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KODAI MATH. J.
20 (1997), 233—240

EIGENVALUE INEQUALITIES AND MINIMAL SUBMANIFOLDS

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Abstract

Let (S^m, g_0) be the unit sphere, (M^n, g) its submanifold, λ_1 the first non-zero eigenvalue of (M^n, g), H the mean curvature vector field of M^n. By Takahashi theorem, if M^n is minimal, then $\lambda_1 \leqq n$. In this paper, we establish some eigenvalue inequalities and use them to prove:

1. If x is mass symmetric and of order $\{k, k+1\}$ for some k such that $\lambda_k \geqq n$ or $\lambda_{k+1} \leqq n$, then φ is minimal and $\lambda_k = n$ or $\lambda_{k+1} = n$.

2. If H is parallel, $\int_M H dv_M = 0$ and $\sigma^2 \leqq \lambda_1$, then $H=0$ or $\sigma^2 = \lambda_1$.

3. If H is parallel and $\lambda_k = n$ for some k, then $H=0$ or $\sigma^2(x) \geqq \lambda_{k+1} - \lambda_k$ for some $x \in M^n$.

4. $\lambda_1 \leqq \dfrac{nV^2}{V^2 - \left(\int_M H dv_M\right)^2}$. Especially, if $\int_M H dv_M = 0$, then $\lambda_1 \leqq n$, and that $\lambda_1 = n$ implies that φ is minimal.

1. Introduction

Let M^n be a compact n-dimensional Riemannian manifold, Δ_M the Laplace-Beltrami operator on M^n. Then $-\Delta_M$ has the discrete spectrum:

$$\operatorname{spec}(\Delta_M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to \infty\}$$

Denote the unit hypersphere in the Euclidean space E^{m+1} by S^m. A. Ros in [2] imbedded S^m into the space of real symmetric matrices and obtained some eigenvalue inequalities for minimal submanifolds of S^m. In this paper, we consider more simple and more natural immersion $x: M^n \xrightarrow{\varphi} S^m \xrightarrow{i} E^{m+1}$, where φ is an isometric immersion, not necessarily minimal, i is the inclusion, and we also get some similar inequalities. Last, we use them to prove the following results:

Keywords. Immersion of k-type; order of an immersion; mean curvature vector (field); Weingarten transformation

Mathematical Subject Classifications (1991) 58G25

Received January 20, 1997; revised May 8, 1997.

1. If x is mass symmetric and of order $\{k, k+1\}$ for some k such that $\lambda_k \geqq n$ or $\lambda_{k+1} \leqq n$, then φ is minimal and $\lambda_k = n$ or $\lambda_{k+1} = n$.

2. If H is parallel, $\int_M H dv_M = 0$ and $\sigma^2 \leqq \lambda_1$, then $H = 0$ or $\sigma^2 = \lambda_1$.

3. If H is parallel and $\lambda_k = n$ for some k, then $H = 0$ or $\sigma^2(x) \geqq \lambda_{k+1} - \lambda_k$ for some $x \in M^n$.

4. $\lambda_1 \leqq \dfrac{nV^2}{V^2 - \left(\int_M H dv_M\right)^2}$. Especially, if $\int_M H dv_M = 0$, then $\lambda_1 \leqq n$, and that $\lambda_1 = n$ implies that φ is minimal.

Here and below, V is the volume of M^n, $H \in E^{m+1}$ the mean curvature vector field of M^n in S^m, λ_1 the first nonzero eigenvalue of Δ_M, σ the length of the second fundamental form of M^n in S^m; equalities for vectors mean that components of both ends equate, and integrations of vectors stand for those of components.

2. Lemmas

Let $x = i \circ \varphi$ be as above and $\langle\ ,\ \rangle$ be the Euclidean inner product. From now on, we use Δ_M denote the Laplacian acting on functions over M^n as well as on vector fields of E^{m+1} which are restricted to M^n. In latter case, it acts on components. Then x has L^2 decomposition: $x = x_0 + \sum_{u \geqq 1} x_u$, where x_0 is a constant vector, $\Delta_M x_u = -\lambda_u x_u$; $\int_M \langle x_u, x_v \rangle dv_M = \begin{cases} 0 & u \neq v, \\ a_u (\geqq 0) & u = v. \end{cases}$ x_0 is called mass center of x. If x_0 is congrucent with the geometric center of S^m, we say that x is mass-symmetric. If $\exists\, u_i \geqq 1$, $i = 1, \dots, k$, such that $x = x_0 + \sum_{i=1}^k x_{u_i}$, then x is called of k-type and $\{u_1, \dots, u_k\}$ is by definition the order of x. It is known that if φ is minimal, then φ is mass symmetric, of 1-type and its order is $\{k\}$ for some k by Takahashi theorem and Lemma 2.4 below. From the decomposition of x we have:

$$\Delta_M x = -\sum_{u \geqq 0} \lambda_u x_u\,;\ \Delta_M^2 x = \sum_{u \geqq 0} \lambda_u^2 x_u\,.$$

