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UNICITY THEOREMS FOR MEROMORPHIC FUNCTIONS

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Abstract

This paper studies the problem of uniqueness of meromorphic functions. In this paper we will improve a result given by K. Tohge.

§1. Introduction

By a "meromorphic function" we will mean a meromorphic function in the complex plane. It is assumed that the reader is familiar with the notations of the Nevanlinna theory that can be found, for instance, in [2] or [4]. Let f and g be two non-constant meromorphic functions and a be a value in the extended complex plane. We say that f and g share a value a IM (ignoring multiplicity), if f and g have the same a-points, and also they share the value a CM (counting multiplicity), if f and g have the same a-points with the same multiplicity. Let k be a positive integer or ∞, we denote by $\bar{E}_{k)}(a,f)$ the set of a-points of f with multiplicity $\leq k$ (ignoring multiplicity), by $N_{k)}(r,1/(f-a))$ the counting function of a-points of f with multiplicity $\leq k$ and by $N_{(2}(r,1/(f-a))$ the counting function of a-points of f with multiplicity ≥ 2 (See [4]). Finally we say a is a Picard exceptional value of f, if $f(z) \neq a$.

In [3] K. Tohge proved the following:

THEOREM 1. *Let f and g be non-constant meromorphic functions that share three values $0, 1, \infty$ CM and f', g' share 0 CM. Then f and g satisfy one of the following*:

$$\text{(i)}\quad f \equiv g,$$

$$\text{(ii)}\quad fg \equiv 1,$$

$$\text{(iii)}\quad (f-1)(g-1) \equiv 1,$$

$$\text{(1.1)}\qquad \text{(iv)}\quad f+g \equiv 1,$$

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(v) $f \equiv cg$,

(vi) $f - 1 \equiv c(g - 1)$,

(vii) $[(c-1)f + 1][(c-1)g - c] \equiv -c$,

where c $(\neq 0, 1)$ *is a constant.*

In this paper, we prove the following theorem which is an improvement of Theorem 1.

THEOREM 2. *Let f and g be non-constant meromorphic functions that share two values* $0, \infty$ CM *and f', g' share the value* 0 IM. *If* $\bar{E}_{k)}(1, f) = \bar{E}_{k)}(1, g)$, *where k is a positive integer or* ∞, *then f and g satisfy one of the identities in* (1.1).

§2. Some lemmas

LEMMA 1 (See [4]). *Let f and g be distinct non-constant entire functions and a_1, a_2 are distinct finite complex numbers. If a_1 is a Picard exceptional value of f, g and f, g share the value a_2* CM, *then* $f = e^{\alpha} + a_1$ *and* $g = (a_1 - a_2)^2 e^{-\alpha} + a_1$, *where α is a non-constant entire function.*

LEMMA 2 (See [4]). *Let f and g be distinct non-constant meromorphic functions and a_1, a_2 be distinct finite complex numbers. If a_1, a_2 are Picard exceptional values of f, g and f, g share the value* ∞ CM, *then*

$$f = \frac{a_1 e^{\alpha} - a_2}{e^{\alpha} - 1}, \quad g = \frac{a_1 e^{-\alpha} - a_2}{e^{-\alpha} - 1},$$

where α is a non-constant entire function.

LEMMA 3 (See [1]). *Let f and g be non-constant meromorphic functions that share three values* $0, 1, \infty$ CM. *If f is a Möbius transformation of g, then f and g satisfy one of the identities in* (1.1).

Proof. Suppose $f \not\equiv g$. Since f is a Möbius transformation of g,

$$g = \frac{af + b}{cf + d}, \tag{2.1}$$

where a, b, c, d are finite complex numbers and $ad - bc \neq 0$. There are three cases.

CASE I. If ∞ is a Picard exceptional value of f, then there are four subcases.

