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ON THE FUNDAMENTAL INEQUALITY FOR NON-DEGENERATE HOLOMORPHIC CURVES

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1. Introduction

Let

$$f: \boldsymbol{C} \longrightarrow P^n(\boldsymbol{C})$$

be a holomorphic curve from $\boldsymbol{C}$ into the n-dimensional complex projective space $P^n(\boldsymbol{C})$, where n is a positive integer, and let

$$(f_1, \dots, f_{n+1}): \boldsymbol{C} \longrightarrow \boldsymbol{C}^{n+1} - \{\boldsymbol{0}\}$$

be a reduced representation of f. We then write f as follows:

$$f=[f_1, \dots, f_{n+1}].$$

We use the following notation:

$$\|f(z)\|=(|f_1(z)|^2+\cdots+|f_{n+1}(z)|^2)^{1/2}$$

and for a vector $\boldsymbol{a}=(a_1, \dots, a_{n+1})$ in $\boldsymbol{C}^{n+1}$

$$(\boldsymbol{a}, f)=a_1f_1+\cdots+a_{n+1}f_{n+1},$$

$$(\boldsymbol{a}, f(z))=a_1f_1(z)+\cdots+a_{n+1}f_{n+1}(z),$$

$$\|\boldsymbol{a}\|=(|a_1|^2+\cdots+|a_{n+1}|^2)^{1/2}.$$

The characteristic function $T(r, f)$ of f is defined as follows (see [11]):

$$T(r, f)=\frac{1}{2\pi}\int_0^{2\pi}\log\|f(re^{i\theta})\|d\theta-\log\|f(0)\|.$$

Further, put

$$U(z)=\max_{1\le j\le n+1}|f_j(z)|,$$

then it is known ([1]) that

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$$(1)\qquad T(r, f)=\frac{1}{2\pi}\int_0^{2\pi}\log U(re^{i\theta})d\theta+O(1).$$

We suppose throughout the paper that f is transcendental; that is to say,

$$\lim_{r\to\infty}\frac{T(r, f)}{\log r}=\infty.$$

We denote by $\rho(f)$ the order of f:

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r, f)}{\log r}$$

and by $S(r, f)$ any quantity satisfying

$$S(r, f)=\begin{cases} O(\log r) & (r\to\infty) \quad \text{if} \quad \rho(f)<\infty \\ O(\log rT(r, f)) & (r\to\infty,\ r\notin E) \quad \text{otherwise}, \end{cases}$$

where E is a subset of $[0, \infty)$ the measure of which is finite.

For meromorphic functions in $|z|<\infty$ we shall use the standard notation and symbols of the Nevanlinna theory of meromorphic functions ([3]).

For $\boldsymbol{a}=(a_1, \dots, a_{n+1})\in \boldsymbol{C}^{n+1}-\{0\}$ such that $(\boldsymbol{a}, f)\neq 0$, we write

$$m(r, \boldsymbol{a}, f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|\boldsymbol{a}\|\|f(re^{i\theta})\|}{|(\boldsymbol{a}, f(re^{i\theta}))|}d\theta$$

and

$$N(r, \boldsymbol{a}, f)=N(r, 1/(\boldsymbol{a}, f)).$$

Then we have

$$(2)\qquad T(r, f)=N(r, \boldsymbol{a}, f)+m(r, \boldsymbol{a}, f)+O(1) \qquad ([11], \text{p. } 76).$$

It is called that the quantity

$$\delta(\boldsymbol{a}, f)=1-\limsup_{r\to\infty}\frac{N(r, \boldsymbol{a}, f)}{T(r, f)}=\liminf_{r\to\infty}\frac{m(r, \boldsymbol{a}, f)}{T(r, f)}$$

is the deficiency of $\boldsymbol{a}$ with respect to f. By (2)

$$0\leqq\delta(\boldsymbol{a}, f)\leqq 1$$

since $m(r, \boldsymbol{a}, f)\geqq 0$ and $N(r, \boldsymbol{a}, f)\geqq 0$ for $r\geqq 1$.

Further, let $\nu(c)$ be the order of zero of $(\boldsymbol{a}, f(z))$ at $z=c$ and for a positive integer k let

$$n_k(r, \boldsymbol{a}, f)=\sum_{|c|\leqq r}\min\{\nu(c), k\}.$$

Then, we put for $r>0$

$$N_k(r, \boldsymbol{a}, f)=\int_0^r\frac{n_k(t, \boldsymbol{a}, f)-n_k(0, \boldsymbol{a}, f)}{t}dt+n_k(0, \boldsymbol{a}, f)\log r$$

and put

$$\delta_k(\boldsymbol{a}, f)=1-\limsup_{r\to\infty}\frac{N_k(r, \boldsymbol{a}, f)}{T(r, f)}.$$

It is easy to see that $\delta(\boldsymbol{a}, f)\leqq\delta_k(\boldsymbol{a}, f)\leqq 1$ by definition.

We say that f is linearly non-degenerate (or simply, non-degenerate) if and only if $f_1, \dots, f_{n+1}$ are linearly independent over $\boldsymbol{C}$ and that f is linearly degenerate when f is not linearly non-degenerate.

It is well-known that f is non-degenerate if and only if the Wronskian $W(f_1, \dots, f_{n+1})$ of $f_1, \dots, f_{n+1}$ is not identically equal to zero. From now on we suppose that f is non-degenerate.

Let X be a subset of $\boldsymbol{C}^{n+1}-\{\boldsymbol{0}\}$ such that $\#X\geqq n+1$. We suppose that X is in general position; that is to say, any $n+1$ elements of X are linearly independent. About 65 years ago, H. Cartan ([1]) proved the following fundamental inequality.

THEOREM A. *For any* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q$ *in* X,

$$(q-n-1)T(r, f)<\sum_{j=1}^{q}N(r, \boldsymbol{a}_j, f)-N(r, 1/W(f_1, \dots, f_{n+1}))+S(r, f).$$

E. I. Nochka generalized this theorem to the case when f is linearly degenerate (see [2], Chapter 3). Our first purpose of this paper is to give an improvement of Theorem A.

Recently, we have introduced the following notion for holomorphic curves in [7], which corresponds to the derivative of meromorphic functions when $n=1$.

DEFINITION A. We call the holomorphic curve induced by the mapping

$$(f_1^{n+1}, \dots, f_n^{n+1}, W(f_1, \dots, f_{n+1})): \boldsymbol{C}\longrightarrow\boldsymbol{C}^{n+1}$$

the derived holomorphic curve of f and express it by f^*.

It is easy to see that f^* is independent of the choice of reduced representation of f ([7]).

