M. ARA
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GEOMETRY OF F-HARMONIC MAPS

MITSUNORI ARA

1. Introduction

Harmonic maps are critical points of the energy functional defined on the space of smooth maps between Riemannian manifolds. There are many studies on harmonic maps. Also, p-harmonic maps and exponentially harmonic maps have been developed. Baird and Eells [BE], and Takeuchi [T] studied some conformal properties of harmonic maps and p-harmonic maps, respectively. They showed that if the dimension of the target manifold is equal to 2 (resp. p), then the fibers of harmonic morphisms (resp. horizontally conformal p-harmonic maps) are minimal submanifolds in the domain manifold. Leung [L], Cheung and Leung [CL], and Koh [K] discussed the stability of harmonic maps, p-harmonic maps and exponentially harmonic maps, respectively.

We would like to construct a unified theory for several varieties of harmonic map. We give the notion of F-harmonic maps, which is a generalization of harmonic maps, p-harmonic maps or exponentially harmonic maps.

In this paper, we discuss some conformal properties and the stability of F-harmonic maps. Our results are extensions of [BE], [T] for conformal properties, and [L], [CL], [K] for the stability. We can see results for harmonic maps, p-harmonic maps or exponentially harmonic maps in a different viewpoint.

Let $F : [0, \infty) \to [0, \infty)$ be a C^2 function such that $F' > 0$ on $(0, \infty)$. For a smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds (M, g) and (N, h), we define the F-energy $E_F(\phi)$ of ϕ by

$$E_F(\phi) = \int_M F\left(\frac{|d\phi|^2}{2}\right) v_g,$$

where $|d\phi|$ denotes the Hilbert-Schmidt norm of the differential $d\phi \in \Gamma(T^*M \otimes \phi^{-1}TN)$ with respect to g and h, and v_g is the volume element of (M, g). It is the energy, the p-energy, the α-energy of Sacks-Uhlenbeck [SU] and the exponential energy when $F(t) = t, (2t)^{p/2}/p$ $(p \geq 4)$, $(1 + 2t)^\alpha$ $(\alpha > 1$, $\dim M = 2)$ and e^t, respectively. We shall say that ϕ is an F-harmonic map if it is a critical point of the F-energy functional, which is a generalization of harmonic maps, p-harmonic maps or exponentially harmonic maps.

This paper is organized as follows. In Section 2, we derive the first variation formula for F-harmonic maps, and have a certain relation between F-harmonic maps and harmonic maps through conformal deformations. In

Section 3, we study the following problem; for a smooth given map $\psi : M \to N$, does there exist an F-harmonic map ϕ is homotopic to ψ? In Section 4, we deal with the stress energy tensor for the F-energy functional, and discuss weakly conformal F-harmonic maps. In Section 5, we discuss horizontally conformal F-harmonic maps, which is a generalization of harmonic morphisms (cf. [B], [BE]) or horizontally conformal p-harmonic maps (cf. [BG], [T]). In Section 6, we derive the second variation formula for F-harmonic maps. In Section 7, we study the stability of F-harmonic maps to unit spheres. One of our main results is as follows.

THEOREM 7.1. *Let $\phi : M \to S^n$ be an F-harmonic map from a compact Riemannian manifold M to the n-dimensional unit sphere S^n. Assume that*

$$(*) \qquad \int_M |d\phi|^2 \left\{ |d\phi|^2 F''\left(\frac{|d\phi|^2}{2}\right) + (2-n)F'\left(\frac{|d\phi|^2}{2}\right) \right\} v_g < 0.$$

Then ϕ is unstable.

In the case of nonconstant harmonic maps, the condition $(*)$ implies that $n > 2$, since $F' = 1, F'' = 0$. Similarly, in the case of nonconstant p-harmonic maps, nonconstant exponentially harmonic maps, the condition $(*)$ implies that $n > p, n - 2 > |d\phi|^2$, respectively. Therefore, Theorem 7.1 is an extension of [L], [CL] and [K] for the stability of harmonic maps, p-harmonic maps and exponentially harmonic maps, respectively.

COROLLARY 7.2. *Assume that* (i) $F'' \leq 0$ *and* $n \geq 3$, *or* (ii) $F'' < 0$ *and* $n = 2$. *Then any stable F-harmonic map from a compact Riemannian manifold M to S^n is constant.*

We remark that Corollary 7.2 (i) is also an extension of the result of [L] for harmonic maps.

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2. The first variation formula

Let $F : [0, \infty) \to [0, \infty)$ be a C^2 function such that $F' > 0$ on $(0, \infty)$. Let $\phi : M \to N$ be a smooth map from an m-dimensional Riemannian manifold (M, g) to a Riemannian manifold (N, h). We call ϕ an F-harmonic map if it is a critical point of the F-energy functional. That is, ϕ is an F-harmonic map if and only if

$$\frac{d}{dt} E_F(\phi_t)|_{t=0} = 0$$

for any compactly supported variation $\phi_t : M \to N$ $(-\varepsilon < t < \varepsilon)$ with $\phi_0 = \phi$.

Let ∇ and ${}^N\nabla$ denote the Levi-Civita connections of M and N, respectively. Let $\tilde{\nabla}$ be the induced connection on $\phi^{-1}TN$ defined by $\tilde{\nabla}_X W = {}^N\nabla_{\phi_* X} W$, where X is a tangent vector of M and W is a section of $\phi^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M. We define the F-tension field $\tau_F(\phi)$ of ϕ by

$$\tau_F(\phi) = \sum_{i=1}^m \left\{ \tilde{\nabla}_{e_i} \left(F'\left(\frac{|d\phi|^2}{2} \right) \phi_* e_i \right) - F'\left(\frac{|d\phi|^2}{2} \right) \phi_* \nabla_{e_i} e_i \right\}$$

$$= F'\left(\frac{|d\phi|^2}{2} \right) \tau(\phi) + \phi_* \left\{ \operatorname{grad}\left(F'\left(\frac{|d\phi|^2}{2} \right) \right) \right\},$$

where $\tau(\phi) = \sum_{i=1}^m (\tilde{\nabla}_{e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i)$ is the tension field of ϕ.

Under the notation above we have the following:

THEOREM 2.1 (The first variation formula).

$$\frac{d}{dt} E_F(\phi_t)|_{t=0} = -\int_M h(V, \tau_F(\phi)) v_g,$$

where $V = d\phi_t/dt|_{t=0}$.

Therefore a smooth map $\phi : M \to N$ is an F-harmonic map if and only if the F-tension field $\tau_F(\phi) = 0$.

Example. (i) Harmonic maps with constant energy density are F-harmonic maps. In particular, in the case where ϕ is a isometric immersion, the following properties are equivalent:

a) ϕ is minimal;

b) ϕ is harmonic;

c) ϕ is F-harmonic.

(ii) In the case where ϕ is a Riemannian submersion, the following properties are equivalent:

a) The fibers of ϕ are minimal submanifolds;

b) ϕ is harmonic;

c) ϕ is F-harmonic.

(iii) The map $\phi : \boldsymbol{R}^m - \{0\} \to S^{m-1}$ defined by $\phi(x) = x/|x|$ is an F-harmonic map.