1. If 1 and 0 are Picard exceptional values of f, then this case is impossible due to the second fundamental theorem.

2. If 1 and 0 are not Picard exceptional values of f, then from (2.1) we get $b = 0$, $c + d = a$ and hence (2.1) becomes

$$cfg + dg = (c + d)f. \tag{2.2}$$

Since $b = 0$ and from $ad - bc \neq 0$ we find $ad \neq 0$ and hence (2.2) becomes

$$(A - 1)fg - Af + g \equiv 0, \tag{2.3}$$

where $A = a/d \neq 0$. If $A = 1$, then from (2.3) we find $f \equiv g$. This is a contradiction. Therefore $A \neq 1$. Thus from (2.3) we find $[(A - 1)f + 1] \cdot [(A - 1)g - A] \equiv -A$, which is (vii).

3. If 0 is a Picard exceptional value of f and 1 is not a Picard exceptional value of f, then by Lemma 1 we find $f = e^{\alpha}$ and $g = e^{-\alpha}$. From this we find $fg \equiv 1$, which is (ii).

4. If 1 is a Picard exceptional value of f and 0 is not a Picard exceptional value of f, then by Lemma 1 we find $f = e^{\alpha} + 1$ and $g = e^{-\alpha} + 1$. From this we find $(f - 1)(g - 1) \equiv 1$, which is (iii).

Case II. If 0 is a Picard exceptional value of f, then there are two subcases.

1. If ∞ and 1 are not Picard exceptional values of f, then from (2.1) we find $c = 0$ and $a + b = d$. Again by (2.1) we get $f - 1 = A(g - 1)$ where A $(\neq 0, 1)$ is a constant, which is (vi).

2. If 1 is a Picard exceptional value of f and ∞ is not a Picard exceptional value of f, then by Lemma 2 we find $f = -1/(e^{\alpha} - 1)$ and $g = -1/(e^{-\alpha} - 1)$. From this we find $f + g \equiv 1$, which is (iv).

Case III. If 1 is a Picard exceptional value of f, then there is only one subcase: If $0, \infty$ are not Picard exceptional values of f, then by (2.1) we find $b = c = 0$ and hence (2.1) becomes $f \equiv Ag$, which is (v). □

§3. Proof of Theorem 2

From the conditions of Theorem 2 we find

$$f = e^{\alpha} g, \tag{3.1}$$

where α is an entire function. If $e^{\alpha} \equiv c$, where c is a nonzero constant, then from this and (3.1) we deduce (v). We now suppose e^{α} is non-constant and hence $\alpha' \not\equiv 0$. Again from (3.1) we have

$$f' = g' e^{\alpha} + g\alpha' e^{\alpha}. \tag{3.2}$$

Let z_0 be a zero for f' of order $p \geq 1$, then the Taylor expansion of f' about z_0 is

$$f'(z) = a_p (z - z_0)^p + \cdots, \quad a_p \neq 0. \tag{3.3}$$

Since f' and g' share the value 0 IM, therefore

$$g'(z) = b_q (z - z_0)^q + \cdots, \quad b_q \neq 0. \tag{3.4}$$

Without loss of generality, we can assume that $p \le q$. From (3.2), (3.3) and (3.4) we find

$$g(z)\alpha'(z) = (z - z_0)^p [a_p e^{-\alpha} - b_q (z - z_0)^{q-p} + O(z - z_0)].$$

From this we find either $\alpha'(z_0) = 0$ or $g(z_0) = 0$. If $g(z_0) = 0$ then $p = q$. Thus we find

$$\overline{N}\left(r, \frac{1}{f'}\right) - N_E'(r, 0) \le N\left(r, \frac{1}{\alpha'}\right) \le T(r, \alpha') + O(1) \le S(r, f) + S(r, g), \tag{3.5}$$

where $N_E'(r, 0)$ denotes the counting function of zeros of f' and g' with same multiplicity, each zero being counted only once. Similarly with respect to g' we find

$$\overline{N}\left(r, \frac{1}{g'}\right) - N_E'(r, 0) = S(r, f) + S(r, g). \tag{3.6}$$

Let z_1 be a zero for $f - 1$ of order $p \ge 2$. Then z_1 is also a zero for f' and hence for g'. From (3.2) we find $\alpha'(z_1) = 0$. From this we find

$$\overline{N}_{(2}\left(r, \frac{1}{f - 1}\right) \le N\left(r, \frac{1}{\alpha'}\right) \le T(r, \alpha') + O(1) \le S(r, f) + S(r, g). \tag{3.7}$$

