j} \mid \alpha_j \in \boldsymbol{Z}_{\geq 0}, \sum_{j=1}^{n}\alpha_j = l\}$, which is an orthogonal basis of $\mathrm{sym}^l(V_1)$. By Weissauer [11], $E_{m+n}^{(2n)}$ is a holomorphic modular form with rational Fourier coefficients. Since $\pi^{-n(k-n-m)}L^{k,l}\tilde{\mathscr{D}}_{m+n}^{k-n-m}$ preserves the rationality of Fourier coefficients, we have

$$g_{R,\xi} \in M_{k,l}^{n}(\mathrm{sym}^l(V_2))_{\boldsymbol{Q}}.$$

From (4.1), for any symmetric, semi-integral, semi-positive matrix R, we obtain

$$(f, g_{R,\xi}) = c(f)a_{R,\xi}(\imath^{-1}(f)) \quad \text{for all } \xi \in X_n,$$

where $a_{R,\xi}(\cdot)$ is the ξ-component of the Fourier coefficient $a_R(\cdot)$. Let $h_{R,\xi}$ be the projection of $g_{R,\xi}$ to $S_{k,l}^{n}(\mathrm{sym}^l(V_2))$. Then we obtain

$$(f, h_{R,\xi}) = c(f)a_{R,\xi}(\imath^{-1}(f)) \quad \text{for all } \xi \in X_n.$$

We fix R and ξ such that $a_{R,\xi}(\imath^{-1}(f)) \neq 0$.

Let $\lambda \in \widehat{\mathbf{T}_C}$ be an eigenvalue on f. Let $\sigma \in \mathrm{Aut}(\boldsymbol{C})$ and $N := \dim_{\boldsymbol{C}} S_{k,l}^n(\lambda)$. Since $f \in S_{k,l}^n(\lambda)_{\boldsymbol{Q}(\lambda)}$, by Takei [10, Theorem 1],

$$f^\sigma \in S_{k,l}^n(\lambda^\sigma)_{\boldsymbol{Q}(\lambda^\sigma)}$$

and there exists an orthogonal basis $\{f_j\}_{j=1}^N$ of $S_{k,l}^n(\lambda)$ such that

$$f_1 = f \quad \text{and} \quad f_j \in S_{k,l}^n(\lambda)_{\boldsymbol{Q}(\lambda)} \quad (j = 1, \dots, N).$$

Let $h(\lambda)$ be the projection of $h_{R,\xi}$ to $S_{k,l}^n(\lambda)$. Since $h_{R,\xi} \in S_{k,l}^n(\mathrm{sym}^l(V_2))_{\boldsymbol{Q}}$,

$$h(\lambda)^\sigma = h(\lambda^\sigma).$$

Writing $h(\lambda) = \sum_{j=1}^N \beta_j f_j$, we have

$$c(f) a_{R,\xi}(\iota^{-1}(f)) = \beta_1(f, f).$$

On the other hand,

$$c(f^\sigma) a_{R,\xi}(\iota^{-1}(f^\sigma)) = \beta_1^\sigma(f^\sigma, f^\sigma).$$

Therefore

$$\left(\frac{c(f)}{(f,f)}\right)^\sigma = \frac{\beta_1^\sigma}{a_{R,\xi}(\iota^{-1}(f^\sigma))} = \frac{c(f^\sigma)}{(f^\sigma, f^\sigma)}.$$

Thus Theorem is proved. □

Acknowledgements. The author would like to thank Professor S. Mizumoto for introducing the subject to the author and giving helpful guidance.

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