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MONOTONE DISCONTINUITY OF LATTICE OPERATIONS IN A QUASILINEAR HARMONIC SPACE

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Abstract

We claim, contrary to the linear case, that the lattice operations among harmonic functions are not necessarily monotone continuous in quasilinear harmonic spaces by showing the existence of a quasilinear harmonic space (X, H) in which there are harmonic functions u_n in $H(X)(n=1,2,\cdots,\infty)$ with the following properties: the least harmonic majorant $u_n \vee 0$ and the greatest harmonic minorant $u_n \wedge 0$ of u_n and 0 exist in $H(X)$ for every $n=1,2,\cdots,\infty$; the sequence $(u_n)_{1 \leq n<\infty}$ is increasing and convergent to u_∞ on X; the sequence $(u_n \wedge 0)_{1 \leq n<\infty}$ converges increasingly to a harmonic function strictly less than $u_\infty \wedge 0$ on X.

1. Introduction

In the theory of $\mathcal{A}$-harmonic functions (including p-harmonic functions) as developed by Heinonen, Kilpeläinen, and Martio in their monograph [2], the order structure and in particular the induced lattice structure of the space of $\mathcal{A}$-harmonic functions (see 5 below) supplement the lack of its linear structure. In this sense the availability of the monotone continuity of lattice operations would greatly enrich the $\mathcal{A}$-harmonic function theory. More specifically, denote by $u \vee v$ ($u \wedge v$, resp.) the least $\mathcal{A}$-harmonic majorant (the greatest $\mathcal{A}$-harmonic minorant, resp.) of two $\mathcal{A}$-harmonic functions u and v on a region Ω of the m-dimensional Euclidean space $\boldsymbol{R}^m$, if it exists. Consider $\mathcal{A}$-harmonic functions u_n $(n=1, 2, \cdots, \infty)$ such that both $u_n \vee 0$ and $u_n \wedge 0$ exist on Ω $(n=1, 2, \cdots, \infty)$. We wish to know whether the following statement is true or not.

2. STATEMENT. *If the sequence $(u_n)_{1 \leq n<\infty}$ is increasing and convergent to u_∞ on Ω, then $(u_n \wedge 0)_{1 \leq n<\infty}$ converges to $u_\infty \wedge 0$ on Ω:*

$$\lim_{n\to\infty} u_n \wedge 0 = u_\infty \wedge 0. \tag{3}$$

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The validity of $\lim_{n\to\infty} u_n \vee 0 = u_\infty \vee 0$ is trivially true and therefore, if the space of $\mathcal{A}$-harmonic functions happens to form a linear space, then the relations $u_n \vee 0 + u_n \wedge 0 = u_n$ $(n=1, 2, \cdots, \infty)$ instantly imply (3). Hence the difficulty of course comes from the nonlinearity of the space of $\mathcal{A}$-harmonic functions. The purpose of this paper is to claim that (3), even if it is true, cannot be proven in general by merely using the so-called quasilinear harmonic structures of $\mathcal{A}$-harmonic functions; more precisely, there is a quasilinear harmonic space (cf. [2, Chap. 16]; see 4 below) and a sequence $(u_n)_{1\leq n<\infty}$ and its limit u_∞ in $H(X)$ as described above such that (3) is invalid (see Theorem 21 below).

4. Quasilinear harmonic spaces

Consider a locally compact, locally connected, and connected Hausdorff space X. To exclude triviality we moreover assume that X is noncompact. Let H be a *sheaf* of continuous functions, i.e. H is a mapping of the family of open sets U of X to a set of subfamilies $H(U)$ of real valued functions on U satisfying the following sheaf axioms:

(S.1) if U is open in X, then $H(U) \subset C(U)$, the space of continuous functions on U;

(S.2) if $U \subset V$ are open in X and $u \in H(V)$, then $u|U \in H(U)$;

(S.3) if U_ι are open in X and $U = \cup_\iota U_\iota$ and u is a function on U, then $u|U_\iota \in H(U_\iota)$ for all ι imply $u \in H(U)$.

We say that an open set U of X is *regular* with respect to a sheaf H of continuous functions if the following conditions are fulfilled:

(R.1) U is relatively compact in X;

(R.2) for each $f \in C(\partial U)$ there is a unique $H_f^U \in H(U) \cap C(\bar{U})$ such that $H_f^U | \partial U = f$;

(R.3) for each pair of f and g in $C(\partial U)$ the condition $f \leqq g$ on ∂U implies that $H_f^U \leqq H_g^U$ on U.

After introducing these terminology we are able to state the following three axioms.

AXIOM A. For every compact set K of X and for every open set U of X with $K \subset U$ there is a regular open set Ω of X such that $K \subset \Omega \subset \bar{\Omega} \subset U$.