(iv) We choose a C^3 function F such that $F'(t) + 2tF''(t)$ is not identically zero. For $m \geq 2$, let $\psi(r)$ $(0 < a < r < b)$ be a solution of the ordinary differential equation

$$\left\{ F'\left(\frac{1}{2}\dot{\psi}^2\right) + F''\left(\frac{1}{2}\dot{\psi}^2\right)\dot{\psi}^2 \right\} \ddot{\psi} + \frac{m-1}{r} F'\left(\frac{1}{2}\dot{\psi}^2\right)\dot{\psi} = 0$$

for a suitable initial condition. Set $A = \{x \in \boldsymbol{R}^m \mid a < |x| < b\}$. Then the map $\phi : A \to \boldsymbol{R}$ defined by $\phi(x) = \psi(|x|)$ is an F-harmonic map.

Proof of Theorem 2.1. Let $\Phi : (-\varepsilon, \varepsilon) \times M \to N$ be defined by $\Phi(t, x) = \phi_t(x)$, where $(-\varepsilon, \varepsilon) \times M$ is equipped with the product metric. We extend the vector fields $\partial/\partial t$ on $(-\varepsilon, \varepsilon)$, X on M naturally on $(-\varepsilon, \varepsilon) \times M$, and denote those also by $\partial/\partial t, X$. Then

$$V = \Phi_* \frac{\partial}{\partial t}\bigg|_{t=0}.$$

We shall use the same notations ∇ and $\tilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon, \varepsilon) \times M$ and the induced connection on $\Phi^{-1}TN$.

We compute

$$\begin{aligned}
\frac{\partial}{\partial t} F\left(\frac{|d\phi_t|^2}{2}\right) &= F'\left(\frac{|d\phi_t|^2}{2}\right) \frac{1}{2} \frac{\partial}{\partial t} |d\phi_t|^2 \\
&= F'\left(\frac{|d\phi_t|^2}{2}\right) \sum_{i=1}^m h(\tilde{\nabla}_{\partial/\partial t} \Phi_* e_i, \Phi_* e_i) \\
&= F'\left(\frac{|d\phi_t|^2}{2}\right) \sum_{i=1}^m h\left(\tilde{\nabla}_{e_i} \Phi_* \frac{\partial}{\partial t}, \Phi_* e_i\right) \\
&= F'\left(\frac{|d\phi_t|^2}{2}\right) \sum_{i=1}^m \left\{ e_i \cdot h\left(\Phi_* \frac{\partial}{\partial t}, \Phi_* e_i\right) - h\left(\Phi_* \frac{\partial}{\partial t}, \tilde{\nabla}_{e_i} \Phi_* e_i\right)\right\},
\end{aligned}$$

where we use that

$$\tilde{\nabla}_{\partial/\partial t} \Phi_* e_i - \tilde{\nabla}_{e_i} \Phi_* \frac{\partial}{\partial t} = \Phi_*\left[\frac{\partial}{\partial t}, e_i\right] = 0$$

for the third equality.

Let X_t be a compactly supported vector field on M such that $g(X_t, Y) = h(\Phi_*(\partial/\partial t), \Phi_* Y)$ for any vector field Y on M. Then

$$\begin{aligned}
(2.1)\quad & \frac{\partial}{\partial t} F\left(\frac{|d\phi_t|^2}{2}\right) \\
&= F'\left(\frac{|d\phi_t|^2}{2}\right) \sum_{i=1}^m e_i \cdot g(X_t, e_i) - F'\left(\frac{|d\phi_t|^2}{2}\right) \sum_{i=1}^m h\left(\Phi_* \frac{\partial}{\partial t}, \tilde{\nabla}_{e_i} \Phi_* e_i\right)
\end{aligned}$$

$$= F'\left(\frac{|d\phi_t|^2}{2}\right)\sum_{i=1}^{m}\{g(\nabla_{e_i}X_t, e_i) + g(X_t, \nabla_{e_i}e_i)\}$$

$$- F'\left(\frac{|d\phi_t|^2}{2}\right)\sum_{i=1}^{m} h\left(\Phi_*\frac{\partial}{\partial t}, \tilde{\nabla}_{e_i}\Phi_* e_i\right)$$

$$= F'\left(\frac{|d\phi_t|^2}{2}\right)\operatorname{div}(X_t) - F'\left(\frac{|d\phi_t|^2}{2}\right)\sum_{i=1}^{m} h\left(\Phi_*\frac{\partial}{\partial t}, \tilde{\nabla}_{e_i}\Phi_* e_i - \Phi_*\nabla_{e_i}e_i\right)$$

$$= \operatorname{div}\left(F'\left(\frac{|d\phi_t|^2}{2}\right)X_t\right) - g\left(X_t, \operatorname{grad}\left(F'\left(\frac{|d\phi_t|^2}{2}\right)\right)\right)$$

$$- h\left(\Phi_*\frac{\partial}{\partial t}, F'\left(\frac{|d\phi_t|^2}{2}\right)\sum_{i=1}^{m}(\tilde{\nabla}_{e_i}\Phi_* e_i - \Phi_*\nabla_{e_i}e_i)\right)$$

$$= \operatorname{div}\left(F'\left(\frac{|d\phi_t|^2}{2}\right)X_t\right) - h\left(\Phi_*\frac{\partial}{\partial t}, \Phi_*\left(\operatorname{grad}\left(F'\left(\frac{|d\phi_t|^2}{2}\right)\right)\right)\right.$$

$$\left. + \sum_{i=1}^{m} F'\left(\frac{|d\phi_t|^2}{2}\right)(\tilde{\nabla}_{e_i}\Phi_* e_i - \Phi_*\nabla_{e_i}e_i)\right)$$

$$= \operatorname{div}\left(F'\left(\frac{|d\phi_t|^2}{2}\right)X_t\right) - h\left(\Phi_*\frac{\partial}{\partial t}, \sum_{i=1}^{m}\left\{\tilde{\nabla}_{e_i}\left(F'\left(\frac{|d\phi_t|^2}{2}\right)\Phi_* e_i\right)\right.\right.$$

$$\left.\left. - F'\left(\frac{|d\phi_t|^2}{2}\right)\Phi_*\nabla_{e_i}e_i\right\}\right).$$

By (2.1) and Green's theorem, we get

$$\frac{d}{dt}E_F(\phi_t)|_{t=0} = \int_M \frac{\partial}{\partial t}F\left(\frac{|d\phi_t|^2}{2}\right)\Bigg|_{t=0} v_g$$

$$= -\int_M h\left(V, \sum_{i=1}^{m}\left\{\tilde{\nabla}_{e_i}\left(F'\left(\frac{|d\phi|^2}{2}\right)\phi_* e_i\right) - F'\left(\frac{|d\phi|^2}{2}\right)\phi_*\nabla_{e_i}e_i\right\}\right)v_g$$

$$= -\int_M h(V, \tau_F(\phi))v_g.$$

Next we give a certain relation between F-harmonic maps and harmonic maps through conformal deformations, which is an extension of [BG, Lemma 1.2] for p-harmonic maps and [H, Theorem 2] for exponentially harmonic maps.