Define:

$$\Psi_k = -\int_M \langle \Delta_M x, x \rangle dv_M - \lambda_k \int_M \langle x, x \rangle dv_M$$

$$\Theta_k = \int_M \langle \Delta_M x, \Delta_M x \rangle dv_M + \lambda_k \int \langle \Delta_M x, x \rangle dv_M$$

$$\Omega_k = -\int_M \langle \Delta_M^2 x, \Delta_M x \rangle dv_M - \lambda_k \int_M \langle \Delta_M x, \Delta_M x \rangle dv_M$$

Then, we have

LEMMA 2.1. *For any integer* $k \geqq 1$,

(i) $\Theta_k - \lambda_{k+1}\Psi_k = \sum_{u \geqq 0} (\lambda_u - \lambda_k)(\lambda_u - \lambda_{k+1}) a_u \geqq 0$

and the equality holds iff x mass symmetric and is of order $\{k, k+1\}$;

(ii) $\Omega_k - \lambda_{k+1}\Theta_k = \sum_{u \geqq 0} \lambda_u (\lambda_u - \lambda_k)(\lambda_u - \lambda_{k+1}) a_u \geqq 0$

and the equality holds iff x is of order $\{k, k+1\}$.

Remark. The two inequalities above holds for $k \geqq 0$.

Let H be the mean curvature vector field of M^n. Then Takahashi theorem says that $\Delta_M x = nH - nx$. Hence $\Delta_M^2 x = n\Delta_M H - n^2 H + n^2 x$. Therefore,

$$\int_M \langle x, x \rangle dv_M = V \, ; \ \int_M \langle \Delta_M x, x \rangle dv_M = -nV \, ;$$

$$\int_M \langle \Delta_M x, \Delta_M x \rangle dv_M = n^2 \int_M \langle H, H \rangle dv_M + n^2 V \, ;$$

$$\int_M \langle \Delta_M^2 x, \Delta_M x \rangle dv_M = n^2 \int_M \langle \Delta_M H, H \rangle dv_M - 2n^3 \int_M \langle H, H \rangle dv_M - n^3 V \, ;$$

$$\Psi_k = (n - \lambda_k) V \, ; \ \Theta_k = n^2 \int_M \langle H, H \rangle dv_M + n(n - \lambda_k) V \, ;$$

$$\Omega_k = -n^2 \int_M \langle \Delta_M H, H \rangle dv_M + n^2 (2n - \lambda_k) \int_M \langle H, H \rangle dv_M + n^2 (n - \lambda_k) V \, .$$

From these formulas, we have:

LEMMA 2.2. *For any* $k \geqq 0$,

(i) $\Theta_k - \lambda_{k+1}\Psi_k = n^2 \int_M H^2 dv_M + (n - \lambda_k)(n - \lambda_{k+1}) V$,

(ii) $\Omega_k - \lambda_{k+1}\Theta_k = -n^2 \int_M \langle \Delta_M H, H \rangle dv_M + n^2 (2n - \lambda_k - \lambda_{k+1}) \int_M H^2 dv_M$
$+ n(n - \lambda_k)(n - \lambda_{k+1}) V$.

Next, use $\nabla^\perp$ to denote the normal connection of φ, σ the length of the second fundamental form of φ. We have

LEMMA 2.3.

$$-\int_M \langle \Delta_M H, H \rangle dv_M \leqq \int_M \langle \nabla^\perp H, \nabla^\perp H \rangle dv_M + \int_M H^2 \sigma^2 dv_M$$

where the equality holds whenever the codimension is one.