AXIOM B. If U is a subregion of X and $(u_j)_{1\leq j<\infty}$ is an increasing sequence in $H(U)$, then either $\lim_{j\to\infty} u_j \equiv +\infty$ on U or $\lim_{j\to\infty} u_j \in H(U)$.

AXIOM C. If $u \in H(U)$ and $\lambda \in \boldsymbol{R}$, the real number field, then $u+\lambda \in H(U)$ and $\lambda u \in H(U)$.

According to [2, p. 319] (see also Laine [3] and Lehtola [4]), the pair (X, H) of the space X as stated above and a sheaf H of continuous functions on X will be referred to as a *quasilinear harmonic space* if Axioms A, B, and

C are satisfied; compare this with various *linear* harmonic spaces (cf. e.g. [1], [5], etc.).

5. $\mathcal{A}$-harmonic functions

Consider an operator $\mathcal{A}$: $\boldsymbol{R}^m \times \boldsymbol{R}^m \rightarrow \boldsymbol{R}^m$ satisfying the following four conditions: (i) $\mathcal{A}(x, h)$ is continuous in h for a.e. fixed x and measurable in x for all fixed h; (ii) there are constants $0<\alpha \leqq \beta<\infty$ such that $\mathcal{A}(x, h) \cdot h \geqq \alpha|h|^p$ and $|\mathcal{A}(x, h)| \leqq \beta|h|^{p-1}$ for a.e. x and for all h; (iii) $(\mathcal{A}(x, h_1)-\mathcal{A}(x, h_2)) \cdot (h_1-h_2)>0$ for a.e. x whenever $h_1 \neq h_2$; (iv) $\mathcal{A}(x, \lambda h)=|\lambda|^{p-2} \lambda \mathcal{A}(x, h)$ for a.e. x and for all h whenever $\lambda \in \boldsymbol{R} \setminus \{0\}$. A continuous weak solution of the quasilinear elliptic equation

$$-\operatorname{div} \mathcal{A}(x, \nabla u(x))=0$$

on U is said to be *$\mathcal{A}$-harmonic* on U and we denote the totality of $\mathcal{A}$-harmonic functions on U by $H_{\mathcal{A}}(U)$. Then $H_{\mathcal{A}}$ is a sheaf of continuous functions on $\boldsymbol{R}^m$. It is one of fundamental results in $\mathcal{A}$-harmonic function theory that $(\boldsymbol{R}^m, H_{\mathcal{A}})$ forms a quasilinear harmonic space (see [2, Chap. 6]).

Many important consequences are expected in the theory of $\mathcal{A}$-harmonic functions if Statement 2 in the introduction is true so that we naturally would like to prove it. However we will show that Statement 2 cannot be proven by using only Axioms A, B, and C in $(\boldsymbol{R}^m, H_{\mathcal{A}})$. In other words, Statement 2 is *false* for some quasilinear harmonic space (X, H). We will construct such an (X, H) in the sequel. Our example is motivated by that of Martio (cf. Laine [3] and [2, p. 319]).

6. An example of a quasilinear harmonic space

Consider a one dimensional subset X of the plane $\boldsymbol{R}^2$ given by

$$X=\{(x, 0): x>0\} \cup\left(\bigcup_{k=1}^{\infty}\{(k, y): y \geqq 0\}\right). \tag{7}$$

With the induced plane topology X is a locally compact, locally connected, connected, and noncompact Hausdorff space. The open intervals $I=\{(x, 0): a<x<b\}$, $0 \leqq a<b<\infty$ with $(k, 0) \notin I$ $(k \in \boldsymbol{N}$, the set of positive integers), $J_k=\{(k, y): a<y<b\}$ $(k \in \boldsymbol{N})$, $0 \leqq a<b \leqq \infty$, and the neighborhoods $T_k=\{(x, 0): k-t<x<k+t\} \cup\{(k, y): 0 \leqq y<t\}$ of the point $(k, 0)$, $0<t<1$ $(k \in \boldsymbol{N})$, form a base for the topology of X. For each open set $U \subset X$ we define $H(U)$ as follows.

(H_1) If U is of the form I above, then $H(U)$ consists of all affine (i.e. linear) functions on U, i.e. $H(U)$ consists of functions u of the form $u(x, 0)=px+q$ for some p and q in $\boldsymbol{R}$.

(H_2) If U is of the form J_k above, then $H(U)$ consists of all affine functions on U, i.e. $H(U)$ consists of functions u of the form $u(k, y)=py+q$ for some p and q in $\boldsymbol{R}$.