PROPOSITION 2.2. *Let $\phi : (M,g) \to (N,h)$ be a smooth map from an m-dimensional Riemannian manifold (M,g) $(m \geq 3)$ to a Riemannian manifold (N,h). In the case where $F'(0) = 0$, we assume that $d\phi_x \neq 0$ for any x in M. Then ϕ is F-harmonic if and only if ϕ is harmonic with respect to the conformally related metric $\tilde{g}$ given by*

$$\tilde{g} = \left\{ F'\left(\frac{|d\phi|^2}{2}\right) \right\}^{2/(m-2)} \cdot g.$$

Proof. Let $\tilde{g}$ be a metric on M, conformally related to g by $\tilde{g} = \lambda^2 \cdot g$ for some positive smooth function λ on M. If $\tau(\phi)$ denotes the tension field of the map $\phi : (M,g) \to (N,h)$, then the tension field $\tilde{\tau}(\phi)$ with respect to the metric $\tilde{g}$ is given by

$$\tilde{\tau}(\phi) = \frac{1}{\lambda^m}\{\lambda^{m-2}\tau(\phi) + \phi_*(\mathrm{grad}(\lambda^{m-2}))\}.$$

Putting $\lambda = \{F'(|d\phi|^2/2)\}^{1/(m-2)}$, we have

$$\left\{ F'\left(\frac{|d\phi|^2}{2}\right) \right\}^{m/(m-2)} \tilde{\tau}(\phi) = \tau_F(\phi).$$

The proposition follows from this equation.

3. Existence of F-harmonic maps

In this section we assume that (M,g) and (N,h) are compact Riemannian manifolds, and $\mathscr{H}$ is a homotopy class of a smooth given map $(M,g) \to (N,h)$. The following result is due to Eells and Ferreira.

THEOREM 3.1 (cf. [EF]). *Suppose that $m = \dim M \geq 3$. Then there is a smooth metric $\tilde{g}$ on M conformally equivalent to g, and a map $\phi \in \mathscr{H}$ such that $\phi : (M,\tilde{g}) \to (N,h)$ is harmonic.*

Yoshida [Y] and Hong [H] gave the p-harmonic version and the exponentially harmonic version of the above theorem, respectively. We would like to derive the F-harmonic version.

THEOREM 3.2. *Suppose that $m = \dim M \geq 3$. Let $F : [0,\infty) \to [0,\infty)$ be a smooth function such that $F' > 0$ on $[0,\infty)$ and $F''(0) \neq 0$. Then there is a* *smooth metric $\tilde{g}$ on M conformally equivalent to g, and a map $\phi \in \mathscr{H}$ such that $\phi : (M, \tilde{g}) \to (N, h)$ is F-harmonic.*

Remark. Theorem 3.2 is an extension of [H, Theorem 1] for exponentially harmonic maps.

In order to prove Theorem 3.2, we introduce some lemmas.

LEMMA 3.3. *Suppose that $m \geq 3$. Let $F : [0, \infty) \to [0, \infty)$ be a smooth function such that $F' > 0$ on $[0, \infty)$ and $(m-2)F'(t) - 2tF''(t) \neq 0$, $F''(t) \neq 0$ on $[0, \varepsilon)$ for some positive constant ε. Then there is a smooth function $\Phi(y)$ on $[0, \varepsilon')$ for some positive constant ε' such that $F'((\Phi(y))^2 y) = (\Phi(y))^{m-2}$.*

Proof. Since $F''(t) \neq 0$ on $[0, \varepsilon)$, $F'(t)$ on $[0, \varepsilon)$ has a smooth inverse function G. So we have

$$G(F'(t)) = t \quad \text{on } [0, \varepsilon),$$

$$G'(F'(t))F''(t) = 1 \quad \text{on } [0, \varepsilon).$$

We shall consider the function

$$y = \frac{G(x^{m-2})}{x^2}.$$

The derivative of y is

$$\frac{dy}{dx} = \frac{1}{x^3}\{(m-2)x^{m-2}G'(x^{m-2}) - 2G(x^{m-2})\}.$$

Thus

$$\frac{dy}{dx}((F'(t))^{1/(m-2)}) = \frac{1}{(F'(t))^{3/(m-2)}F''(t)}\{(m-2)F'(t) - 2tF''(t)\} \neq 0$$

for $t \in [0, \varepsilon)$.

Hence we have

$$\frac{dy}{dx} \neq 0 \quad \text{on } x \text{ between } (F'(0))^{1/(m-2)} \quad \text{and} \quad (F'(\varepsilon))^{1/(m-2)}.$$

Therefore, there is a smooth function $\Phi(y)$ on $[0, \varepsilon')$ for some positive constant ε such that $y = G((\Phi(y))^{m-2})/(\Phi(y))^2$, and $F'((\Phi(y))^2 y) = (\Phi(y))^{m-2}$.

We can see the following lemma in [H].

LEMMA 3.4 (cf. [H, Lemma 2]). *Suppose that the map* $\phi : (M, g) \to (N, h)$ *is harmonic. Then for any* $\varepsilon > 0$, *there exists a smooth metric* $\tilde{g}$ *conformally equivalent to g such that* $\phi : (M, \tilde{g}) \to (N, h)$ *is harmonic and* $|d\phi|_{\tilde{g}} \leq \varepsilon$, *where* $|d\phi|_{\tilde{g}}$ *is the Hilbert-Schmidt norm with respect to* $\tilde{g}$ *and h.*

We prove the following theorem in a way analogous to [H, Theorem 3].

THEOREM 3.5. *Suppose that* $m = \dim M \geq 3$, *and that* $\phi \in \mathscr{H}$ *and* $\phi : (M, g) \to (N, h)$ *is harmonic. Let F be a smooth function such that* $F' > 0$ *on* $[0, \infty)$ *and* $F''(0) \neq 0$. *Then there is a smooth metric* $\hat{g}$ *on M conformally equivalent to g such that* $\phi : (M, \hat{g}) \to (N, h)$ *is F-harmonic.*

Proof. Since $m \geq 3$, $F'(0) \neq 0$ and $F''(0) \neq 0$, there exists a positive constant ε such that $(m-2)F'(t) - 2tF''(t) \neq 0$, $F''(t) \neq 0$ on $[0, \varepsilon)$. By Lemma 3.4, we can suppose that $\phi : (M, g) \to (N, h)$ is harmonic and $|d\phi|_g^2/2 < \varepsilon'$ where ε' is given in Lemma 3.3.

Since $\phi : (M, g) \to (N, h)$ is harmonic,

$$\tau(\phi) = 0.$$

Write $\hat{g} = \lambda^{-2} g$ for a smooth positive function $\lambda : M \to \boldsymbol{R}$. We have

$$0 = \tau(\phi) = \frac{1}{\lambda^m}\{\lambda^{m-2}\hat{\tau}(\phi) + \phi_*(\mathrm{grad}_{\hat{g}}(\lambda^{m-2}))\}.$$

Since $|d\phi|_g^2/2 < \varepsilon'$, we can define the above λ by

$$\lambda = \Phi\left(\frac{|d\phi|_g^2}{2}\right) > 0,$$

where $\Phi(y)$ is given in Lemma 3.3. This yields that

$$\lambda^{m-2} = F'\left(\lambda^2 \frac{|d\phi|_g^2}{2}\right) = F'\left(\frac{|d\phi|_{\hat{g}}^2}{2}\right).$$

Therefore we have

$$\hat{\tau}_F(\phi) = F'\left(\frac{|d\phi|_{\hat{g}}^2}{2}\right)\hat{\tau}(\phi) + \phi_*\left(\mathrm{grad}_{\hat{g}}\left(F'\left(\frac{|d\phi|_{\hat{g}}^2}{2}\right)\right)\right) = 0.$$

This proves Theorem 3.5.