Proof. Without loss of generality, we let $M^n \subset S^m$, and $\varphi = i$, the inclusion. Let ∇ and $\bar{\nabla}$ be the Riemannian connections of M^n and S^m respectively. On M^n, we take a local field of orthonormal frame: $e_1, \ldots, e_n$. Then by Gauss and Weingarten formulas we get:

$$\frac{1}{2}\Delta_M\langle H, H\rangle = \frac{1}{2}\sum_{j=1}^{n}\nabla_{e_j}\nabla_{e_j}\langle H, H\rangle = \frac{1}{2}\sum_{j=1}^{n}\nabla_{e_j}(\bar{\nabla}_{e_j}\langle H, H\rangle)$$

$$= \sum_{j=1}^{n}\nabla_{e_j}\langle\bar{\nabla}_{e_j}H, H\rangle = \sum_{j=1}^{n}\nabla_{e_j}\langle\nabla^{\perp}_{e_j}H, H\rangle$$

$$= \sum_{j=1}^{n}\bar{\nabla}_{e_j}\langle\nabla^{\perp}_{e_j}H, H\rangle$$

$$= \sum_{j=1}^{n}(\langle\bar{\nabla}_{e_j}(\nabla^{\perp}_{e_j}H), H\rangle + \langle\nabla^{\perp}_{e_j}H, \bar{\nabla}_{e_j}H\rangle)$$

$$= \sum_{j=1}^{n}\langle\nabla^{\perp}_{e_j}\nabla^{\perp}_{e_j}H, H\rangle + \sum_{j=1}^{n}\langle\nabla^{\perp}_{e_j}H, \nabla^{\perp}_{e_j}H\rangle$$

$$\triangleq \langle\Delta^{\perp}H, H\rangle + \langle\nabla^{\perp}H, \nabla^{\perp}H\rangle$$

Define $\Delta_M v = \sum_{j=1}^{n}(D_{e_j}D_{e_j}v - D_{\nabla_{e_j}e_j}v)$, where D is the flat connection of E^{m+1}, $v = v(x) \in E^{m+1}$, $n \in M^n$. It is easy to verify that Δ_M defined here is exactly the Laplacian acting on vectors by on their components which is defined at the begining of this section. We can choose $\{e_i, i=1, 2, \ldots, n\}$ such that $\nabla_{e_i}e_j = 0$. Hence we have

$$\Delta_M H = \sum_{j=1}^{n}D_{e_j}D_{e_j}H = \sum_{j=1}^{n}D_{e_j}(\bar{\nabla}_{e_j}H + \bar{B}(e_j, H))$$

$$= \sum_{j=1}^{n}D_{e_j}\bar{\nabla}_{e_j}H \quad \text{because } S^m \text{ is totally umbilical}$$

$$= \sum_{j=1}^{n}[\bar{\nabla}_{e_j}\bar{\nabla}_{e_j}H + \bar{B}(e_j, \bar{\nabla}_{e_j}H)]$$

where, $\bar{B}$ and $\bar{A}$ are the second fundamental form and Weingarten transformation of S^m respectively. In the following, we use B and A stand for corresponding quantities of M^n in S^m. Then

$$\sum_{j=1}^{n}\bar{\nabla}_{e_j}(\bar{\nabla}_{e_j}H) = \sum_{j=1}^{n}\bar{\nabla}_{e_j}(-A_H e_j + \nabla^{\perp}_{e_j}H)$$

$$= \sum_{j=1}^{n} -(\nabla_{e_j}(A_H e_j) + B(A_H e_j, e_j)) + \sum_{j=1}^{n}(-A_{\nabla^{\perp}_{e_j}H}e_j + \nabla^{\perp}_{e_j}\nabla^{\perp}_{e_j}H)$$

$$= -\sum_{j=1}^{n}[\nabla_{e_j}(A_H e_j) + A_{\nabla^{\perp}_{e_j}H}e_j] + \sum_{j=1}^{n}(-B(A_H e_j, e_j) + \nabla^{\perp}_{e_j}\nabla^{\perp}_{e_j}H).$$

From the above calculations, we have

$$\langle \Delta_M H, H \rangle = -\sum_{j=1}^{n} \langle B(A_H e_j, e_j), H \rangle + \langle \Delta^{\perp} H, H \rangle$$

$$= -\sum_{j=1}^{n} \langle A_H e_j, A_H e_j \rangle + \langle \Delta^{\perp} H, H \rangle$$

$$\triangleq -\langle A_H, A_H \rangle + \langle \Delta^{\perp} H, H \rangle$$

where $\Delta^{\perp} = \sum_{j=1}^{n} \nabla_{e_j}^{\perp} \nabla_{e_j}^{\perp}$. So $\langle \Delta^{\perp} H, H \rangle = \langle \Delta_M H, H \rangle + \langle A_H, A_H \rangle$. Hence we have

$$\frac{1}{2} \Delta_M \langle H, H \rangle = \langle \Delta_M H, H \rangle + \langle \nabla^{\perp} H, \nabla^{\perp} H \rangle + \langle A_H, A_H \rangle .$$