(H_3) If U is of the form T_k above, then $H(U)$ consists of continuous functions u such that

$$u(k, 0)=\frac{1}{2}\left(\max_{\partial U} u+\min_{\partial U} u\right) \tag{8}$$

and u is affine on the intervals from $(k, 0)$ to the boundary points of U.

(H_4) If $U=\{(x, 0): a<x<b\} \cup(\cup_{k\in N, a<k<b}\{(k, y): 0\leqq y<c_k\})$ for $0\leqq a<b \leqq\infty$ and $0<c_k\leqq\infty$ $(k\in \boldsymbol{N}, a<k<b)$, then $H(U)$ consists of continuous functions u such that $u|T_k\in H(T_k)$ $(k\in \boldsymbol{N}, a<k<b)$ for $T_k=\{(x, 0): k-t_k<x<k+t_k\}\cup \{(k, y): 0\leqq y<t_k\}$ with $t_k=\min(k-a, b-k, c_k, 1)$ and u is affine on intervals which are connected components of $\{(x, 0): a<x<b\}\setminus\{(k, 0): k\in \boldsymbol{N}, a<k<b\}$ and on intervals $\{(k, y): 0<y<c_k\}$ $(k\in \boldsymbol{N}, a<k<b)$.

(H_5) If U is an arbitrary open set in X, then each connected component V of U is of the form in (H_1), in (H_2), or in (H_4) and we let $u\in H(U)$ if and only if $u\in H(V)$ for each such component V.

It is clear that H is a sheaf of continuous functions on X, i.e. H satisfies (S.1)-(S.3). Concerning the Axiom C, the stability of H under the addition of scalars is clear. To see the stability of H under the multiplication by scalars we only have to observe that, in case (H_3), $\max_{\partial U}\lambda f=\lambda\min_{\partial U} f$ and $\min_{\partial U}\lambda f=\lambda\max_{\partial U} f$ for scalars $\lambda<0$. Axiom B is easily seen to hold. Thus (X, H) is concluded to be a quasilinear harmonic space if Axiom A is assured. It is achieved by establishing the following assertion.

9. PROPOSITION. *Every open subset U of X in* (7) *which is relatively compact in X or more generally in $X\cup\{(0, 0)\}$ with the relative plane topology is regular with respect to H on X determined by* (H_1)-(H_5).

Proof. In order to show that U is regular it suffices to show that each connected component V of U is regular. Any connected component V of U is one of the following forms: the form in (H_1), in (H_2), or in (H_4). The regularity of V of the form in (H_1) or in (H_2) is entirely clear. Hence we only have to consider the case V is of the form in (H_4) with $b<\infty$. Therefore we only have to prove that, if U is of the form

$$U=\{(x, 0): a<x<b\}\cup\left(\bigcup_{k=1}^{n}\{(m+k, y): 0\leqq y<d^k\}\right),$$

where $0\leqq a<b<\infty$, $m\leqq a<m+1\leqq m+n<b\leqq m+n+1$ $(m\in \boldsymbol{N}\cup\{0\}, n\in \boldsymbol{N})$, and $0<d^k<\infty$ $(1\leqq k\leqq n)$, then U is regular. We set $p_0=(a, 0)$, $p_k=(m+k, d^k)$ $(1\leqq k\leqq n)$, $p_{n+1}=(b, 0)$ and also $q_k=(m+k, 0)$ $(1\leqq k\leqq n)$. Then $\partial U=\{p_j: 0\leqq j\leqq n+1\}$ considered in $X\cup\{(0, 0)\}$. We are to find a unique $H_f^U\in H(U)\cap C(\bar{U})$ such that $H_f^U|\partial U=f$ on ∂U for any $f\in C(\partial U)$ and to show that $f\leqq g$ on ∂U implies $H_f^U \leqq H_g^U$ on U. For the purpose we observe that $f\in C(\partial U)$ may be identified with a vector $a=(a^0, a^1, \cdots, a^n, a^{n+1})\in \boldsymbol{R}^{n+2}$ by $f(p_j)=a^j$ $(j=0, 1, \cdots, n, n+1)$. Then

H_f^U is completely deterimined by the vector $x=(x^1, \cdots, x^n)\in \boldsymbol{R}^n$ such that $H_f^U(q_k)=x^k$ $(1\leqq k\leqq n)$. It is easy to see by (H_1)-(H_5) and especially by (H_3) that the condition $H_f^U\in H(U)$ is valid if and only if x satisfies the following simultaneous quasilinear equations

$$\text{(10)}\quad \max\left(\frac{x^{i-1}-x^i}{c^{i-1}}, \frac{a^i-x^i}{d^i}, \frac{x^{i+1}-x^i}{c^i}\right)+\min\left(\frac{x^{i-1}-x^i}{c^{i-1}}, \frac{a^i-x^i}{d^i}, \frac{x^{i+1}-x^i}{c^i}\right) =0$$