Similarly with respect to g we find

$$\overline{N}_{(2}\left(r, \frac{1}{g - 1}\right) = S(r, f) + S(r, g). \tag{3.8}$$

We denote by $N_{1)}(r, 1)$ the counting function of common simple 1-points of f and g. Noting $\overline{E}_{k)}(1, f) = \overline{E}_{k)}(1, g)$, from (3.7) and (3.8), we have

$$\overline{N}\left(r, \frac{1}{f - 1}\right) - N_{1)}(r, 1) = S(r, f) + S(r, g), \tag{3.9}$$

and

$$\overline{N}\left(r, \frac{1}{g - 1}\right) - N_{1)}(r, 1) = S(r, f) + S(r, g). \tag{3.10}$$

Set

$$\Delta_1 = \frac{f''}{f'} - \frac{g''}{g'}. \tag{3.11}$$

From the fundamental estimate of logarithmic derivative it follows that

$$m(r, \Delta_1) = S(r, f) + S(r, g). \tag{3.12}$$

From (3.11) we find

$$N(r,\Delta_1) \le \overline{N}\left(r,\frac{1}{f'}\right) - N_E'(r,0) + \overline{N}\left(r,\frac{1}{g'}\right) - N_E'(r,0). \tag{3.13}$$

From (3.5), (3.6), (3.12) and (3.13) we find

$$T(r,\Delta_1) = S(r,f) + S(r,g). \tag{3.14}$$

Let z_∞ be a simple pole of f, then from (3.11) we find

$$\Delta_1(z_\infty) = 0. \tag{3.15}$$

If $\Delta_1 \equiv 0$, then from (3.11) we find

$$f = ag + b, \tag{3.16}$$

where a $(\neq 0)$, b are constants. If $b = 0$, then from (3.16) we deduce (v). We now suppose $b \neq 0$. Since $\overline{E}_{k)}(1,f) = \overline{E}_{k)}(1,g)$, therefore, if $\overline{N}_{k)}(r, 1/(f-1)) \not\equiv 0$, then from (3.16) we find $a + b = 1$. From this and (3.16) we deduce (vi). We now suppose

$$\overline{N}_{k)}\left(r,\frac{1}{f-1}\right) = \overline{N}_{k)}\left(r,\frac{1}{g-1}\right) \equiv 0. \tag{3.17}$$

From (3.1) and (3.16) we find

$$f - 1 = \frac{(b-1)e^\alpha + a}{e^\alpha - a} \quad \text{and} \quad g - 1 = \frac{a + b - e^\alpha}{e^\alpha - a}. \tag{3.18}$$

From this and (3.17) we find $b = 1$ and $a = -1$. From this and (3.18) we deduce (iv). We now suppose $\Delta_1 \not\equiv 0$. From (3.14) and (3.15) we find

$$N_{1)}(r,f) \le N\left(r,\frac{1}{\Delta_1}\right) \le T(r,\Delta_1) + O(1) \le S(r,f) + S(r,g). \tag{3.19}$$

Set

$$\Delta_2 = \frac{f'}{f-1} - \frac{f'}{f} - \frac{g'}{g-1} + \frac{g'}{g}. \tag{3.20}$$

From the fundamental estimate of logarithmic derivative it follows that

$$m(r,\Delta_2) = S(r,f) + S(r,g). \tag{3.21}$$

From (3.9), (3.10), (3.20) and (3.21) we find

$$T(r,\Delta_2) = S(r,f) + S(r,g). \tag{3.22}$$

Let z_∞' be a pole for f of order $p \ge 2$. Then from (3.20) we find

$$\Delta_2(z_\infty') = 0. \tag{3.23}$$

If $\Delta_2 \equiv 0$, then from (3.20) we find

$$\frac{f-1}{f} = c\frac{g-1}{g}, \tag{3.24}$$

where c is a nonzero constant. From this it is easy to see f and g share the value 1 CM. And hence from (3.24) and Lemma 3 we find f and g satisfy one of the identities in (1.1). We now suppose $\Delta_2 \not\equiv 0$. From (3.22) and (3.23) we find

$$N_{(2}(r,f) \leq 2N\left(r,\frac{1}{\Delta_2}\right) \leq 2T(r,\Delta_2) + O(1) \leq S(r,f) + S(r,g). \tag{3.25}$$