Therefore,

$$-\int_M \langle \Delta_M H, H \rangle dv_M = \int_M \langle \nabla^{\perp} H, \nabla^{\perp} H \rangle dv_M + \int_M \langle A_H, A_H \rangle dv_M .$$

Take a local unit normal vector field e_{n+1} of M^n in TS^m such that e_{n+1} is parallel with H, i.e. $H = |H| e_{n+1}$. Then

$$\langle A_H, A_H \rangle = \sum_{j=1}^{n} \langle A_H e_j, A_H e_j \rangle = H^2 \sum_{j=1}^{n} \langle A_{n+1} e_j, A_{n+1} e_j \rangle$$

$$= H^2 \| A_{n+1} \|^2 \leqq H^2 \sigma^2$$

and equality holds if the codimension of M^n in S^m is one. Now, the proof of Lemma 2.3 is complete.

In next section, when we discuss the upper bound of λ_1, we need the following two lemmas.

LEMMA 2.4. *M^n is mass symmetric iff* $\int_M H dv_M = 0$.

Proof. By Takahashi theorem and the L^2 decompositions of x and H, we have

$$\Delta x = n(H - x) = n(H_0 - x_0) + \sum_{u \geqq 1} n(H_u - x_u) .$$

Integrating both ends and noting that when $u \geqq 1$, $\int_M H_u dv_M = \int_M x_u dv_M = 0$, we can obtain: $\int_M H_0 dv_M = \int_M x_0 dv_M$, i.e. $H_0 = x_0$. Hence $\int_M H dv_M = \int_M H_0 dv_M = \int_M x_0 dv_M = x_0 V$. Lemma 2.4 follows.

LEMMA 2.5. $\left(\int_M H dv_M\right)^2 \leqq V^2$, *and the equality is true iff x is constant.*

Proof. $\left(\int_M H dv_M\right)^2 = \left(\int_M H_0 dv_M\right)^2 = \left(\int_M x_0 dv_M\right)^2 = \langle x_0, x_0 \rangle V^2$

$$= \int_M \langle x_0, x_0 \rangle dv_M \cdot V \leqq \int_M \langle x, x \rangle dv_M \cdot V = V^2 ,$$

and the equality holds iff $x = x_0$.

3. Eigenvalue inequalities and their corollaries

By Takahashi theorem, if φ is minimal, then φ is mass symmetric and of order $\{k\}$ for some k, and $\lambda_k=n$. The following theorem shows that the inverse is true.

THEOREM 3.1. *If $i\circ\varphi$ is mass symmetric and of order $\{k, k+1\}$ for some k such that $\lambda_k\geqq n$ or $\lambda_{k+1}\leqq n$, then φ is minimal (hence is of 1-type by Takahashi theorem) and $\lambda_k=n$ or $\lambda_{k+1}=n$.*

Proof. By Lemmas 2.1-(i) and 2.2-(i) we have

$$n^2\int_M H^2 dv_M+(n-\lambda_k)(n-\lambda_{k+1})V=0$$

from which the result follows.

THEOREM 3.2. *For any $k\geqq 0$*

$$n\int_M\langle\nabla^\perp H, \nabla^\perp H\rangle dv_M+n\int_M H^2(\sigma^2+2n-\lambda_k-\lambda_{k+1})dv_M$$
$$+(n-\lambda_k)(n-\lambda_{k+1})V\geqq 0$$

where the equality for $k\geqq 1$ implies that φ is of order $\{k, k+1\}$.

Proof. From Lemmas 2.1-(ii), 2.2-(ii) and 2.3 we can reach the inequality we need. The equality shows that $i\circ\varphi$ is of order $\{k, k+1\}$ by Lemma 2.1-(ii).

COROLLARY 3.1. *Let φ have parallel mean curvature normal and $\lambda_k=n$ for some k. Then M^n is minimal or $\sigma^2(x)\geqq\lambda_{k+1}-\lambda_k$ for some $x\in M^n$.*

THEOREM 3.3. *Let x_0 be the mass center of $x=i\circ\varphi$, then for any real number t, we have*

$$(n-\lambda_1)(n-t)^2V+2n^2(n-t)\int_M H^2 dv_M+n^2\int_M\langle\nabla^\perp H, \nabla^\perp H\rangle dv_M$$
$$+n^2\int_M H^2(\sigma^2-\lambda_1)dv_M+\lambda_1 t^2\int_M\langle x_0, x_0\rangle dv_M\geqq 0.$$