Let $d(z)$ be an entire function such that the functions

$$f_j^{n+1}/d\ (j=1, \dots, n) \quad\text{and}\quad W(f_1, \dots, f_{n+1})/d$$

are entire functions without common zeros. Then,

$$f^*=[f_1^{n+1}/d, \dots, f_n^{n+1}/d, W(f_1, \dots, f_{n+1})/d].$$

We proved the following in [7].

THEOREM B. (a) $T(r, f^*)\leqq(n+1)T(r, f)-N(r, 1/d)+S(r, f)$,

(b) f^* *is transcendental,*

(c) $\rho(f^*)=\rho(f)$,

(d) f^* *is not always non-degenerate.*

Further, we introduced the following subset of X in [8], which corresponds to the pole of meromorphic functions when $n=1$.

$$X(0)=\{\boldsymbol{a}=(a_1, \dots, a_{n+1})\in X : a_{n+1}=0\}.$$

It is easy to see that $\#X(0)\leqq n$ as X is in general position.

Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}$ be the standard basis of $\boldsymbol{C}^{n+1}$. Then, we have

THEOREM C. *For any* $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q \in X-X(0)$ $(1\leqq q<\infty)$,

$$\sum_{j=1}^{q} m(r, \boldsymbol{a}_j, f)\leqq m(r, \boldsymbol{e}_{n+1}, f^*)+S(r, f)$$

(see Theorem 1 in [8] and [9]).

When $X(0)$ is empty, we can easily obtain Theorem A from Theorem B, (a) and Theorem C, but Theorem C does not contain Theorem A when $X(0)$ is not empty. It is desirable for us to give a result which contains Theorem A. To that end, we shall introduce some new notions in Section 2, and in Section 3 we shall give a refinement of Theorem A and an improvement of the defect relation. In Section 4 we shall give an improvement of the second main theorem for moving targets obtained by M. Ru and W. Stoll ([4]), which is the second purpose of this paper.

2. Preliminaries and lemmas

Let $f=[f_1, \dots, f_{n+1}]$, $T(r, f)$, X and $X(0)$ etc. be as in Section 1.

DEFINITION 1. We put

$$u(z)=\max_{1\leqq j\leqq n} |f_j(z)|$$

and

$$t(r, f)=\frac{1}{2\pi}\int_0^{2\pi}\log u(re^{i\theta})d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log u(e^{i\theta})d\theta.$$

It is easy to see the following properties of $t(r, f)$.

PROPOSITION 1. (a) *$t(r, f)$ is independent of the choice of reduced representation of f.*

(b) $t(r, f)\leqq T(r, f)+O(1)$.

(c) $N(r, 1/f_j)\leqq t(r, f)+O(1)$ $(j=1, \dots, n)$.

As an improvement of Theorem B, (a) (=Lemma 3([7])), we can prove the following

LEMMA 1. $T(r, f^*)\leqq T(r, f)+nt(r, f)-N(r, 1/d)+S(r, f)$.

Proof. From the inequality

$$\begin{aligned}\|f^*(z)\|^2&=\{|f_1(z)|^{2(n+1)}+\cdots+|f_n(z)|^{2(n+1)}+|W(f_1, \dots, f_{n+1})(z)|^2\}/|d(z)|^2\\ &\leqq\frac{U(z)^2}{|d(z)|^2}\Big\{|f_1(z)|^{2n}+\cdots+|f_n(z)|^{2n}\\ &\quad+|f_1(z)\cdots f_n(z)|^2\frac{|W(f_1, \dots, f_{n+1})(z)|^2}{|f_1(z)\cdots f_{n+1}(z)|^2}\Big\}\\ &\leqq\frac{U(z)^2}{|d(z)|^2}u(z)^{2n}\Big\{n+\frac{|W(f_1, \dots, f_{n+1})(z)|^2}{|f_1(z)\cdots f_{n+1}(z)|^2}\Big\}\end{aligned}$$

and from the fact that

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|W(f_1, \dots, f_{n+1})(re^{i\theta})|}{|f_1(re^{i\theta})\cdots f_{n+1}(re^{i\theta})|}d\theta=S(r, f)$$

(see [1], p. 12–p. 15), we easily obtain our lemma by (1).

DEFINITION 2. We put

$$\omega=\liminf_{r\to\infty}\frac{t(r, f)}{T(r, f)}\quad\text{and}\quad\Omega=\limsup_{r\to\infty}\frac{t(r, f)}{T(r, f)}.$$

PROPOSITION 2. $0\leqq\omega\leqq\Omega\leqq1$.

Suppose now that $X(0)$ is not empty and that $X(0)=\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_\nu\}$ $(1\leqq\nu\leqq n)$. Put

$$(\boldsymbol{b}_j, f)=G_j\qquad(j=1, \dots, \nu).$$

DEFINITION 3. We express the holomorphic curve induced by the mapping

$$(G_1\cdots G_\nu f_1^{n+1-\nu}, \dots, G_1\cdots G_\nu f_n^{n+1-\nu}, W(f_1, \dots, f_{n+1})): \boldsymbol{C}\longrightarrow\boldsymbol{C}^{n+1}$$

by f_ν^*.

It is easy to see that f_ν^* is independent of the choice of reduced representation of f as in the case of f^*.

Let d_ν be an entire function such that the functions

$$G_1\cdots G_\nu f_j^{n+1-\nu}/d_\nu\,(j=1, \dots, n)\quad\text{and}\quad W(f_1, \dots, f_{n+1})/d_\nu$$

are entire functions without common zeros. Then we have the following

LEMMA 2.

$$T(r, f_\nu^*)\leqq T(r, f)+(n-\nu)t(r, f)+\sum_{j=1}^{\nu}N(r, 1/G_j)-N(r, 1/d_\nu)+S(r, f).$$

Proof. We suppose without loss of generality that $\boldsymbol{b}_1, \dots, \boldsymbol{b}_\nu, \boldsymbol{e}_{\nu+1}, \dots, \boldsymbol{e}_n, \boldsymbol{e}_{n+1}$ are linearly independent because $\boldsymbol{b}_1, \dots, \boldsymbol{b}_\nu$ are linearly independent vectors in $X(0)$.

Now, put $\Pi=G_1\cdots G_\nu$ and $l=n-\nu$. Then, we have the inequality

$$\begin{aligned}
&\|f_\nu^*(z)\|^2\\
&=\{|\Pi(z)|^2|f_1(z)|^{2(l+1)}+\cdots+|\Pi(z)|^2|f_n(z)|^{2(l+1)}+|W(z)|^2\}/|d_\nu(z)|^2\\
&=\frac{|\Pi(z)|^2}{|d_\nu(z)|^2}\left\{|f_1(z)|^{2(l+1)}+\cdots+|f_n(z)|^{2(l+1)}+\frac{|f_{\nu+1}(z)\cdots f_{n+1}(z)W(z)|^2}{|\Pi(z)f_{\nu+1}(z)\cdots f_{n+1}(z)|^2}\right\}\\
&\leqq\frac{|\Pi(z)|^2}{|d_\nu(z)|^2}U(z)^2u(z)^{2(n-\nu)}\left(n+\frac{|W(z)|^2}{|\Pi(z)f_{\nu+1}(z)\cdots f_{n+1}(z)|^2}\right),
\end{aligned}$$

where $W=W(f_1, \dots, f_{n+1})$. From this inequality we easily obtain our lemme as in the proof of Lemma 1 since

$$W(f_1, \dots, f_{n+1})=c_\nu W(G_1, \dots, G_\nu, f_{\nu+1}, \dots, f_{n+1}) \qquad (c_\nu\neq 0, \text{ constant}).$$

As in the case of (b) and (c) of Theorem B, we can prove the following properties of f_ν^*.