Proof of Theorem 3.2. Combining Theorems 3.1 and 3.5, we can prove Theorem 3.2.

THEOREM 3.6. *Suppose that* $m = \dim M \geq 3$. *Let* $F : [0, \infty) \to [0, \infty)$ *be a smooth function such that* $F' > 0$ *on* $(0, \infty)$ *and* $(m-2)F'(t) - 2tF''(t) \neq 0$, $F''(t) \neq 0$ *on* $(0, \varepsilon)$ *for some positive constant* ε. *Then there is a smooth metric* $\tilde{g}$ *on* M_+ *conformally equivalent to g, and a map* $\phi \in \mathscr{H}$ *such that* $\phi : (M_+, \tilde{g}) \to (N, h)$ *is F-harmonic, where* $M_+ = \{x \in M; |d\phi(x)| \neq 0\}$.

Remark. (i) Theorem 3.6 is an extension of [Y] for p-harmonic maps.
(ii) The assumption $(m-2)F'(t) - 2tF''(t) \neq 0$ on $(0, \varepsilon)$ for some positive constant ε in this theorem is not satisfied in the case of m-harmonic maps.

Proof. We can prove this theorem in a way analogous to Theorem 3.2.

4. The F-stress energy tensor

Let $\phi : (M, g) \to (N, h)$ be a smooth map from an m-dimensional Riemannian manifold (M, g) to a Riemannian manifold (N, h). The stress energy tensor $S_F(\phi)$ of ϕ associated to the F-energy functional E_F (which we call, the F-stress energy tensor of ϕ, in short) is given by

$$S_F(\phi) = F\left(\frac{|d\phi|^2}{2}\right) \cdot g - F'\left(\frac{|d\phi|^2}{2}\right) \cdot \phi^* h$$

(cf. [B, Chapter 3]).

PROPOSITION 4.1 (cf. [B, Chapter 3]). *Under the notation above,*

$$(\operatorname{div} S_F(\phi))(X) = -h(\tau_F(\phi), \phi_* X)$$

for any vector field X on M.

Therefore, if ϕ is an F-harmonic map, then $\operatorname{div} S_F(\phi) \equiv 0$. Conversely, if ϕ is a submersion almost everywhere and $\operatorname{div} S_F(\phi) \equiv 0$, then ϕ is an F-harmonic map.

This proposition is included in [B, Chapter 3]. But here, we give its elementary proof.

Proof. Let ∇ and ${}^N\nabla$ denote the Levi-Civita connections of M and N, respectively. Let $\tilde{\nabla}$ be the induced connection on $\phi^{-1}TN$. We choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ on M with $\nabla_{e_i} e_j|_x = 0$ at a point $x \in M$.

Let X be a vector field on M. At x, we compute

$$\begin{aligned}
(\operatorname{div} S_F(\phi))(X) &= \sum_{i=1}^m (\nabla_{e_i} S_F(\phi))(e_i, X) \\
&= \sum_{i=1}^m \{e_i(S_F(\phi)(e_i, X)) - S_F(\phi)(\nabla_{e_i} e_i, X) - S_F(\phi)(e_i, \nabla_{e_i} X)\} \\
&= \sum_{i=1}^m \left\{ e_i\left(F\left(\frac{|d\phi|^2}{2}\right) g(e_i, X)\right) - e_i\left(F'\left(\frac{|d\phi|^2}{2}\right) h(\phi_* e_i, \phi_* X)\right) \right. \\
&\quad \left. - F\left(\frac{|d\phi|^2}{2}\right) g(e_i, \nabla_{e_i} X) + F'\left(\frac{|d\phi|^2}{2}\right) h(\phi_* e_i, \phi_* \nabla_{e_i} X) \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m} \left[F'\left(\frac{|d\phi|^2}{2}\right) \sum_{j=1}^{m} h(\tilde{\nabla}_{e_i}\phi_* e_j, \phi_* e_j) \cdot g(e_i, X) \right. \\
&\quad - e_i\left(F'\left(\frac{|d\phi|^2}{2}\right)\right) h(\phi_* e_i, \phi_* X) \\
&\quad - F'\left(\frac{|d\phi|^2}{2}\right) \{h(\tilde{\nabla}_{e_i}\phi_* e_i, \phi_* X) + h(\phi_* e_i, \tilde{\nabla}_{e_i}\phi_* X)\} \\
&\quad \left. + F'\left(\frac{|d\phi|^2}{2}\right) h(\phi_* e_i, \phi_* \nabla_{e_i} X) \right] \\
&= F'\left(\frac{|d\phi|^2}{2}\right) \sum_{j=1}^{m} h(\tilde{\nabla}_X \phi_* e_j, \phi_* e_j) \\
&\quad - h\left(\phi_*\left(\operatorname{grad}\left(F'\left(\frac{|d\phi|^2}{2}\right)\right)\right), \phi_* X\right) \\
&\quad - F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^{m} h(\tilde{\nabla}_{e_i}\phi_* e_i, \phi_* X) \\
&\quad - F'\left(\frac{|d\phi|^2}{2}\right) \sum_{i=1}^{m} h(\tilde{\nabla}_{e_i}\phi_* X - \phi_* \nabla_{e_i} X, \phi_* e_i).
\end{aligned}
$$

At x, we have

$$
\sum_{i=1}^{m} \tilde{\nabla}_{e_i}\phi_* e_i = \sum_{i=1}^{m} (\tilde{\nabla}_{e_i}\phi_* e_i - \phi_* \nabla_{e_i} e_i) = \tau(\phi),
$$

and

$$
\begin{aligned}
\tilde{\nabla}_{e_i}\phi_* X - \phi_* \nabla_{e_i} X &= \tilde{\nabla}_X \phi_* e_i + \phi_*[e_i, X] - \phi_* \nabla_{e_i} X \\
&= \tilde{\nabla}_X \phi_* e_i + \phi_*(\nabla_{e_i} X - \nabla_X e_i) - \phi_* \nabla_{e_i} X \\
&= \tilde{\nabla}_X \phi_* e_i.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
(\operatorname{div} S_F(\phi))(X) &= -F'\left(\frac{|d\phi|^2}{2}\right) h(\tau(\phi), \phi_* X) - h\left(\phi_*\left(\operatorname{grad}\left(F'\left(\frac{|d\phi|^2}{2}\right)\right)\right), \phi_* X\right) \\
&= -h(\tau_F(\phi), \phi_* X).
\end{aligned}
$$

Proposition 4.2. *Let $\phi : (M,g) \to (N,h)$ be a weakly conformal F-harmonic map from an m-dimensional Riemannian manifold (M,g) to a Riemannian manifold (N,h). Assume that the zeros of $(m-2)F'(t) - 2tF''(t)$ are isolated. Then ϕ is a homothetic map.*

Remark. (i) If $F'' \leq 0$ and $m \geq 3$, or $F'' \neq 0$ and $m = 2$, then the assumption for F in this proposition is satisfied.

(ii) This proposition is an extension of [BE, Example 3.3] for harmonic maps and [T, Corollary 4] for p-harmonic maps.