Proof. Set $F_t=-\Delta_M x-t(x-x_0)$. Then $\int_M F_t dv_M=0$. By the minimal principle for λ_1 we have

$$(*)\qquad -\int_M\langle\Delta_M F_t, F_t\rangle dv_M\geqq\lambda_1\int_M\langle F_t, F_t\rangle dv_M.$$

On the other hand,

$$F_t = -nH + nx - t(x - x_0), \qquad \Delta_M F_t = -n\Delta_M H + n(n-t)(H - x),$$

$$\int_M \langle x, x_0 \rangle dv_M = \int_M \langle x_0, x_0 \rangle dv_M, \int_M \langle \Delta_M x, x_0 \rangle dv_M = 0,$$

$$\int_M \langle \Delta_M H, x_0 \rangle dv_M = 0, \int_M \langle H, x_0 \rangle dv_M = \int_M \langle x_0, x_0 \rangle dv_M.$$

Therefore, we reach

$$\int_M \langle \Delta_M F_t, F_t \rangle dv_M = n^2 \int_M \langle \Delta_M H, H \rangle dv_M - 2n^2(n-t) \int_M H^2 dv_M - n(n-t)^2 V,$$

$$\int_M \langle F_t, F_t \rangle dv_M = n^2 \int_M H^2 dv_M + (n-t)^2 V - t^2 \int_M \langle x_0, x_0 \rangle dv_M.$$

Substituting the last two equalities into (*) and using Lemma 2.3, the theorem follows.

For minimal submanifolds of a unit sphere, if $\sigma^2 \leqq n$, then $\sigma^2 = 0$ or n. For non-minimal ones, we also have a similar result (see (ii) below).

COROLLARY 3.2. *Suppose that $x = i \circ \varphi$ is mass symmetric or equivalently that $\int_M H dv_M = 0$.*

(i) *If $\lambda_1 = n$, then φ is minimal.*

(ii) *If $\nabla^\perp H = 0$, $\sigma^2 \leqq \lambda_1$, then $\sigma^2 = \lambda_1$ unless $H = 0$.*

Proof. Under the assumption of (i), using Theorem 3.3 (taking $t > n$), we have

$$\int_M H^2 dv_M \leqq \frac{1}{2(n-t)} \left\{ -\int_M \langle \nabla^\perp H, \nabla^\perp H \rangle dv_M - \int_M H^2 (\sigma^2 - \lambda_1) dv_M \right\}.$$

Let $t \to \infty$ we have $H = 0$.

For the proof of (ii), we use Theorem 3.3 (take $t = n$) and Lemma 2.4. We get $H^2 \int_M (\sigma^2 - \lambda_1) dv_M \geqq 0$. Then the (ii) follows.

COROLLARY 3.3. $\lambda_1 \leqq \dfrac{nV^2}{V^2 - \left(\int_M H dv_M \right)^2}$.

Especially, if $\int_M H dv_M = 0$, then $\lambda_1 \leqq n$. Furthermore, $\lambda_1 = n$ implies $H = 0$.

Proof. Let the both ends of the inequality in Theorem 3.3 be divided by $(n-t)^2$, then let t go to infinity. The inequality is obtained. The rest follows from Corollary 3.2-(i).

Remark. Let M be an n-dimensional compact submanifold of the unit hypersphere S^m of an Euclidean $m+1$-space with lower order p and upper order q. B. Y. Chen proved the following two statements (see [1] p. 144 Corollary 6.13):

(1) If M is mass symmetric, then $\lambda_1 \leqq \lambda_p \leqq n$. In particular, $\lambda_p = n$ iff M is of 1-type and of order $\{p\}$.

(2) If M is of finite type, then $\lambda_q \geqq n$. In particular, $\lambda_q = n$ iff M is of 1-type and of order $\{q\}$.

Because 1-type is 2-type, by Theorem 3.1 we know that M in (1) and (2) above is in fact minimal (if M is also of mass symmetric in (2)).

REFERENCES

[1] B.Y. CHEN, A report on submanifolds of finite type, Soochow J. Math., **22** (1996), 117-337.

[2] A. ROS, Eigenvalue inequalities for minimal submanifolds and P-manifolds, Math. Z., **187** (1984), 393-404.

[3] Y.B. SHEN, On eigenvalue inequalities of minimal and Kaehlerian submanifolds, Acta Math. Sinica A, **34** (1991), 551-560 (in Chinese).

[4] T. TAKAHASHI, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan, **18** (1966), 380-385.

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