$(i=1, \cdots, n)$, where $x^0=a^0$ and $x^{n+1}=a^{n+1}$. Here $c^0=(m+1)-a$, $c^k=1$ $(k=1, \cdots, n-1)$, and $c^n=b-(m+n)$. Thus we only have to prove that there is a unique solution $x=(x^1, \cdots, x^n)$ of (10) for any $a=(a^0, \cdots, a^{n+1})$ and that if a is replaced by $\bar{a}=(\bar{a}^0, \cdots, \bar{a}^{n+1})$ with $a^j\leqq\bar{a}^j$ $(j=0, \cdots, n+1)$, then the corresponding $\bar{x}=(\bar{x}^1, \cdots, \bar{x}^n)$ satisfies $x^i\leqq\bar{x}^i$ $(i=1, \cdots, n)$. This is assured by Proposition 26 in Appendix 24 at the end of this paper. □

11. An increasing sequence in $H(X)$

We take the quasilinear harmonic space (X, H) constructed above in 6. For two functions u and v in $H(X)$ we denote by $u\vee v$, if it exists, the function in $H(X)$ with the following two conditions: (i) $u\vee v\geqq u$ and v on X; (ii) if $h\in H(X)$ and $h\geqq u$ and v on X, then $h\geqq u\vee v$ on X. Similarly $u\wedge v$, if it exists, is characterized as the function in $H(X)$ by the following two conditions: (i) $u\wedge v\leqq u$ and v on X; (ii) if $h\in H(X)$ and $h\leqq u$ and v on X, then $h\leqq u\wedge v$ on X. In short, $u\vee v$ ($u\wedge v$, resp.) is the least (greatest, resp.) harmonic majorant (minorant, resp.) of u and v on X. Hence $u\wedge v=-((-u)\vee(-v))$ and $u\vee v=-((-u)\wedge(-v))$.

We define a sequence $(u_n)_{1\leq n<\infty}$ of functions $u_n\in H(X)$ and a function $u_\infty\in H(X)$ as follows. First on $\{(x, 0): x\geqq 0\}$ we set

$$\text{(12)}\qquad u_n(x, 0)=\begin{cases} x & (0<x\leqq n), \\ -x+2n & (n\leqq x<\infty); \end{cases}$$

and next on $\bigcup_{k=1}^\infty\{(k, y): 0\leqq y<\infty\}$ we set

$$\text{(13)}\qquad u_n(k. y)=\begin{cases} y+k & (0\leqq y<\infty;\ 1\leqq k\leqq n), \\ -k+2n & (0\leqq y<\infty;\ n+1\leqq k<\infty). \end{cases}$$

It is easy to see that $u_n\in H(X)$ and u_n has the boundary value zero at the ideal boundary point $(0, 0)$ of X $(n=1, 2, \cdots)$. The definition of u_∞ is simpler:

$$\text{(14)}\qquad u_\infty(x, 0)=x \qquad (0<x<\infty);$$

$$\text{(15)}\qquad u_\infty(k, y)=y+k \qquad (0\leqq y<\infty;\ k\in\boldsymbol{N}).$$

It is also easy to see that $u_\infty \in H(X)$, u_∞ has the boundary value zero at the ideal boundary point (0, 0) of X, and $u_\infty > 0$ on X. Obviously

(16) $$u_n \leqq u_{n+1}\ (n \in \boldsymbol{N}) \quad \text{and} \quad u_\infty = \lim_{n\to\infty} u_n \quad \text{on} \quad X.$$

Finally consider the function v on X defined by

(17) $$v(x, 0) = -x \qquad (0 < x < \infty);$$

(18) $$v(k, y) = -k \qquad (0 \leqq y < \infty;\ k \in \boldsymbol{N}).$$

Obviously $v \in H(X)$, v has boundary volue zero at the ideal boundary point (0, 0) of X, and $v < 0$ on X. We will show in 23 below that

(19) $$u_n \wedge 0 = v \qquad (n = 1, 2, \cdots)$$

on X so that we have

(20) $$\lim_{n\to\infty} u_n \wedge 0 = v < 0 = u_\infty \wedge 0$$

on X. Clearly $u_n \vee 0$ exists in $H(X)$ since $u_n \leqq u_\infty = u_\infty \vee 0\ (n \in \boldsymbol{N})$). Therefore we have established the following result.