Proof. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field on M. As ϕ is weakly conformal, there is a nonnegative smooth function λ^2 on M such that $\phi^*h = \lambda^2 g$. So we have $S_F(\phi) = (F(m\lambda^2/2) - \lambda^2 F'(m\lambda^2/2)) \cdot g$. Let X be a vector field on M. Since ϕ is F-harmonic, by Proposition 4.1 we have

$$
\begin{aligned}
0 &= \operatorname{div} S_F(\phi)(X) \\
&= \sum_{i=1}^m \nabla_{e_i}\left(\left(F\left(\frac{m}{2}\lambda^2\right) - \lambda^2 F'\left(\frac{m}{2}\lambda^2\right)\right) \cdot g\right)(e_i, X) \\
&= \sum_{i=1}^m e_i\left(F\left(\frac{m}{2}\lambda^2\right) - \lambda^2 F'\left(\frac{m}{2}\lambda^2\right)\right) \cdot g(e_i, X) \\
&= \sum_{i=1}^m \left\{F'\left(\frac{m}{2}\lambda^2\right)\frac{m}{2}e_i(\lambda^2) - e_i(\lambda^2)F'\left(\frac{m}{2}\lambda^2\right) - \lambda^2 F''\left(\frac{m}{2}\lambda^2\right)\frac{m}{2}e_i(\lambda^2)\right\} g(e_i, X) \\
&= \frac{1}{2}\left\{(m-2)F'\left(\frac{m}{2}\lambda^2\right) - m\lambda^2 F''\left(\frac{m}{2}\lambda^2\right)\right\} \cdot X(\lambda^2).
\end{aligned}
$$

Thus λ^2 is constant, and ϕ is homothetic.

5. Horizontally conformal F-harmonic maps

Let $\phi : (M,g) \to (N,h)$ be a smooth map between Riemannian manifolds (M,g) and (N,h). For each $x \in M$ satisfying $d\phi_x \neq 0$, set $V_x = \operatorname{Ker} d\phi_x$ and let H_x be the orthogonal complement of V_x in T_xM. We call V_x the vertical space at x, and H_x the horizontal space at x. For $X \in T_xM$, we may decompose $X = X^H + X^V$, where $X^H \in H_x$ and $X^V \in V_x$.

We say that ϕ is horizontally conformal if there exists a positive smooth function λ on M such that $h(\phi_* X, \phi_* Y) = \lambda^2 \cdot g(X, Y)$ for all $X, Y \in H_x$ and $x \in M$. The function λ is called the dilation of ϕ.

Theorem 5.1. *Let $\phi : (M,g) \to (N,h)$ be a horizontally conformal F-harmonic map with dilation λ from an m-dimensional Riemannian manifold (M,g) to an n-dimensional Riemannian manifold (N,h), where $m > n$. Assume that the* *zeros of* $(n-2)F'(t) - 2tF''(t)$ *are isolated. Then the following properties are equivalent*:

(i) *The fibers of* ϕ *are minimal submanifolds*;

(ii) $\mathrm{grad}(\lambda^2)$ *is vertical*;

(iii) *the horizontally distribution has mean curvature* $\mathrm{grad}(\lambda^2)/2\lambda^2$.

Remark. (i) If $F'' \leq 0$ and $n \geq 3$, or $F'' \neq 0$ and $n = 2$, then the assumption for F in this theorem is satisfied.

(ii) This theorem is an extension of [BE, Theorem 5.2] for harmonic morphisms and [T, Proposition 7] for horizontally conformal p-harmonic maps.

Proof. For $x \in M$, we choose a local orthonormal frame field $\{e_i\}_{i=1}^m$ near x with $e_1, \dots, e_n$ horizontal and $e_{n+1}, \dots, e_m$ vertical. As ϕ is horizontally conformal with dilation λ, we have $|d\phi|^2 = n\lambda^2$ and $S_F(\phi) = F(n\lambda^2/2) \cdot g - F'(n\lambda^2/2) \cdot \phi^* h$. Since ϕ is F-harmonic, by Proposition 4.1 we have

$$
\begin{aligned}
(5.1) \qquad 0 &= (\operatorname{div} S_F(\phi))(e_j) \\
&= \sum_{i=1}^m (\nabla_{e_i} S_F(\phi))(e_i, e_j) \\
&= \sum_{i=1}^m \{e_i(S_F(\phi)(e_i, e_j)) - S_F(\phi)(\nabla_{e_i} e_i, e_j) - S_F(\phi)(e_i, \nabla_{e_i} e_j)\} \\
&= \sum_{i=1}^m \left\{ e_i\left(F\left(\frac{n}{2}\lambda^2\right)\right) \cdot g(e_i, e_j) - e_i\left(F'\left(\frac{n}{2}\lambda^2\right) \cdot h(\phi_* e_i, \phi_* e_j)\right) \right. \\
&\quad \left. + F'\left(\frac{n}{2}\lambda^2\right) \cdot h(\phi_* \nabla_{e_i} e_i, \phi_* e_j) + F'\left(\frac{n}{2}\lambda^2\right) \cdot h(\phi_* e_i, \phi_* \nabla_{e_i} e_j) \right\} \\
&= \frac{n}{2} F'\left(\frac{n}{2}\lambda^2\right) e_j(\lambda^2) - \sum_{i=1}^n e_i\left(F'\left(\frac{n}{2}\lambda^2\right) \cdot h(\phi_* e_i, \phi_* e_j)\right) \\
&\quad + F'\left(\frac{n}{2}\lambda^2\right) \sum_{i=1}^m \{h(\phi_* \nabla_{e_i} e_i, \phi_* e_j) + h(\phi_* e_i, \phi_* \nabla_{e_i} e_j)\}.
\end{aligned}
$$

For j $(1 \leq j \leq n)$ we have

$$
\begin{aligned}
(5.2) \qquad 0 &= \sum_{i=1}^n e_i g(e_i, e_j) \\
&= \sum_{i=1}^n \{g(\nabla_{e_i} e_i, e_j) + g(e_i, \nabla_{e_i} e_j)\}
\end{aligned}
$$

$$= \sum_{i=1}^{n} \{ g((\nabla_{e_i} e_i)^H, e_j) + g(e_i, (\nabla_{e_i} e_j)^H) \}$$

$$= \frac{1}{\lambda^2} \sum_{i=1}^{n} \{ h(\phi_*(\nabla_{e_i} e_i)^H, \phi_* e_j) + h(\phi_* e_i, \phi_*(\nabla_{e_i} e_j)^H) \}$$

$$= \frac{1}{\lambda^2} \sum_{i=1}^{n} \{ h(\phi_* \nabla_{e_i} e_i, \phi_* e_j) + h(\phi_* e_i, \phi_* \nabla_{e_i} e_j) \}.$$