From (3.19) and (3.25) we find

$$N(r,f) = S(r,f) + S(r,g). \tag{3.26}$$

Set

$$\Delta_3 = \frac{f''}{f'} - 2\frac{f'}{f} - \frac{g''}{g'} + 2\frac{g'}{g}. \tag{3.27}$$

Similar to the above, from (3.5), (3.6) and (3.27) it is easy to see that

$$T(r,\Delta_3) = S(r,f) + S(r,g). \tag{3.28}$$

Let z_0 be a simple zero of f. Then from (3.27) we find

$$\Delta_3(z_0) = 0. \tag{3.29}$$

If $\Delta_3 \equiv 0$, then from (3.27) we easily arrive at that f and g satisfy one of identities in (1.1). We now suppose $\Delta_3 \not\equiv 0$. Then from (3.28) and (3.29) we find

$$N_{1)}\left(r,\frac{1}{f}\right) = S(r,f) + S(r,g). \tag{3.30}$$

Set

$$\Delta_4 = \frac{f'}{f-1} - \frac{g'}{g-1}. \tag{3.31}$$

Again by a similar way as the above, we find from (3.9), (3.10) and (3.31) that

$$T(r,\Delta_4) = S(r,f) + S(r,g). \tag{3.32}$$

Let z_0' be a zero for f of order $p \geq 2$. Then from (3.31) we find

$$\Delta_4(z_0') = 0. \tag{3.33}$$

If $\Delta_4 \equiv 0$, then from (3.31) we easily arrive at that f and g satisfy one of the identities in (1.1). We now suppose $\Delta_4 \not\equiv 0$. Then from (3.32) and (3.33) we find

$$N_{(2}\left(r,\frac{1}{f}\right) = S(r,f) + S(r,g). \tag{3.34}$$

And hence from (3.30) and (3.34) we find

$$N\left(r,\frac{1}{f}\right)=S(r,f)+S(r,g). \tag{3.35}$$

Set

$$\Delta_5=\frac{f''}{f'}-2\frac{f'}{f-1}-\frac{g''}{g'}+2\frac{g'}{g-1}. \tag{3.36}$$

Similar to the above we find from (3.5), (3.6), (3.9), (3.10) and (3.26) that

$$T(r,\Delta_5)=S(r,f)+S(r,g). \tag{3.37}$$

Let z_1 be a common simple 1-point of f and g. Then from (3.36) we find

$$\Delta_5(z_1)=0. \tag{3.38}$$

If $\Delta_5\equiv 0$, then from (3.36) we easily arrive at that f and g satisfy one of the identities in (1.1). We now suppose $\Delta_5\not\equiv 0$. Then from (3.37) and (3.38), we find

$$\begin{aligned}N_{1)}(r,1)\le N\left(r,\frac{1}{\Delta_5}\right)&\le T(r,\Delta_5)+O(1)\\&\le S(r,f)+S(r,g).\end{aligned}$$

From this and (3.9), we have

$$\overline{N}\left(r,\frac{1}{f-1}\right)=S(r,f)+S(r,g). \tag{3.39}$$

Similarly, we get

$$\overline{N}\left(r,\frac{1}{g-1}\right)=S(r,f)+S(r,g). \tag{3.40}$$

Thus from (3.26), (3.35), (3.39), (3.40) and the second fundamental theorem for f and g we find

$$\begin{aligned}T(r,f)+T(r,g)&\le 2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)\\&\quad+2\overline{N}(r,f)+S(r,f)+S(r,g)\\&\le S(r,f)+S(r,g),\end{aligned}$$

this is impossible. And so the proof of Theorem 2 is finished. □

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