21. THEOREM. *There is a quasilinear harmonic space* (X, H) *which carries an increasing sequence* $(u_n)_{1 \leqq n < \infty}$ *in* $H(X)$ *with a limit* u_∞ *in* $H(X)$ *such that* $u_n \vee 0$ *and* $u_n \wedge 0$ *exist in* $H(X)$ $(1 \leqq n \leqq \infty)$ *and*

$$\lim_{n\to\infty} u_n \wedge 0 < u_\infty \wedge 0$$

on X; *therefore the lattice operations on quasilinear harmonic spaces are not necessarily monotone continuous.*

22. *Remark.* If $(u_n)_{1 \leqq n < \infty} \uparrow u_\infty$ in $H(X)$ in a quasilinear harmonic space (X, H), then $u_n \vee 0 \uparrow u_\infty \vee 0$ is always true but $u_n \wedge 0 \uparrow u_\infty \wedge 0$ may not be true as we saw above. As the dual assertion, if $(u_n)_{1 \leqq n < \infty} \downarrow u_\infty$ in $H(X)$, then $u_n \wedge 0 \downarrow u_\infty \wedge 0$ is always true but $u_n \vee 0 \downarrow u_\infty \vee 0$ may not be true. What we say "monotone discontinuity" should be understood in this sense. On the other hand, as stated in the introduction, if (X, H) is *linear* in the sense that every $H(U)$ forms a linear space, then the "monotone continuity" of lattice operations $\wedge$ and $\vee$ is true.

23. Proof of (19)

Fix an $n \in \boldsymbol{N}$. Clearly $v < u_n$ and $v < 0$ on X so that $v \leqq u_n \wedge 0 =: h$ on X. Hence $v \leqq h \leqq u_n \cap 0 := \min(u_n, 0)$ on X. By the fact that v and u_n have boundary value zero at (0, 0), we see that h is continuous at (0, 0) and $h(0, 0) = 0$. Observe that h is linear (i.e. affine) in each interval $\{(x, 0) : k-1 \leqq x \leqq k\}\ (k \in \boldsymbol{N})$.

Similarly, h and $u_n \cap 0$ are linear on the half-line $\{(k, y): 0 \leqq y < \infty\}$ and in particular the slopes of v and $u_n \cap 0$ are zero. Hence the slope of h on $\{(k, y): 0 \leqq y < \infty\}$ must be zero. This implies that the slope of h on $\{(x, 0): k-1 \leqq x \leqq k\}$ and that on $\{(x, 0): k \leqq x \leqq k+1\}$ must coincide with each other. Therefore h is linear on the half-line $\{(x, 0): 0 \leqq x < \infty)$ and its slope must be -1 since otherwise h and the part of $u_n \cap 0$ which is a half-line with slope -1 must intersect, contradicting $h \leqq u_n \cap 0$ on $\{(x, 0): 0 \leqq x < \infty\}$. This proves that $h=v$. □

24. Appendix. Simultaneous quasilinear equation

In the proof of Proposition 9 it occured the need to solve the following type of simultaneous quasilinear equations which may also be in use in the theory of finite networks (cf. e.g. [6]):

$$\max\left(\frac{x^{i-1}-x^i}{c^{i-1}}, \frac{a^i-x^i}{d^i}, \frac{x^{i+1}-x^i}{c^i}\right)+\min\left(\frac{x^{i-1}-x^i}{c^{i-1}}, \frac{a^i-x^i}{d^i}, \frac{x^{i+1}-x^i}{c^i}\right) = 0 \tag{25}$$

($i=1, \cdots, n$), where $x^0=a^0$ and $x^{n+1}=a^{n+1}$. Here $n \in \boldsymbol{N}$; $c=(c^0, c^1, \cdots, c^n) \in \boldsymbol{R}^{n+1}$ and $d=(d^1, \cdots, d^n) \in \boldsymbol{R}^n$ are arbitrary vectors of strictly positive components but fixed once and for all for each $n \in \boldsymbol{N}$; $a=(a^0, a^1, \cdots, a^n, a^{n+1}) \in \boldsymbol{R}^{n+2}$ is a known vector but may vary; $x=(x^1, \cdots, x^n) \in \boldsymbol{R}^n$ is an unknown vector to be sought. We will look into the solvability of the equations (25) and study the dependence of the solution x on the given vector a.

For convenience we say that, for a given vector $a=(a^0, \cdots, a^{n+1}) \in \boldsymbol{R}^{n+2}$, a vector $x=(x^1, \cdots, x^n) \in \boldsymbol{R}^n$ is an *a-system* for $c=(c^0, \cdots, c^n) \in \boldsymbol{R}^{n+1}$ and $d=(d^1, \cdots, d^n) \in \boldsymbol{R}^n$ if x satisfies the equations (25). For two vectors $\zeta_k=(\zeta_k^1, \cdots, \zeta_k^m) \in \boldsymbol{R}^m$ ($k=1, 2$) we write $\zeta_1 \leqq \zeta_2$ ($\zeta_1 < \zeta_2$, resp.) to mean $\zeta_1^i \leqq \zeta_2^i$ ($\zeta_1^i < \zeta_2^i$, resp.) ($i=1, \cdots, m$). The purpose of this appendix is to maintain and prove the following result.