By (5.1) and (5.2), for j $(1 \leq j \leq n)$

$$0 = \frac{n}{2} F'\left(\frac{n}{2}\lambda^2\right) e_j(\lambda^2) - e_j\left(\lambda^2 F'\left(\frac{n}{2}\lambda^2\right)\right)$$

$$+ F'\left(\frac{n}{2}\lambda^2\right) \sum_{i=n+1}^{m} \{ h(\phi_* \nabla_{e_i} e_i, \phi_* e_j) + h(\phi_* e_i, \phi_* \nabla_{e_i} e_j) \}$$

$$= \frac{n}{2} F'\left(\frac{n}{2}\lambda^2\right) e_j(\lambda^2) - F'\left(\frac{n}{2}\lambda^2\right) e_j(\lambda^2) - \frac{n}{2}\lambda^2 F''\left(\frac{n}{2}\lambda^2\right) e_j(\lambda^2)$$

$$+ F'\left(\frac{n}{2}\lambda^2\right) \sum_{i=n+1}^{m} h(\phi_*(\nabla_{e_i} e_i)^H, \phi_* e_j)$$

$$= \frac{1}{2}\left\{ (n-2) F'\left(\frac{n}{2}\lambda^2\right) - n\lambda^2 F''\left(\frac{n}{2}\lambda^2\right) \right\} e_j(\lambda^2)$$

$$+ F'\left(\frac{n}{2}\lambda^2\right) \sum_{i=n+1}^{m} \lambda^2 g((\nabla_{e_i} e_i)^H, e_j)$$

$$= \frac{1}{2}\left\{ (n-2) F'\left(\frac{n}{2}\lambda^2\right) - n\lambda^2 F''\left(\frac{n}{2}\lambda^2\right) \right\} e_j(\lambda^2)$$

$$+ \lambda^2 F'\left(\frac{n}{2}\lambda^2\right) \sum_{i=n+1}^{m} g(\nabla_{e_i} e_i, e_j).$$

The mean curvature vector H_1 of the fiber of ϕ is given by

$$H_1 = \frac{1}{m-n} \sum_{j=1}^{n} \sum_{i=n+1}^{m} g(\nabla_{e_i} e_i, e_j) e_j.$$

Thus we get

$$0 = \frac{1}{2}\left\{ (n-2) F'\left(\frac{n}{2}\lambda^2\right) - n\lambda^2 F''\left(\frac{n}{2}\lambda^2\right) \right\} (\mathrm{grad}(\lambda^2))^H + (m-n)\lambda^2 F'\left(\frac{n}{2}\lambda^2\right) H_1.$$

From this equation we can see that (i) is equivalent to (ii).

Using (5.1) for j $(n+1 \leq j \leq m)$, we have

$$0 = F'\left(\frac{n}{2}\lambda^2\right)\left\{\frac{n}{2}e_j(\lambda^2) + \sum_{i=1}^{n} h(\phi_* e_i, \phi_* \nabla_{e_i} e_j)\right\}.$$

For i $(1 \leq i \leq n)$,

$$\begin{aligned} h(\phi_* e_i, \phi_* \nabla_{e_i} e_j) &= h(\phi_* e_i, \phi_*(\nabla_{e_i} e_j)^H) = \lambda^2 g(e_i, (\nabla_{e_i} e_j)^H) \\ &= \lambda^2 g(e_i, \nabla_{e_i} e_j) = -\lambda^2 g(\nabla_{e_i} e_i, e_j). \end{aligned}$$

So we have

$$0 = \frac{n}{2}e_j(\lambda^2) - \lambda^2 \sum_{i=1}^{n} g(\nabla_{e_i} e_i, e_j).$$

Hence the mean curvature H_2 of the horizontally distribution is given by

$$\begin{aligned} H_2 &= \frac{1}{n}\sum_{j=n+1}^{m}\sum_{i=1}^{n} g(\nabla_{e_i} e_i, e_j)e_j = \frac{1}{2\lambda^2}\sum_{j=n+1}^{m} e_j(\lambda^2)e_j \\ &= \frac{(\mathrm{grad}(\lambda^2))^V}{2\lambda^2} = \frac{\mathrm{grad}(\lambda^2) - (\mathrm{grad}(\lambda^2))^H}{2\lambda^2}. \end{aligned}$$

From this equation we can see that (ii) is equivalent to (iii).

6. The second variation formula

In this section, we calculate the second variation of the F-energy functional.

Let $\phi : (M, g) \to (N, h)$ be a smooth map from an m-dimensional Riemannian manifold (M, g) to a Riemannian manifold (N, h).

THEOREM 6.1 (The second variation formula). *Let $\phi : M \to N$ be an F-harmonic map. Let $\phi_{s,t} : M \to N$ $(-\varepsilon < s, t < \varepsilon)$ be a compactly supported two-parameter variation such that $\phi_{0,0} = \phi$, and set $V = \partial\phi_{s,t}/\partial t|_{s,t=0}$, $W = \partial\phi_{s,t}/\partial s|_{s,t=0}$. Then*

$$\begin{aligned} \frac{\partial^2}{\partial s \partial t} E_F(\phi_{s,t})|_{s,t=0} &= \int_M F''\left(\frac{|d\phi|^2}{2}\right)\langle\tilde{\nabla}V, d\phi\rangle\langle\tilde{\nabla}W, d\phi\rangle v_g \\ &+ \int_M F'\left(\frac{|d\phi|^2}{2}\right)\cdot\left\{\langle\tilde{\nabla}V, \tilde{\nabla}W\rangle - \sum_{i=1}^{m} h({}^N R(V, \phi_* e_i)\phi_* e_i, W)\right\} v_g, \end{aligned}$$

where $\langle\,,\rangle$ *is the inner product on* $T^*M \otimes \phi^{-1}TN$ *and* ${}^N R$ *is the curvature tensor of* N.

We put

$$I(V, W) = \frac{\partial^2}{\partial s \partial t} E_F(\phi_{s,t})|_{s,t=0}.$$

An F-harmonic map ϕ is called stable if $I(V, V) \geq 0$ for any compactly supported vector field V along ϕ.

Proof. Let $\Phi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M \to N$ be defined by $\Phi(s, t, x) = \phi_{s,t}(x)$, where $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M$ is equipped with the product metric. We extend the vector fields $\partial/\partial t$ on $(-\varepsilon, \varepsilon)$, $\partial/\partial s$ on $(-\varepsilon, \varepsilon)$, X on M naturally on $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M$, and denote those also by $\partial/\partial t, \partial/\partial s, X$. Then

$$V = \Phi_* \frac{\partial}{\partial t}\bigg|_{s,t=0}, \quad W = \Phi_* \frac{\partial}{\partial s}\bigg|_{s,t=0}. \tag{6.1}$$

We shall use the same notations ∇ and $\tilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times M$ and the induced connection on $\Phi^{-1}TN$.