PROPOSITION 3. (a) f_ν^* *is transcendental,*
(b) $\rho(f_\nu^*)=\rho(f)$.

To prove this proposition we use the relation

$$\begin{aligned}
\frac{W(f_1, \dots, f_{n+1})}{G_1\cdots G_\nu f_1^{n+1-\nu}}&=\frac{f_1}{G_1}\cdots\frac{f_1}{G_\nu}\cdot\frac{W(f_1, \dots, f_{n+1})}{f_1^{n+1}}\\
&=\frac{f_1}{G_1}\cdots\frac{f_1}{G_\nu}\cdot W\left(\left(\frac{f_2}{f_1}\right)', \dots, \left(\frac{f_{n+1}}{f_1}\right)'\right)
\end{aligned}$$

and the fact that for $G=a_1f_1+\cdots+a_nf_n$

$$\frac{G}{f_1}=a_1+\sum_{j=2}^n a_j\cdot\frac{f_j}{f_1} \quad\text{is}\quad \begin{cases}\text{rational if so are } f_j/f_1\ (j=2, \dots, n);\\ \text{of order } <\rho(f) \text{ if } \rho(f_j/f_1)<\rho(f)\ (j=2, \dots, n).\end{cases}$$

3. Fundamental inequality

Let $f=[f_1, \dots, f_{n+1}]$, $T(r, f)$, X and $X(0)$ etc. be as in Section 1. Suppose that $X(0)$ is not empty and that

$$X(0)=\{\boldsymbol{b}_1, \dots, \boldsymbol{b}_\nu\} \qquad (1\leqq\nu\leqq n).$$

We put

$$(\boldsymbol{b}_j, f)=G_j \qquad (j=1, \dots, \nu).$$

We suppose without loss of generality that $\boldsymbol{b}_1, \dots, \boldsymbol{b}_\nu, \boldsymbol{e}_{\nu+1}, \dots, \boldsymbol{e}_n, \boldsymbol{e}_{n+1}$ are linearly independent as in the proof of Lemma 2.

THEOREM 1. *For any* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q \in X - X(0)$ $(1 \leqq q < \infty)$,

$$\sum_{j=1}^{q} m(r, \boldsymbol{a}_j, f) \leqq m(r, \boldsymbol{e}_{n+1}, f_\nu^*) + S(r, f)$$

$$\leqq T(r, f) + (n-\nu)t(r, f) + \sum_{j=1}^{\nu} N(r, 1/G_j) - N(r, 1/W) + S(r, f),$$

where $W = W(f_1, \ldots, f_{n+1})$.

Proof. We have only to prove this theorem for $q \geqq n+1$. Put

$$(\boldsymbol{a}_j, f) = F_j \qquad (j = 1, \ldots, q).$$

For any $z(\neq 0)$ arbitrarily fixed, let

$$|F_{j_1}(z)| \leqq |F_{j_2}(z)| \leqq \cdots \leqq |F_{j_q}(z)| \qquad (1 \leqq j_1, \ldots, j_q \leqq q).$$

Then there is a positive constant K such that

(3) $$\|f(z)\| \leqq K |F_{j_p}(z)| \qquad (p = n+1, \ldots, q),$$

(4) $$|F_{j_p}(z)| \leqq K \|f(z)\| \qquad (p = 1, \ldots, q)$$

and since the $n+1$-th elements of vectors a_j are different from zero,

(5) $$|f_{n+1}(z)| \leqq K\{u(z) + |F_{j_p}(z)|\} \qquad (p = 1, \ldots, q).$$

(From now on we denote by K a positive constant, which may be different from each other when it appears in different places.)

(i) The case when $u(z) \leqq |F_{j_1}(z)|$.

Since $\|f(z)\| \leqq K |F_{j_1}(z)|$ in this case by (5), we have

(6) $$\prod_{j=1}^{q} \frac{\|\boldsymbol{a}_j\| \|f(z)\|}{|(\boldsymbol{a}_j, f(z))|} \leqq K.$$

(ii) The case when $|F_{j_1}(z)| < u(z)$.

In this case, by using (5) we have

$$\|f(z)\| \leqq K u(z)$$

and by (3) we obtain

(7) $$\prod_{j=1}^{q} \frac{\|\boldsymbol{a}_j\| \|f(z)\|}{|(\boldsymbol{a}_j, f(z))|} \leqq K \prod_{p=1}^{n} \frac{u(z)}{|F_{j_p}(z)|} = K \frac{|G_1(z) \cdots G_\nu(z)| u(z)^n}{|G_1(z) \cdots G_\nu(z) F_{j_1}(z) \cdots F_{j_n}(z)|} = (*).$$

Here, we put

$$G_i = H_i \ (i = 1, \ldots, \nu), \qquad F_{j_p} = H_{\nu+p} \ (p = 1, \ldots, n)$$

and let

$$|H_{i_1}(z)|\leqq|H_{i_2}(z)|\leqq\cdots\leqq|H_{i_{n+\nu}}(z)|$$

Then for $k=2,\ldots,\nu$

$$u(z)\leqq K|H_{i_{n+k}}(z)|$$

and we have for $W=W(f_1,\ldots,f_{n+1})$ and $\Pi=G_1\cdots G_\nu$

$$(7')\qquad (*)\leqq K\frac{|\Pi(z)|u(z)^{n+1-\nu}}{|H_{i_1}(z)\cdots H_{i_{n+1}}(z)|}$$

$$=K\frac{|\Pi(z)|u(z)^{n+1-\nu}}{|W(z)|}\cdot\frac{|W(z)|}{|H_{i_1}(z)\cdots H_{i_{n+1}}(z)|}$$

$$=K\frac{|\Pi(z)|u(z)^{n+1-\nu}}{|W(z)|}\cdot\frac{|W(H_{i_1},\ldots,H_{i_{n+1}})(z)|}{|H_{i_1}(z)\cdots H_{i_{n+1}}(z)|}$$

since $H_{i_1},\ldots,H_{i_{n+1}}$ are linearly independent and

$$W(H_{i_1},\ldots,H_{i_{n+1}})=cW(f_1,\ldots,f_{n+1})\qquad (c\neq 0,\ \text{constant}).$$