26. PROPOSITION. *For any vector $a \in \boldsymbol{R}^{n+2}$ there exists a unique a-system $x \in \boldsymbol{R}^n$ (: unique existence); if x_k is the a_k-system ($k=1, 2$), then $a_1 \leqq a_2$ implies $x_1 \leqq x_2$ (: monotoneity).*

Proof. For any vector $\zeta=(\zeta^1, \cdots, \zeta^m) \in \boldsymbol{R}^m$ and a scalar $\lambda \in \boldsymbol{R}$ we denote by $\zeta+\lambda$ the vector $(\zeta^1+\lambda, \cdots, \zeta^m+\lambda) \in \boldsymbol{R}^m$. Observe that $x+\lambda$ is an $(a+\lambda)$-system if and only if x is an a-system. Hence in proving the above proposition we may and thus will assume without loss of generality that $a \geqq 0$. We start with proving the following

27. *Assertion.* (i) For any $a=(a^0, a^1, a^2) \in \boldsymbol{R}^3$ ($a \geqq 0$) there exists a unique a-system $x=\varphi(a)=\varphi(a^0, a^1, a^2) \in \boldsymbol{R}^1$ for $c=(c^0, c^1)$ and $d=(d^1)$; (ii) $\varphi(a_1) \leqq \varphi(a_2)$ if $a_1 \leqq a_2$; (iii) the function $x=\varphi(a)$ is continuous in a; (iv) the graph of the function $a^i \mapsto x=\varphi(a^0, a^1, a^2)$ ($a^i \geqq 0$) in (a^i, x)-plane is a polygonal line consisting

of at most 3 line segments and a half-line each of which has the form $x=Aa^i+B$ with the slope $A=A(\{a^0, a^1, a^2\}\setminus\{a^i\})\in[0, 1)$ and $B=B(\{a^0, a^1, a^2\}\setminus\{a^i\})\in[0, \infty)$ $(i=0, 1, 2)$.

In this case the equations (25) defining a-system $(a=(a^0, a^1, a^2))$ $x=x^1=\varphi(a)$ for $c=(c^0, c^1)$ and $d=(d^1)$ is reduced to a single quasilinear equation

$$\max\left(\frac{a^0-x^1}{c^0}, \frac{a^1-x^1}{d^1}, \frac{a^2-x^1}{c^1}\right)+\min\left(\frac{a^0-x^1}{c^0}, \frac{a^1-x^1}{d^1}, \frac{a^2-x^1}{c^1}\right)=0. \tag{28}$$

We will express $x=x^1=\varphi(a)$ explicitly below in terms of a by solving (28) concretely, which proves (i) in Assertion 27 instantly. By observing the dependence of $\varphi(a)$ upon a in the explicit formula of $\varphi(a)$ below we can at once see the validity of not only (ii) but also (iii) in Assertion 27. In the proof of (iv) in Assertion 27 we only have to treat the $i=0$ case since the other cases can be similarly taken care of. For all these purposes in mind we now try to solve (28) concretely in such a fashion that $x=x^1$ is viewed as the continuous piecewise linear function of a^0. For the purpose we may assume that $a^1\leqq a^2$ without loss of generality. For simplicity we set $\alpha=((c^1+c^0)a^1+(d^1-c^0)a^2)/(d^1+c^1)$ and $\beta=((c^1-c^0)a^1+(d^1+c^0)a^2)/(d^1+c^1)$, and in the case $d^1\neq c^1$, $\gamma=((c^1+c^0)a^1-(d^1+c^0)a^2)/(c^1-d^1)$. By the conventional assumption $a^1\leqq a^2$ we see that always $\alpha\leqq\beta$. It is easy to solve (28) concretely in a unique fashion as follows: if $d^1=c^1$, then $\alpha\leqq\beta$ and x^1, as the function of a^0, is the restriction to $[0, \infty)$ of the function in a^0 given by

$$x^1=\begin{cases}(c^1a^0+c^0a^2)/(c^0+c^1) & (-\infty<a^0\leqq\alpha),\\ (c^1a^1+d^1a^2)/(c^1+d^1) & (\alpha\leqq a^0\leqq\beta),\\ (d^1a^0+c^0a^1)/(c^0+d^1) & (\beta\leqq a^0<\infty);\end{cases} \tag{29}$$

if $d^1<c^1$, then $\gamma\leqq\alpha\leqq\beta$ and x^1, as the function of a^0, is the restriction to $[0, \infty)$ of the function in a^0 given by