Using (2.1) we have

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial t} E_F(\phi_{s,t})|_{s,t=0} &= \frac{\partial}{\partial s} \int_M \frac{\partial}{\partial t} F\left(\frac{|d\phi_{s,t}|^2}{2}\right) v_g \bigg|_{s,t=0} \\
&= -\int_M \frac{\partial}{\partial s} h\left(\Phi_* \frac{\partial}{\partial t}, \sum_{i=1}^m \left\{\tilde{\nabla}_{e_i}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_* e_i\right)\right.\right. \\
&\qquad \left.\left. - F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_* \nabla_{e_i} e_i\right\}\right)\bigg|_{s,t=0} v_g \\
&= -\int_M h\left(\Phi_* \frac{\partial}{\partial t}, \sum_{i=1}^m \left\{\tilde{\nabla}_{\partial/\partial s}\tilde{\nabla}_{e_i}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_* e_i\right)\right.\right. \\
&\qquad \left.\left. - \tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_* \nabla_{e_i} e_i\right)\right\}\right)\bigg|_{s,t=0} v_g,
\end{aligned} \tag{6.2}$$

where we use the F-harmonicity for the last equality. We compute

$$(6.3)\quad h\left(\Phi_*\frac{\partial}{\partial t},\sum_{i=1}^{m}\left\{\tilde\nabla_{\partial/\partial s}\tilde\nabla_{e_i}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*e_i\right)-\tilde\nabla_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*\nabla_{e_i}e_i\right)\right\}\right)$$
$$=h\left(\Phi_*\frac{\partial}{\partial t},\sum_{i=1}^{m}\left\{\tilde\nabla_{e_i}\tilde\nabla_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*e_i\right)\right.\right.$$
$$+{}^NR\left(\Phi_*\frac{\partial}{\partial s},\Phi_*e_i\right)\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*e_i\right)$$
$$\left.\left.-\tilde\nabla_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*\nabla_{e_i}e_i\right)\right\}\right),$$

where we use $[\partial/\partial s, e_i]=0$.

The first term in the right-hand side of (6.3) is

$$(6.4)\quad h\left(\Phi_*\frac{\partial}{\partial t},\sum_{i=1}^{m}\tilde\nabla_{e_i}\tilde\nabla_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*e_i\right)\right)$$
$$=\sum_{i=1}^{m}e_i\cdot h\left(\Phi_*\frac{\partial}{\partial t},\tilde\nabla_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*e_i\right)\right)$$
$$-\sum_{i=1}^{m}h\left(\tilde\nabla_{e_i}\Phi_*\frac{\partial}{\partial t},\tilde\nabla_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*e_i\right)\right).$$

The second term in the right-hand side of (6.4) is

$$(6.5)\quad \sum_{i=1}^{m}h\left(\tilde\nabla_{e_i}\Phi_*\frac{\partial}{\partial t},\tilde\nabla_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*e_i\right)\right)$$
$$=\sum_{i=1}^{m}h\left(\tilde\nabla_{e_i}\Phi_*\frac{\partial}{\partial t},F''\left(\frac{|d\phi_{s,t}|^2}{2}\right)\sum_{j=1}^{m}h(\tilde\nabla_{\partial/\partial s}\Phi_*e_j,\Phi_*e_j)\Phi_*e_i\right.$$
$$\left.+F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\tilde\nabla_{\partial/\partial s}\Phi_*e_i\right)$$
$$=F''\left(\frac{|d\phi_{s,t}|^2}{2}\right)\sum_{i=1}^{m}h\left(\tilde\nabla_{e_i}\Phi_*\frac{\partial}{\partial t},\Phi_*e_i\right)\cdot\sum_{j=1}^{m}h\left(\tilde\nabla_{e_j}\Phi_*\frac{\partial}{\partial s},\Phi_*e_j\right)$$
$$+F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\sum_{i=1}^{m}h\left(\tilde\nabla_{e_i}\Phi_*\frac{\partial}{\partial t},\tilde\nabla_{e_i}\Phi_*\frac{\partial}{\partial s}\right).$$

Let X_1 and X_2 be compactly supported vector fields on M such that

$$g(X_1, Y) = F''\left(\frac{|d\phi|^2}{2}\right)\langle\tilde{\nabla} W, d\phi\rangle \cdot h(\phi_* Y, V),$$

$$g(X_2, Y) = F'\left(\frac{|d\phi|^2}{2}\right)h(\tilde{\nabla}_Y W, V)$$

for any vector field Y on M, respectively. For the first term in the right-hand side of (6.4) and the last term in the right-hand side of (6.3), we have

$$(6.6)\quad \sum_{i=1}^m e_i \cdot h\left(\tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_* e_i\right), \Phi_*\frac{\partial}{\partial t}\right)$$
$$- \sum_{i=1}^m h\left(\tilde{\nabla}_{\partial/\partial s}\left(F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\Phi_*\nabla_{e_i} e_i\right), \Phi_*\frac{\partial}{\partial t}\right)$$
$$= \sum_{i=1}^m e_i \cdot h\left(F''\left(\frac{|d\phi_{s,t}|^2}{2}\right)\sum_{j=1}^m h(\tilde{\nabla}_{\partial/\partial s}\Phi_* e_j, \Phi_* e_j)\Phi_* e_i\right.$$
$$\left. + F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\tilde{\nabla}_{\partial/\partial s}\Phi_* e_i, \Phi_*\frac{\partial}{\partial t}\right)$$
$$- \sum_{i=1}^m h\left(F''\left(\frac{|d\phi_{s,t}|^2}{2}\right)\sum_{j=1}^m h(\tilde{\nabla}_{\partial/\partial s}\Phi_* e_j, \Phi_* e_j)\Phi_*\nabla_{e_i} e_i\right.$$
$$\left. + F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)\tilde{\nabla}_{\partial/\partial s}\Phi_*\nabla_{e_i} e_i, \Phi_*\frac{\partial}{\partial t}\right)$$
$$= \sum_{i=1}^m e_i \cdot \left\{F''\left(\frac{|d\phi_{s,t}|^2}{2}\right)\sum_{j=1}^m h\left(\tilde{\nabla}_{e_j}\Phi_*\frac{\partial}{\partial s}, \Phi_* e_j\right) \cdot h\left(\Phi_* e_i, \Phi_*\frac{\partial}{\partial t}\right)\right\}$$
$$- \sum_{i=1}^m F''\left(\frac{|d\phi_{s,t}|^2}{2}\right)\sum_{j=1}^m h\left(\tilde{\nabla}_{e_j}\Phi_*\frac{\partial}{\partial s}, \Phi_* e_j\right) \cdot h\left(\Phi_*\nabla_{e_i} e_i, \Phi_*\frac{\partial}{\partial t}\right)$$
$$+ \sum_{i=1}^m e_i\left\{F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)h\left(\tilde{\nabla}_{e_i}\Phi_*\frac{\partial}{\partial s}, \Phi_*\frac{\partial}{\partial t}\right)\right\}$$
$$- \sum_{i=1}^m F'\left(\frac{|d\phi_{s,t}|^2}{2}\right)h\left(\tilde{\nabla}_{\nabla_{e_i} e_i}\Phi_*\frac{\partial}{\partial s}, \Phi_*\frac{\partial}{\partial t}\right).$$

When $s = t = 0$, (6.6) becomes

$$(6.7)\quad \sum_{i=1}^{m} e_i \cdot g(X_1, e_i) - \sum_{i=1}^{m} g(X_1, \nabla_{e_i} e_i) + \sum_{i=1}^{m} e_i \cdot g(X_2, e_i) - \sum_{i=1}^{m} g(X_2, \nabla_{e_i} e_i)$$

$$= \sum_{i=1}^{m} g(\nabla_{e_i} X_1, e_i) + \sum_{i=1}^{m} g(\nabla_{e_i} X_2, e_i)$$

$$= \operatorname{div}(X_1) + \operatorname{div}(X_2).$$

By Green's theorem the integral of (6.7) vanishes. The theorem follows from (6.1)–(6.7).