From (6), (7) and (7') we obtain the inequality

$$\sum_{j=1}^{q}\log\frac{\|\boldsymbol{a}_j\|\|f(z)\|}{|(\boldsymbol{a}_j,f(z))|}\leqq\log^+\frac{|\Pi(z)|u(z)^{n+1-\nu}}{|W(z)|}$$

$$+\sum_{(j_1,\ldots,j_{n+1})}\log^+\frac{|W(H_{i_1},\ldots,H_{i_{n+1}})(z)|}{|H_{i_1}(z)\cdots H_{i_{n+1}}(z)|}+\log^+K,$$

where $\sum_{(j_1,\ldots,j_{n+1})}$ is the summation taken over all combinations $(j_1,\ldots,j_{n+1})$ chosen from $\{1,\ldots,q\}$ which appear in the above argument when we vary z in $0<|z|<\infty$, and integrating this inequality from 0 to 2π with respect to θ, where $z=re^{i\theta}$, we obtain the inequality

$$\sum_{j=1}^{q}m(r,\boldsymbol{a}_j,f)\leqq m(r,\boldsymbol{e}_{n+1},f_\nu^*)+S(r,f)=(**)$$

since, by applying (1) to f_ν^* and to the following equality

$$\log^+\frac{|\Pi(z)|u(z)^{n+1-\nu}}{|W(z)|}=\log\max\left\{\frac{|\Pi(z)|u(z)^{n+1-\nu}}{|d_\nu(z)|},\ \frac{|W(z)|}{|d_\nu(z)|}\right\}-\log\frac{|W(z)|}{|d_\nu(z)|}$$

$$=\log\frac{1}{|d_\nu(z)|}\max\{|\Pi(z)||f_1(z)|^{n+1-\nu},\ldots,|\Pi(z)||f_n(z)|^{n+1-\nu},|W(z)|\}$$

$$-\log\frac{|W(z)|}{|d_\nu(z)|},$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|\Pi(re^{i\theta})|u(re^{i\theta})^{n+1-\nu}}{|W(re^{i\theta})|}d\theta=T(r,f_\nu^*)-N(r,1/(W/d_\nu))+O(1)$$

$$=m(r,\boldsymbol{e}_{n+1},f_\nu^*)+O(1)$$

by using $N(r, \boldsymbol{e}_{n+1}, f_\nu^*)=N(r, 1/(W/d_\nu))$ and we have for each $(j_1, \dots, j_{n+1})$

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|W(H_{\iota_1}, \dots, H_{\iota_{n+1}})(re^{i\theta})|}{|H_{\iota_1}(re^{i\theta})\cdots H_{\iota_{n+1}}(re^{i\theta})|}d\theta=S(r, f)$$

as in the proof of Lemma 1 (see [1], p. 12-p. 15), and by Lemma 2

$$(**)\leqq T(r, f)+(n-\nu)t(r, f)+\sum_{j=1}^{\nu}N(r, 1/G_j)-N(r, 1/W)+S(r, f).$$

THEOREM 2. *Let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q$ be any vectors of X such that the number of elements of the set $X(0)\cap\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ is equal to μ $(0\leqq\mu\leqq n)$. Then, we have*

$$\sum_{j=1}^{q}m(r, \boldsymbol{a}_j, f)\leqq(\mu+1)T(r, f)+(n-\mu)t(r, f)-N(r, 1/W)+S(r, f),$$

where $W=W(f_1, \dots, f_{n+1})$.

Proof. We easily obtain this theorem from Theorem C and Lemma 1 for $\mu=0$. When $1\leqq\mu\leqq n$, put

$$X(0)\cap\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}=\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_\mu\}$$

and

$$(\boldsymbol{a}_j, f)=G_j \qquad (j=1, \dots, \mu).$$

Then, from Theorem 1 we have

$$\sum_{j=\mu+1}^{q}m(r, \boldsymbol{a}_j, f)\leqq T(r, f)+(n-\mu)t(r, f)+\sum_{j=1}^{\mu}N(r, 1/G_j)-N(r, 1/W)+S(r, f).$$

Adding $\sum_{j=1}^{\mu}m(r, \boldsymbol{a}_j, f)$ to both sides of this inequality we have this theorem as

$$m(r, \boldsymbol{a}_j, f)+N(r, 1/G_j)=T(r, f)+O(1) \qquad (j=1, \dots, \mu).$$

Remark 1. $(\mu+1)T(r, f)+(n-\mu)t(r, f)\leqq(n+1)T(r, f)+O(1)$

since $t(r, f)\leqq T(r, f)+O(1)$, and so Theorem 2 is an improvement of Theorem A.

COROLLARY 1 (Defect relation). *Under the same circumstances as in Theorem* 2,

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j, f)\leqq\mu+1+(n-\mu)\Omega.$$

In fact, from Theorem 2 we obtain the inequality

$$(q-\mu-1)T(r, f)\leqq\sum_{j=1}^{q}N(r, \boldsymbol{a}_j, f)+(n-\mu)t(r, f)-N(r, 1/W)+S(r, f)$$

$$\leqq\sum_{j=1}^{q}N_n(r, \boldsymbol{a}_j, f)+(n-\mu)t(r, f)+S(r, f)$$

by (2) for the first inequality and by the method used in [1], p. 14 for the second inequality, which reduces to our corollary as usual.

Remark 2. (i) $\mu+1+(n-\mu)\Omega\leqq n+1$ and the equality holds if and only if $\mu=n$ or $\Omega=1$.

(ii) If $\rho(f)$ is finite, we can change Ω to ω in Corollary 1.

The number "$\mu+1+(n-\mu)\Omega$" increases with μ $(0\leqq\mu\leqq n)$ when $\Omega<1$. If μ increases to n when q tends to ∞, the bound "$\mu+1+(n-\mu)\Omega$" of this corollary increases to $n+1$ for any $\Omega<1$. There exist, however, examples of X for which μ does not increase to n even when q tends to ∞ and examples of holomorphic curves with $\Omega<1$ and, by using the following notion introduced in [10], we obtain a refinement of the defect relation as follows.

DEFINITION B ([10], Definition 1). We say that

(i) X is maximal (in the sense of general position) if and only if for any Y in general position such that $X\subset Y\subset \boldsymbol{C}^{n+1}$, $X=Y$.

(ii) X is ν-maximal if X is maximal and $\#X(0)=\nu$.

PROPOSITION 4. *For any* ν $(1\leqq\nu\leqq n)$, *there is a* ν-*maximal subset of* $\boldsymbol{C}^{n+1}$ *in the sense of general position* ([10], Theorem 1).