$$x^1=\begin{cases}(d^1a^0+c^0a^1)/(c^0+d^1) & (-\infty<a^0\leqq\gamma),\\ (c^1a^0+c^0a^2)/(c^0+c^1) & (\gamma\leqq a^0\leqq\alpha),\\ (c^1a^1+d^1a^2)/(c^1+d^1) & (\alpha\leqq a^0\leqq\beta),\\ (d^1a^0+c^0a^1)/(c^0+d^1) & (\beta\leqq a^0<\infty);\end{cases} \tag{30}$$

if $d^1>c^1$, then $\alpha\leqq\beta\leqq\gamma$ and x^1, as the function of a^0, is the restriction to $[0, \infty)$ of the function in a^0 given by

$$x^1=\begin{cases}(c^1a^0+c^0a^2)/(c^0+c^1) & (-\infty<a^0\leqq\alpha),\\ (c^1a^1+d^1a^2)/(c^1+d^1) & (\alpha\leqq a^0\leqq\beta),\\ (d^1a^0+c^0a^1)/(c^0+d^1) & (\beta\leqq a^0\leqq\gamma),\\ (c^1a^0+c^0a^2)/(c^0+c^1) & (\gamma\leqq a^0<\infty).\end{cases} \tag{31}$$

The validity of (i)-(iv) in Assertion 27 can now be instantly verified by observing the relations (29), (30), or (31).

By the mathematical induction on $n\in\boldsymbol{N}$ we will prove the following assertion which is also sufficient for the proof of Proposition 26.

32. *Assertion.* (i) For any $a\in\boldsymbol{R}^{n+2}$ $(a\geqq 0)$ there exists a unique a-system $x=\varphi(a\,;\,n)$; (ii) $\varphi(a_1\,;\,n)\leqq\varphi(a_2\,;\,n)$ if $a_1\leqq a_2$; (iii) $x=\varphi(a\,;\,n)$ is continuous in a; (iv) writing $a=(a^0, a')\in\boldsymbol{R}^{n+2}$ $(a'\in\boldsymbol{R}^{n+1}, a'\geqq 0)$ and $\varphi(a\,;\,n)=(\varphi^1(a\,;\,n), \cdots, \varphi^n(a\,;\,n))$, the graph of the function $a^0\mapsto x^1=\varphi^1(a^0, a'\,;\,n)$ $(a^0\geqq 0)$ in (a^0, x^1)-plane is a polygonal line consisting of a finite number of line segments and a half-line each of which has the form $x^1=Aa^0+B$ with the slope $A=A(a')\in[0, 1)$ and $B=B(a')\in[0, \infty)$.

Assertion 32 for $n=1$ is clearly a part of Assertion 27. Take an arbitrary $n\in\boldsymbol{N}$ with $n\geqq 2$ and assume that Assertion 32 is true for $n-1$. We produce from the pair of $c=(c^0, \cdots, c^n)\in\boldsymbol{R}^{n+1}$ and $d=(d^1, \cdots, d^n)\in\boldsymbol{R}^n$ two new pairs of $c_1=(c^0, c^1)\in\boldsymbol{R}^2$ and $d_1=(d^1)\in\boldsymbol{R}^1$ and of $c_2=(c^1, \cdots, c^n)\in\boldsymbol{R}^n$ and $d_2=(d^2, \cdots, d^n)\in\boldsymbol{R}^{n-1}$. Let $x^1\geqq 0$ and $x^2\geqq 0$ be variables but fixed for the time being. From a vector $a=(a^0, a^1, \cdots, a^{n+1})\in\boldsymbol{R}^{n+2}$ we also produce two vectors $a_1=(a^0, a^1, x^2)\in\boldsymbol{R}^3$ and $a_2=(x^1, a^2, a^3, \cdots, a^{n+1})\in\boldsymbol{R}^{n+1}$. First take the unique a_1-system $x_1=(x^1)$ for c_1 and d_1. We set $x^1=\varphi^1(a^0, a^1, x^2\,;\,1)$. Observe that the function $x^2\mapsto x^1=\varphi^1(a^0, a^1, x^2\,;\,1)$ enjoys the properties (i)-(iv) in Assertion 27 for c_1 and d_1. Next, by the assumption of the induction that Assertion 32 is true for $n-1$, there exists the unique a_2-system $x_2=(x^2, \cdots, x^n)$ for c_2 and d_2 and the functions $a_2\mapsto x_2=\varphi(a_2\,;\,n-1)=(\varphi^1(a_2\,;\,n-1), \cdots, \varphi^{n-1}(a_2\,;\,n-1))$ and $x^1\mapsto x^2=\varphi^1(x^1, a^2, \cdots, a^{n+1}\,;\,n-1)$ enjoy the properties (i)-(iv) in Assertion 32 for $n-1$ and for c_2 and d_2. Describe the graphs of functions $x^2\mapsto x^1=\varphi^1(a^0, a^1, x^2\,;\,1)$ and $x^1\mapsto x^2=\varphi^1(x^1, a^2, \cdots, a^{n+1}\,;\,n-1)$ in the (x^1, x^2)-plane. By properties (ii)-(iv) in Assertions 27 and 32, and especially by the fact that graphs of these two functions are polygonal lines each segment or half-line of which has nonnegative slope less than 1 as functions of x^1 or x^2 and that $\varphi^1(a^0, a^1, 0\,;\,1)\geqq 0$ and $\varphi^1(0, a^2, \cdots, a^{n+1}\,;\,n-1)\geqq 0$, a simple geometric consideration assures that these two graphs intersect at only one point (x^1, x^2) in the first quadrant of (x^1, x^2)-plane which varies continuously and increasingly along with $(a^0, a^1, \cdots, a^{n+1})$. Hence in particular we can see that the simultaneous equations