THEOREM 6.2. *Let $\phi : M \to N$ be an F-harmonic map from a Riemannian manifold M to a Riemannian manifold N. Assume that $F'' \geq 0$ and N has nonpositive curvature. Then ϕ is stable.*

Proof. It follows immediately from Theorem 6.1.

Remark. Theorem 6.2 is an extension of the well known fact for harmonic maps (cf. [EL]).

7. Stability of F-harmonic maps to S^n

We consider S^n as a submanifold in $\boldsymbol{R}^{n+1}$. Let ${}^R\nabla$ and ${}^S\nabla$ denote the Levi-Civita connections on $\boldsymbol{R}^{n+1}$ and S^n, respectively.

For a vector V in $\boldsymbol{R}^{n+1}$ at $x \in S^n$, we decompose $V = V^\top + V^\perp$, where $V^\top$ is the tangential part to S^n and $V^\perp = \langle V, x\rangle x$ is the normal part to S^n.

Let B denote the second fundamental form of S^n in $\boldsymbol{R}^{n+1}$. Then for tangent vectors X and Y of S^n at x, $B(X, Y) = -\langle X, Y\rangle x$. For a normal vector field W on S^n, the shape operator A^W corresponding to W is defined by

$$A^W(X) = -({}^R\nabla_X W)^\top,$$

where X is a tangent vector of S^n. Then it satisfies

$$\langle A^W(X), Y\rangle = \langle B(X, Y), W\rangle = -\langle X, Y\rangle\langle x, W\rangle$$

for tangent vectors X and Y of S^n at x.

THEOREM 7.1. *Let $\phi : M \to S^n$ be an F-harmonic map from a compact Riemannian manifold M to the n-dimensional unit sphere S^n. Assume that*

$$\int_M |d\phi|^2 \left\{ |d\phi|^2 F''\left(\frac{|d\phi|^2}{2}\right) + (2 - n)F'\left(\frac{|d\phi|^2}{2}\right) \right\} v_g < 0.$$

Then ϕ is unstable.

Proof. We use the above notation. We assume that M is m-dimensional. Let $\{e_i\}_{i=1}^m, \tilde{\nabla}$ and ${}^S R$ denote a local orthonormal frame field on M, the induced connection on $\phi^{-1}TS^n$ and the curvature tensor of S^n, respectively. Let $\{V_a\}_{a=1}^{n+1}$ be a parallel orthonormal frame field in $\boldsymbol{R}^{n+1}$. We shall consider the second variation

$$I(V_a^\top, V_a^\top) = \int_M F''\left(\frac{|d\phi|^2}{2}\right)\left(\sum_{i=1}^m \langle \tilde{\nabla}_{e_i} V_a^\top, \phi_* e_i\rangle\right)^2 v_g \tag{7.1}$$
$$+ \int_M F'\left(\frac{|d\phi|^2}{2}\right)\sum_{i=1}^m \{|\tilde{\nabla}_{e_i} V_a^\top|^2 - \langle {}^S R(V_a^\top, \phi_* e_i)\phi_* e_i, V_a^\top\rangle\} v_g.$$

Now we discuss at $x = \phi(p)$. As V_a is parallel in $\boldsymbol{R}^{n+1}$,

$$\begin{aligned}\tilde{\nabla}_{e_i} V_a^\top &= {}^S\nabla_{\phi_* e_i} V_a^\top = ({}^R\nabla_{\phi_* e_i} V_a^\top)^\top = ({}^R\nabla_{\phi_* e_i}(V_a - V_a^\perp))^\top \\ &= -({}^R\nabla_{\phi_* e_i} V_a^\perp)^\top = A^{V_a^\perp}(\phi_* e_i).\end{aligned}$$

So we have

$$\langle \tilde{\nabla}_{e_i} V_a^\top, \phi_* e_i\rangle = \langle A^{V_a^\perp}(\phi_* e_i), \phi_* e_i\rangle = -|\phi_* e_i|^2 \langle x, V_a^\perp\rangle = -|\phi_* e_i|^2\langle x, V_a\rangle,$$

and

$$\sum_{a=1}^{n+1}\left(\sum_{i=1}^m \langle \tilde{\nabla}_{e_i} V_a^\top, \phi_* e_i\rangle\right)^2 = |d\phi|^4. \tag{7.2}$$

We have also

$$\begin{aligned}|\tilde{\nabla}_{e_i} V_a^\top|^2 &= |A^{V_a^\perp}(\phi_* e_i)|^2 = \sum_{b=1}^{n+1}\langle A^{V_a^\perp}(\phi_* e_i), V_b\rangle^2 = \sum_{b=1}^{n+1}\langle A^{V_a^\perp}(\phi_* e_i), V_b^\perp\rangle^2 \\ &= \sum_{b=1}^{n+1}\langle \phi_* e_i, V_b^\top\rangle^2\langle x, V_a^\perp\rangle^2 = \sum_{b=1}^{n+1}\langle \phi_* e_i, V_b\rangle^2\langle x, V_a\rangle^2 = |\phi_* e_i|^2\langle x, V_a\rangle^2,\end{aligned}$$

and

$$\sum_{a=1}^{n+1}\sum_{i=1}^m |\tilde{\nabla}_{e_i} V_a^\top|^2 = |d\phi|^2. \tag{7.3}$$

Since

$$\begin{aligned}\langle {}^S R(V_a^\top, \phi_* e_i)\phi_* e_i, V_a^\top\rangle &= |\phi_* e_i|^2 \cdot |V_a^\top|^2 - \langle \phi_* e_i, V_a^\top\rangle^2 \\ &= |\phi_* e_i|^2 \cdot |V_a^\top|^2 - \langle \phi_* e_i, V_a\rangle^2,\end{aligned}$$

we have

$$(7.4)\quad \sum_{a=1}^{n+1}\sum_{i=1}^{m}\langle {}^{S}R(V_a^{\top},\phi_* e_i)\phi_* e_i, V_a^{\top}\rangle = |d\phi|^2\cdot\sum_{a=1}^{n+1}|V_a^{\top}|^2 - |d\phi|^2 = (n-1)|d\phi|^2.$$

By (7.1)–(7.4) we get

$$(7.5)\quad \sum_{a=1}^{n+1} I(V_a^{\top},V_a^{\top}) = \int_M |d\phi|^2\left\{|d\phi|^2 F''\left(\frac{|d\phi|^2}{2}\right) + (2-n)F'\left(\frac{|d\phi|^2}{2}\right)\right\}v_g.$$

By (7.5) and the assumption, we have

$$\sum_{a=1}^{n+1} I(V_a^{\top},V_a^{\top}) < 0,$$

and ϕ is unstable.

COROLLARY 7.2. *Assume that* (i) $F''\le 0$ *and* $n\ge 3$, *or* (ii) $F''<0$ *and* $n=2$. *Then any stable F-harmonic map from a compact Riemannian manifold M to* S^n *is constant.*

Remark. The assumption $F''\le 0$ in this corollary is not satisfied in the case of the p-energy, the exponential energy and the α-energy.

Proof. Suppose that ϕ is not constant. Then by the assumption and Theorem 7.1, ϕ is unstable, which is a contradiction. Thus ϕ is constant.

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GRADUATE SCHOOL OF INFORMATION SCIENCES
TOHOKU UNIVERSITY
KATAHIRA, SENDAI 980-8577, JAPAN
e-mail: ara@ims.is.tohoku.ac.jp