COROLLARY 2 (Defect relation). *Let* X *be a* ν-*maximal subset of* $\boldsymbol{C}^{n+1}$ *in the sense of general position. Then, we have*

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f)\leqq\nu+1+(n-\nu)\Omega .$$

In fact, when $\#\{\boldsymbol{a}\in X: \delta_n(\boldsymbol{a}, f)>0\}<\infty$, there is nothing to prove by Corollary 1. When $\#\{\boldsymbol{a}\in X: \delta_n(\boldsymbol{a}, f)>0\}=\infty$, it is countable by Corollary 1. Let

$$\{\boldsymbol{a}\in X: \delta_n(\boldsymbol{a}, f)>0\}=\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots\},$$

and without loss of generality we put

$$X(0)\cap\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots\}=\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_p\} \qquad (0\leqq p\leqq\nu).$$

Then, by Corollary 1, for any $q>0$

$$\sum_{j=1}^{q}\delta_n(\boldsymbol{a}_j, f)\leqq p+1+(n-p)\Omega\leqq\nu+1+(n-\nu)\Omega$$

and letting q tend to ∞ we have

$$\sum_{\boldsymbol{a}\in X}\delta_n(\boldsymbol{a}, f)=\sum_{j=1}^{\infty}\delta_n(\boldsymbol{a}_j, f)\leqq\nu+1+(n-\nu)\Omega$$

since ν is independent of q.

We here give some examples of f for which $\Omega<1$. Let a_j $(j=1, \dots, n)$ be real numbers satisfying $0<a_1<a_2<\cdots<a_{n-1}<a_n$.

Example 1. A holomorphic curve for which $\Omega<1$.
We consider the following holomorphic curve

$$f=[1, e^{a_1 z}, e^{a_2 z}, \dots, e^{a_n z}].$$

Then, for $z=re^{i\theta}(r>0)$,

$$U(z)=\begin{cases} 1 & \left(\frac{\pi}{2}\leqq\theta\leqq\frac{3}{2}\pi\right) \\ \exp(ra_n\cos\theta) & \left(0\leqq\theta<\frac{\pi}{2},\ \frac{3}{2}\pi<\theta\leqq 2\pi\right) \end{cases}$$

and by (1) we have

$$T(r, f)=\frac{a_n}{\pi}r+O(1).$$

On the other hand

$$u(z)=\begin{cases} 1 & \left(\frac{\pi}{2}\leqq\theta\leqq\frac{3}{2}\pi\right) \\ \exp(ra_{n-1}\cos\theta) & \left(0\leqq\theta<\frac{\pi}{2},\ \frac{3}{2}\pi<\theta\leqq 2\pi\right) \end{cases}$$

and we have

$$t(r, f)=\frac{a_{n-1}}{\pi}r+O(1).$$

We have $\omega=\Omega=a_{n-1}/a_n$, which is smaller than 1.

Example 2. A holomorphic curve for which $\Omega=0$.
We consider the following holomorphic curve

$$f=[1, e^{a_1 z}, \dots, e^{a_{n-1} z}, e^{z^2}].$$

Then, by a simple calculation we have

$$T(r, f)\geqq\frac{r^2}{4\pi}+O(1)$$

and $t(r, f)$ is the same as that given in Example 1, so that $\Omega=0$.

4. Extension of the second fundamental theorem

Let $f=[f_1, \dots, f_{n+1}]$, $\|f(z)\|$, $T(r, f)$ and $U(z)$ be as in Section 1. We set

$$\Gamma=\{a : \text{meromorphic in } |z|<\infty,\ T(r, a)=S_0(r, f)\},$$

where $S_0(r, f)$ is any quantity satisfying

$$S_0(r, f)=o(T(r, f)) \qquad (r\to\infty).$$

Note that Γ is a field. Further we set

$$\Gamma^+=\{\beta(|a_1|+|a_2|+\cdots+|a_m|)^k : a_j\in\Gamma;\ \beta>0, \text{ constant};\ m,\ k\in\boldsymbol{N}\},$$

where N is the set of positive integers. Observe that

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+(|a_1(re^{i\theta})|+\cdots+|a_m(re^{i\theta})|)d\theta\leqq\sum_{j=1}^m m(r,\ a_j)+O(1)$$

$$\leqq\sum_{j=1}^m T(r,\ a_j)+O(1)=S_0(r,\ f).$$

From now on, we use $K(z)$ as a representative for any functions of Γ^+ for brevity, and so $K(z)$ may be different from each other when it appears in different places. Note that

$$\int_0^{2\pi}\log^+K(re^{i\theta})d\theta=S_0(r,\ f).$$

From now on throughout the section we suppose that f is linearly non-degenerate over Γ. Let

$$\boldsymbol{S}_0(f)=\left\{\boldsymbol{A}=[a_1,\ \dots,\ a_{n+1}] : \begin{array}{l}\text{holomorphic curve from } \boldsymbol{C} \text{ into } P^n(\boldsymbol{C}),\\ T(r,\ \boldsymbol{A})=S_0(r,\ f)\end{array}\right\}$$

and let X be a subset of $\boldsymbol{S}_0(f)$. We suppose that $\#X\geqq n+1$ and X is in general position; that is to say, for any $n+1$ elements

$$\boldsymbol{A}_j=[a_{1j},\ \dots,\ a_{n+1j}]\qquad(j=1,\ \dots,\ n+1)$$

of X, $\det(a_{ij})$ is not identically equal to zero (see [10], §4). This is independent of the choice of reduced representations of $\boldsymbol{A}_j\in X$. It is clear that

$$\boldsymbol{S}_0(f)\supset P^n(\boldsymbol{C}).$$

Put

$$X(0)=\{\boldsymbol{A}=[a_1,\ \dots,\ a_{n+1}]\in X:\ a_{n+1}=0\}.$$

DEFINITION C ([10], Definition 2). We say that X is ν-maximal in the sense of general position if and only if it satisfies the following conditions (i) and (ii):

(i) X is maximal in the sense of general position; that is to say, for any subset Y of $\boldsymbol{S}_0(f)$ in general position such that $X\subset Y\subset\boldsymbol{S}_0(f)$, $X=Y$;

(ii) $\#X(0)=\nu$.

Remark 3. $0\leqq\nu\leqq n$.

PROPOSITION 5. *For any ν $(1\leqq\nu\leqq n)$, there is a ν-maximal subset of $\boldsymbol{S}_0(f)$ in the sense of general position* ([10], Theorem 2).

We use the following notation in this section. For any $\boldsymbol{A}=[a_1,\ \dots,\ a_{n+1}]$

of $S_0(f)$, we set

$$(A,\ f)=a_1 f_1+\cdots+a_{n+1}f_{n+1}$$

and

$$(A,\ f)(z)=a_1(z)f_1(z)+\cdots+a_{n+1}(z)f_{n+1}(z)$$

Then we have the following (see [8], Proposition 2):

LEMMA 3. (a) $a_i/a_j\in\Gamma$ *if* $a_j\neq 0$.
(b) $(A,\ f)\neq 0$.