$$(33)\qquad \begin{cases} x^1=\varphi^1(a^0, a^1, x^2\,;\,1) \\ x^2=\varphi^1(x^1, a^2, \cdots, a^{n+1}\,;\,n-1) \end{cases}$$

have a unique solution (x^1, x^2) which is the above point of the intersection. Using this particular x^1 we see that the first component of the $(x^1, a^2, \cdots, a^{n+1})$-system x_2 is the above particular x^2. Let $x_2=(x^2, x^3, \cdots, x^n)$. Then $x=(x^1, x^2, x^3, \cdots, x^n)$ is the unique a-system $(a=(a^0, a^1, \cdots, a^{n+1}))$ for c and d. This proves (i) in Assertion 32 for n and for c and d. We set $x=\varphi(a\,;\,n)=(\varphi^1(a\,;$

$n), \cdots, \varphi^n(a; n))$. The above geometric observation on the unique intersecting point of the two graphs of functions in (33) yields the properties (ii) and (iii) in Assertion 32 for n and for c and d.

By the assumption of the induction for the second equation of (33) there exist two scalars A and B depending only on $a^2, \cdots, a^{n+1}$ and thus independent of x^1 and a^0 with $0 \leqq A < 1$ and $0 \leqq B < \infty$ such that

$$x^2 = Ax^1 + B \tag{34}$$

on $[0, \infty)$ sectionally, i.e. A and B may vary accoding to a subdivision of the x^1-interval $[0, \infty)$. The same is true of the first equation of (33); more precisely, by the concrete representation of $x^1 = \varphi^1(a^0, a^1, x^2; 1)$ given by (29), (30), or (31), we see that x^1 is given by one of the following forms (35)-(37) below sectionally:

$$x^1 = (c^1 a^0 + c^0 x^2)/(c^0 + c^1); \tag{35}$$

$$x^1 = (c^1 a^1 + d^1 x^2)/(c^1 + d^1); \tag{36}$$

$$x^1 = (d^1 a^0 + c^0 a^1)/(c^0 + d^1). \tag{37}$$

First in the case where the first two components (x^1, x^2) of the a-system ($a = (a^0, \cdots, a^{n+1})$) $x = (x^1, x^2, \cdots, x^n)$ is given as the intersection of (34) and (35), replacing x^2 in (35) by that in (34) we have

$$x^1 = \frac{c^1}{(1-A)c^0 + c^1} a^0 + \frac{c^0}{(1-A)c^0 + c^1} B$$

so that x^1 is linear in a^0 sectionally with the slope $c^1/((1-A)c^0 + c^1) \in (0, 1)$ and $c^0 B/((1-A)c^0 + c^1) \in [0, \infty)$. Next in the case of (34) and (36), the simultaneous equations (34) and (36) produce

$$x^1 = 0 \cdot a^0 + \frac{c^1 a^1 + d^1 B}{c^1 + (1-A)d^1}$$

so that x^1 is linear (and in fact constant) in a^0 sectionally with the slope $0 \in [0, 1)$ and $(c^1 a^1 + d^1 B)/(c^1 + (1-A)d^1) \in [0, \infty)$. Last in the case of (34) and (37), the formula (37) itself indicates that x^1 is linear in a^0 sectionally with the slope $d^1/(c^0 + d^1) \in (0, 1)$ and $c^0 a^1/(c^0 + d^1) \in [0, \infty)$. Thus the property (iv) in Assertion 32 for n and for c and d has also been shown to hold.

This completes the induction for the proof of Assertion 32 and hence the proof of Proposition 26 is now over. □

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