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KODAI MATH. J.
19 (1996), 218—233

ON THE BIFURCATION SET OF A POLYNOMIAL FUNCTION AND NEWTON BOUNDARY, II

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1. Introduction

1.1. Let $f: \boldsymbol{C}^n \to \boldsymbol{C}$ be a polynomial function and let us denote by B_f the *bifurcation set* of f, i.e. B_f is the smallest subset $\Gamma \subseteqq \boldsymbol{C}$ such that the restriction $f: \boldsymbol{C}^n \setminus f^{-1}(\Gamma) \to \boldsymbol{C} \setminus \Gamma$ is a locally trivial fibration. It is well known that B_f is a finite set (see for example [13], [3], [11]) containing not only the set Σ_f of critical values of f, but also some *extra* values, corresponding to the so called "critical points at infinity". The problem of describing the bifurcation set B_f was considered by several authors, see for example: [3], [1], [10], [2], [12], [7]. In this note we would like to prove that certain values, given in [7] as possible elements of B_f, really belong to the bifurcation set of a *Newton nondegenerate* polynomial f.

1.2. We recall now some definitions and notations. Let $f: \boldsymbol{C}^n \to \boldsymbol{C}$ be a polynomial function. We shall assume that $f(0)=0$ If

$$f(z) := \sum_{\nu \in \boldsymbol{N}^n} a_\nu z^\nu,$$

we denote:

$$\mathrm{supp}(f) := \{\nu \in \boldsymbol{N}^n \mid a_\nu \neq 0\} \subseteqq \boldsymbol{R}^n,$$

$\overline{\mathrm{supp}(f)}$:= the convex closure in $\boldsymbol{R}^n$ of $\mathrm{supp}(f)$,

$\tilde{\Gamma}_-(f)$:= the convex closure in $\boldsymbol{R}^n$ of $\{0\} \cup \mathrm{supp}(f)$.

For $\Delta \subseteqq \boldsymbol{R}^n$ we put

$$f_\Delta := \sum_{\nu \in \Delta} a_\nu z^\nu$$

and we say that f is *nondegenerate* on Δ if the system of equations

$$\frac{\partial f_\Delta}{\partial z_1}(z) = \cdots = \frac{\partial f_\Delta}{\partial z_n}(z) = 0$$

has no solutions in $(\boldsymbol{C} \setminus \{0\})^n$. We say that f is *Newton nondegenerate* if for every compact face Δ of $\tilde{\Gamma}_-(f)$, with $0 \notin \Delta$, we have that f is nondegenerate

Received April 5, 1995; revised August 4, 1995.

on Δ. Here and below, by face we shall understand face of *any* dimension. f is called *convenient* if the intersection of supp(f) with each coordinate axis is non-empty.

A closed face Δ of $\overline{\mathrm{supp}(f)}$ is called *bad* if:

(i) the affine subvariety of dimension$=\dim\Delta$ spaned by Δ contains the origin, and

(ii) there exists a hyperplane $H\subseteqq \boldsymbol{R}^n$ with equation $a_1x_1+\cdots+a_nx_n=0$, where $x_1, \cdots, x_n$ are the coordinates in $\boldsymbol{R}^n$, such that:

($\mathrm{ii_a}$) there exist i and j with $a_i\cdot a_j<0$

($\mathrm{ii_b}$) $H\cap\overline{\mathrm{supp}(f)}=\Delta$.

More geometrically, condition ($\mathrm{ii_a}$) says that H contains from the interior of the positive octant of $\boldsymbol{R}^n$.