We put for A of X

$$m(r,\ A,\ f)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\|A(re^{i\theta})\|\|f(re^{i\theta})\|}{|(A,\ f)(re^{i\theta})|}\,d\theta,$$

$$N(r,\ A,\ f)=N(r,\ 1/(A,\ f))$$

and

$$\delta(A,\ F)=\liminf_{r\to\infty}\frac{m(r,\ A,\ f)}{T(r,\ f)}.$$

PROPOSITION 6. (a) $m(r,\ A,\ f)+N(r,\ A,\ f)=T(r,\ f)+S_0(r,\ f)$.

(b) $0\leqq\delta(A,\ f)=1-\limsup_{r\to\infty}\dfrac{N(r,\ A,\ f)}{T(r,\ f)}\leqq 1$.

These are trivial by definition.

For any $A=[a_1,\ \dots,\ a_{n+1}]$ and $B=[b_1,\ \dots,\ b_{n+1}]$ of $S_0(f)$ such that $a_j\neq 0$, $b_k\neq 0$, put $(A,\ f)=F$ and $(B,\ f)=G$. Then, we have the following lemma.

LEMMA 4. $T\left(r,\ \dfrac{F/a_j}{G/b_k}\right)\leqq 2nT(r,\ f)+S_0(r,\ f)$. ([8], Lemma 6)

For $A=[a_1,\ \dots,\ a_{n+1}]$ of X, let a_{j_0} be the first element not identically equal to zero. Then, we put

$$\tilde{A}=(a_1/a_{j_0},\ \dots,\ a_{n+1}/a_{j_0})=(g_1,\ \dots,\ g_{n+1}),$$

$$\tilde{X}=\{\tilde{A}:\ A\in X\},\quad \tilde{X}(0)=\{\tilde{A}:\ A\in X(0)\},\quad \|\tilde{A}(z)\|=\|A(z)\|/|a_{j_0}(z)|$$

and for $(A,\ f)=F$

$$\frac{(A,\ f)}{a_{j_0}}=(\tilde{A},\ f)=\tilde{F}=\sum_{j=1}^{n+1}g_j f_j.$$

Then, it is clear that $\tilde{X}$ is in general position; that is to say, for any $n+1$ elements

$$\tilde{A}_i=(g_{i1},\ \dots,\ g_{in+1})\qquad (i=1,\ \dots,\ n+1)$$

of $\tilde{X}$,

$$\det(g_{ij})\neq 0$$

and $g_j \in \Gamma$ by Lemma 3, (a).

Let f and X be those given above in this section. Then, we have the following extension of the second fundamental theorem.

THEOREM 3. *Let* $\boldsymbol{A}_1, \dots, \boldsymbol{A}_q$ *be any elements in* $X - X(0)$ $(1 \leqq q < \infty)$ *and let* $\boldsymbol{B}_1, \dots, \boldsymbol{B}_\mu$ *be in* $X(0)$ $(0 \leqq \mu \leqq n)$. *Then, for any positive number* ε,

$$\sum_{j=1}^{q} m(r, \boldsymbol{A}_j, f) \leqq (1+\varepsilon) T(r, f) + (n-\mu) t(r, f) + \sum_{l=1}^{\mu} N(r, \boldsymbol{B}_l, f) + S(r, f).$$

Proof. We suppose without loss of generality that $q \geqq n+1$. Put for $j=1, \dots, q$

$$\boldsymbol{A}_j = [a_{j1}, \dots, a_{jn+1}], \qquad \tilde{\boldsymbol{A}}_j = (g_{j1}, \dots, g_{jn+1}), \qquad (\boldsymbol{A}_j, f) = F_j$$

and for $l=1, \dots, \mu$

$$\boldsymbol{B}_l = [a_{q+l1}, \dots, a_{q+ln+1}], \qquad \tilde{\boldsymbol{B}}_l = (g_{q+l1}, \dots, g_{q+ln+1}), \qquad (\boldsymbol{B}_l, f) = G_l.$$

We may suppose without loss of generality that $\tilde{\boldsymbol{B}}_1, \dots, \tilde{\boldsymbol{B}}_\mu, \boldsymbol{e}_{\mu+1}, \dots, \boldsymbol{e}_n, \boldsymbol{e}_{n+1}$ are linearly independent over Γ since $\tilde{\boldsymbol{B}}_1, \dots, \tilde{\boldsymbol{B}}_\mu$ are linearly independent over Γ and belong to $\tilde{X}(0)$.

For any integer p, let $V(p)$ be the vector space generated by

$$\left\{ \prod_{k=1}^{n+1} \prod_{j=1}^{q+\mu} g_{jk}^{p(j,k)} : \sum_{k=1}^{n+1} \sum_{j=1}^{q+\mu} p(j, k) \leqq p,\ p(j, k) \geqq 0 \text{ and integer} \right\}$$

over $\boldsymbol{C}$ and

$$d(p) = \dim V(p).$$

Then, $V(p)$ is a subspace of $V(p+1)$ and

$$(8) \qquad \liminf_{p \to \infty} d(p+1)/d(p) = 1$$

by the reduction to absurdity since $d(p) \leqq \binom{(n+1)(q+\mu)+p}{p}$ (see [5], [6]).

Let

$$b_1, \dots, b_{d(p)}, b_{d(p)+1}, \dots, b_{d(p+1)}$$

be a basis of $V(p+1)$ such that

$$b_1, \dots, b_{d(p)}$$

form a basis of $V(p)$. Then, it is clear that the functions

$$\{b_t f_k : t=1, \dots, d(p+1);\ k=1, \dots, n+1\}$$

are linearly independent over $\boldsymbol{C}$. We put for convenience

$$W = W(b_1 f_1, b_2 f_1, \dots, b_{d(p+1)} f_{n+1}).$$

Note that $N(r, W)=S_0(r, f)$.

Let z be a point of $\boldsymbol{C}-\{0\}$ where none of $\{\tilde{F}_j\}_{j=1}^q$ has poles. We rearrange $\{\tilde{F}_j(z)\}_{j=1}^q$ as follows:

$$|\tilde{F}_{j_1}(z)|\leqq\tilde{F}_{j_2}(z)|\leqq\cdots\leqq|\tilde{F}_{j_n}(z)|\leqq\cdots\leqq|\tilde{F}_{j_q}(z)|,$$

where $1\leqq j_1, \ldots, j_q\leqq q$.

We have for $k\geqq n+1$

$$\|f(z)\|\leqq K(z)|\tilde{F}_{j_k}(z)| \tag{9}$$

and for $k=1, \ldots, q$

$$|\tilde{F}_{j_k}(z)|\leqq K(z)\|f(z)\| \tag{10}$$

We then have the following from (9):

$$\begin{aligned}(11)\qquad &\left(\prod_{j=1}^q\frac{\|\boldsymbol{A}_j(z)\|\|f(z)\|}{|(\boldsymbol{A}_j, f)(z)|}\right)^{d(p)}=\left(\prod_{j=1}^q\frac{\|\tilde{\boldsymbol{A}}_j(z)\|\|f(z)\|}{|\tilde{F}_j(z)|}\right)^{d(p)}\\ &=\left(\prod_{j=1}^q\|\tilde{\boldsymbol{A}}_j(z)\|\right)^{d(p)}\left(\prod_{k=1}^n\frac{\|f(z)\|}{|\tilde{F}_{j_k}(z)|}\right)^{d(p)}\left(\prod_{k=n+1}^q\frac{\|f(z)\|}{|\tilde{F}_{j_k}(z)|}\right)^{d(p)}\\ &\leqq K(z)\left(\prod_{k=1}^n\frac{\|f(z)\|}{|\tilde{F}_{j_k}(z)|}\right)^{d(p)}.\end{aligned}$$

We note that by Lemma 3, (a)

$$|f_{n+1}(z)|\leqq K(z)\{|\tilde{F}_{j_k}(z)|+u(z)\}\qquad (k=1, \ldots, q). \tag{12}$$

since $a_{j n+1}\neq 0$ for any $\boldsymbol{A}_j\in X-X(0)$.

(I) The case when $u(z)\leqq|\tilde{F}_{j_1}(z)|$.

In this case we have from (12)

$$\|f(z)\|\leqq K(z)|\tilde{F}_{j_k}(z)|\qquad (k=1, \ldots, n)$$

and we have

$$\left(\prod_{k=1}^n\frac{\|f(z)\|}{|\tilde{F}_{j_k}(z)|}\right)^{d(p)}\leqq K(z). \tag{13}$$

(II) The case when $|\tilde{F}_{j_1}(z)|<u(z)$.

In this case we have from (12) for $k=1$

$$\|f(z)\|\leqq K(z)u(z)$$

and we have for $\Pi_\mu=|G_1\cdots G_\mu|$ and $\tilde{\Pi}_\mu=|\tilde{G}_1\cdots\tilde{G}_\mu|(\mu\geqq 1)$, $\Pi_0=\tilde{\Pi}_0=1$

$$(14)\qquad \left(\prod_{k=1}^{n}\frac{\|f(z)\|}{|\tilde{F}_{j_k}(z)|}\right)^{d(p)}\leqq K(z)\frac{u(z)^{(n+1)d(p)}}{(\prod_{k=1}^{n+1}|\tilde{F}_{j_k}(z)|)^{d(p)}}$$

$$=K(z)\frac{(\tilde{\Pi}_\mu(z)u(z)^{n+1})^{d(p)}}{(\tilde{\Pi}_\mu(z)\prod_{k=1}^{n+1}|\tilde{F}_{j_k}(z)|)^{d(p)}}=(*)$$

by (10). Here, we put

$$\tilde{G}_i=\tilde{H}_i\ (i=1,\ \dots,\ \mu),\qquad \tilde{F}_{j_k}=\tilde{H}_{\mu+k}\ (k=1,\ \dots,\ n+1)$$

and let

$$|\tilde{H}_{i_1}(z)|\leqq|\tilde{H}_{i_2}(z)|\leqq\cdots\leqq|\tilde{H}_{i_{n+1+\mu}}(z)|.$$

Then, for $k=2,\ \dots,\ \mu+1$

$$u(z)\leqq K(z)|\tilde{H}_{i_{n+k}}(z)|$$

and we have

$$(14')\qquad (*)\leqq K(z)\frac{\{\tilde{\Pi}_\mu(z)u(z)^{n+1-\mu}\}^{d(p)}\{\prod_{k=n+2}^{n+\mu+1}K(z)|\tilde{H}_{i_k}(z)|\}^{d(p)}}{\prod_{k=1}^{n+1+\mu}|\tilde{H}_{i_k}(z)|^{d(p)}}$$

$$\leqq K(z)\frac{\{\tilde{\Pi}_\mu(z)u(z)^{n+1-\mu}\}^{d(p)}}{\prod_{k=1}^{n+1}|\tilde{H}_{i_k}(z)|^{d(p)}}.$$

Now $\tilde{H}_{i_1},\ \dots,\ \tilde{H}_{i_{n+1}}$ are linearly independent over Γ and it is easy to see that

$$\{b_1\tilde{H}_{i_1},\ b_2\tilde{H}_{i_1},\ \dots,\ b_{d(p)}\tilde{H}_{i_{n+1}}\}$$

are linearly independent over $\boldsymbol{C}$. Since $\tilde{H}_j=(\tilde{\boldsymbol{A}}_j,\ f)$ or $\tilde{H}_j=(\tilde{\boldsymbol{B}}_j,\ f)$, these functions can be represented as linear combinations of

$$\{b_t f_k:\ 1\leqq t\leqq d(p+1),\ 1\leqq k\leqq n+1\}$$

with constant coefficients:

$$(b_1\tilde{H}_{i_1},\ b_2\tilde{H}_{i_1},\ \dots,\ b_{d(p)}\tilde{H}_{i_{n+1}})=(b_1f_1,\ b_2f_1,\ \dots,\ b_{d(p+1)}f_{n+1})D_1$$

where D_1 is an $(n+1)d(p+1)\times(n+1)d(p)$ matrix whose elements are constants. The rank of D_1 is equal to $(n+1)d(p)$. Let D_2 be an $(n+1)d(p+1)\times(n+1)\{d(p+1)-d(p)\}$ matrix consisting of constant elements such that the $(n+1)d(p+1)\times(n+1)d(p+1)$ matrix

$$D=[D_1D_2]$$

is regular. Put

$$(K_1,\ \dots,\ K_L)=(b_1f_1,\ b_2f_1,\ \dots,\ b_{d(p+1)}f_{n+1})D_2,$$

where $L=(n+1)\{d(p+1)-d(p)\}$, then

$$(b_1\tilde{H}_{i_1},\ \dots,\ b_{d(p)}\tilde{H}_{i_{n+1}},\ K_1,\ \dots,\ K_L)=(b_1f_1,\ \dots,\ b_{d(p+1)}f_{n+1})D,$$

from which we obtain

$$W(b_1\tilde{H}_{\iota_1}, \ldots, K_1, \ldots, K_L)=(\det D)W, \tag{15}$$

where $W=W(b_1 f_1, \ldots, b_{d(p+1)} f_{n+1})$. We put

$$W(b_1\tilde{H}_{\iota_1}, \ldots, K_1, \ldots, K_L)=W(j_1, \ldots, j_{n+1})$$

as $\tilde{H}_{\iota_1}, \ldots, \tilde{H}_{\iota_{n+1}}$ are determined after $(j_1, \ldots, j_{n+1})$.

We then have from (15)

$$\begin{aligned}\frac{1}{\{\prod_{k=1}^{n+1}|\tilde{H}_{\iota_k}(z)|\}^{d(p)}} &= \frac{|W(j_1, \ldots, j_{n+1})(z)|}{|W(z)|\,|\det D|}\cdot\frac{1}{\{\prod_{k=1}^{n+1}|\tilde{H}_{\iota_k}(z)|\}^{d(p)}} \\ &= \frac{1}{|\det D|}\cdot\frac{1}{|W(z)|}\cdot\frac{|W(j_1, \ldots, j_{n+1})(z)|}{\{\prod_{k=1}^{n+1}|\tilde{H}_{\iota_k}(z)|\}^{d(p)}} \\ &\leqq K(z)\frac{\{u(z)\}^L}{|W(z)|}\cdot\frac{|W(j_1, \ldots, j_{n+1})(z)|}{|b_1\tilde{H}_{\iota_1}(z)\cdot b_2\tilde{H}_{\iota_1}(z)\cdots K_L(z)|}\end{aligned} \tag{16}$$

since $|K_j(z)|\leqq K(z)\|f(z)\|(j=1, \ldots, L)$ and $\|f(z)\|\leqq K(z)u(z)$ in this case.

Further, by using the following inequalities for $j=1, \ldots, L$

$$T(r, K_j/b_1\tilde{H}_{\iota_1})\leqq 2nT(r, f)+S_0(r, f),$$

which we can prove as in Lemma 4 since $b_t\in\Gamma(1\leqq t\leqq d(p+1))$, and by Lemma 4, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{|W(j_1, \ldots, j_{n+1})(re^{i\theta})|}{|b_1\tilde{H}_{\iota_1}(re^{i\theta})\cdots K_L(re^{i\theta})|}d\theta=S(r, f) \tag{17}$$

as usual (see [1], p. 12–p. 15).

From (11), (13), (14), (14′) and (16), we obtain

$$\begin{aligned}d(p)\sum_{j=1}^{q}\log\frac{\|A_j(z)\|\|f(z)\|}{|(A_j, f)(z)|} &\leqq \log^+\frac{\tilde{\Pi}_\mu(z)^{d(p)}u(z)^{(n+1)d(p+1)-\mu d(p)}}{|W(z)|} \\ &\quad+\log^+K(z)+\sum_{(j_1,\ldots,j_{n+1})}\log^+\frac{|W(j_1, \ldots, j_{n+1})(z)|}{|b_1\tilde{H}_{\iota_1}(z)\cdots K_L(z)|},\end{aligned} \tag{18}$$

where $\sum_{(j_1,\ldots,j_{n+1})}$ is the summation taken over all combinations $(j_1, \ldots, j_{n+1})$ chosen from $\{1, \ldots, q\}$ which appear in the above argument when we vary z in $0<|z|<\infty$.

Integrating both sides of (18) with respect to θ from 0 to $2\pi (z=re^{i\theta})$, we obtain the inequality

$$\begin{aligned}d(p)\sum_{j=1}^{q}m(r, A_j, f) &\leqq \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{\tilde{\Pi}_\mu(re^{i\theta})^{d(p)}u(re^{i\theta})^{\alpha(p)}}{|W(re^{i\theta})|}d\theta \\ &\quad+S_0(r, f)+S(r, f)\end{aligned} \tag{19}$$

by (17), where $\alpha(p)=(n+1)d(p+1)-\mu d(p)$. Here,

$$(20)\qquad \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{\tilde{\Pi}_\mu(re^{i\theta})^{d(p)}u(re^{i\theta})^{\alpha(p)}}{|W(re^{i\theta})|}d\theta$$

$$\leqq\frac{1}{2\pi}\int_0^{2\pi}\log\{\tilde{\Pi}_\mu^{2d(p)}(|f_1|^{2\alpha(p)}+\cdots+|f_n|^{2\alpha(p)})+|W|^2\}^{1/2}(re^{i\theta})d\theta$$

$$-\frac{1}{2\pi}\int_0^{2\pi}\log|W(re^{i\theta})|\,d\theta+O(1)$$

and as in the proof of Lemma 1 (when $\mu=0$) or Lemma 2 (when $\mu>0$)

$$(21)\qquad \frac{1}{2\pi}\int_0^{2\pi}\log(\tilde{\Pi}_\mu^{2d(p)}(|f_1|^{2\alpha(p)}+\cdots+|f_n|^{2\alpha(p)})+|W|^2\}^{1/2}(re^{i\theta})d\theta$$

$$\leqq\{(n+1)d(p+1)-nd(p)\}\,T(r,f)+(n-\mu)d(p)t(r,f)$$

$$+d(p)\sum_{l=1}^{\mu}N(r,\boldsymbol{B}_l,f)+S_0(r,f)+S(r,f).$$

For any positive number ε, let p be so large that

$$d(p+1)/d(p)<1+\varepsilon/(n+1)$$

by (8). Then, from (19), (20) and (21) we obtain

$$(22)\qquad \sum_{j=1}^{q}m(r,\boldsymbol{A}_j,f)\leqq(1+\varepsilon)T(r,f)+(n-\mu)t(r,f)+\sum_{l=1}^{\mu}N(r,\boldsymbol{B}_l,f)+S(r,f)$$

since $N(r,W)=S_0(r,f)$.

As direct consequences of this theorem we have the followings as in Theorem 2, Corollaries 1 and 2.

COROLLARY 3 (Defect relation). *Under the same assumption as in Theorem 3, we have*

$$\sum_{j=1}^{q}\delta(\boldsymbol{A}_j,f)+\sum_{l=1}^{\mu}\delta(\boldsymbol{B}_l,f)\leqq\mu+1+(n-\mu)\Omega.$$

COROLLARY 4. *Let X be a ν-maximal subset of $\boldsymbol{S}_0(f)$ and ε be any positive number. Then,*

(I) *For any $\boldsymbol{A}_1,\dots,\boldsymbol{A}_q$ in X*

$$\sum_{j=1}^{q}m(r,\boldsymbol{A}_j,f)\leqq(\nu+1+\varepsilon)T(r,f)+(n-\nu)t(r,f)+S(r,f).$$

(II) $\displaystyle\sum_{\boldsymbol{A}\in X}\delta(\boldsymbol{A},f)\leqq\nu+1+(n-\nu)\Omega.$

Remark 4. $\nu+1+(n-\nu)\Omega\leqq n+1$ and the equality holds if and only if $\Omega=1$ or $\nu=n$.

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