We denote by $\mathscr{B}$ the set of bad faces of $\overline{\mathrm{supp}(f)}$. For $\Delta\in\mathscr{B}$, we define:

$$\Sigma_\Delta := \{f_\Delta(z^0)\,|\,z^0\in(\boldsymbol{C}\setminus\{0\})^n \text{ and } \operatorname{grad} f_\Delta(z^0)=0\}.$$

In [7] it is proved the following

1.3. THEOREM. *Suppose that f is a polynomial Newton nondegenerate, not convenient and $f(0)=0$. Then*

$$B_f\subseteqq\{0\}\cup\Sigma_f\cup\bigcup_{\Delta\in\mathscr{B}}\Sigma_\Delta.$$

Also, in [7] it is conjectured that for each $\Delta\in\mathscr{B}$, we have $\Sigma_\Delta\subseteqq B_f$. Our aim is to prove that certain values in Σ_Δ belong to B_f, for any $n\geqq 2$. Note that for $n=2$, this is proved in [7]. We intend to use toric varieties and the extra assumptions we need give us that some "critical points at infinity" are isolated. In the next Section we recalled the construction of toric varieties. In Section 3 we study an interplay between the orbits at infinity of a toric variety and the fibers of f. In Section 4 we derive our main result and in the last Section we give some remarks and examples.

Acknowledgements. During the work to this paper I got hospitality and support from the Department of Pure Mathematics of the Bordeaux I University. Several discussions with Professors M. Oka and A. Dimca were very useful.

2. Toric varieties

2.1. For the construction of toric varieties we use here, we refer to [4] and [8]. Let $(\boldsymbol{R}^n)^*$ denote the dual space of $\boldsymbol{R}^n$. For $a\in(\boldsymbol{R}^n)^*$ we denote by Δ^a the face of $\tilde{\Gamma}_-(f)$ where the function

$$\tilde{\Gamma}_-(f)\ni x\longmapsto\langle a,\ x\rangle:=a(x)\in\boldsymbol{R}$$

takes its minimal value which we denote by d^a. We define an equivalence relation on $(\boldsymbol{R}^n)^*$ by:

$$a \sim b \text{ if and only if } \Delta^a = \Delta^b.$$

We shall denote by σ_+ the positive octant of $(\boldsymbol{R}^n)^*$. Note that for any a in the interior of σ_+, we have:

$$d^a = 0 \text{ and } \Delta^a = \{0\} \subseteqq \tilde{\Gamma}_-(f) \subseteqq \boldsymbol{R}^n.$$

Hence all the vectors in the interior of the positive octant of $(\boldsymbol{R}^n)^*$ are equivalent.

2.2. The equivalence classes of this equivalence relation are polyhedral cones. The set of the closures of all such equivalence classes is a conic polyhedron in $(\boldsymbol{R}^n)^*$, denoted by $\tilde{\Gamma}_-(f)^*$. Next we take K, a unimodular simplicial subdivision of $\tilde{\Gamma}_-(f)^*$, such that σ_+ is one of the cones of K. The dimension of a cone $\sigma \in K$ will be the dimension of the vector space generated by σ in $(\boldsymbol{R}^n)^*$, and the interior of σ will mean the interior in this vector subspace.

For a in the interior of σ, we put $\Delta^\sigma := \Delta^a$; this does not depend on the choice of a, by construction of K. Note that if $\dim \sigma = k$, then $\dim \Delta^\sigma \leqq n-k$, and if $\sigma \subseteqq \sigma'$, then $\Delta^\sigma \supseteqq \Delta^{\sigma'}$.

Choosing $\omega \in \Delta^\sigma$, we obtain that for any $a \in \sigma$, the minimal value of the function

$$\tilde{\Gamma}_-(f) \ni x \longmapsto \langle a, x \rangle \in \boldsymbol{R}$$

is equal to $\langle a, \omega \rangle$. This means that the support function of f is linear on σ. Note also that when $0 \in \Delta^\sigma$ we shall take $\omega = 0$.

2.3. The tori of dimension n, $(\boldsymbol{C} \setminus \{0\})^n$, is a group with multiplication on components as a law. For $q = (q_1, \cdots, q_n) \in \boldsymbol{Z}^n$, the character χ^q is defined by

$$\chi^q : (\boldsymbol{C} \setminus \{0\})^n \longrightarrow \boldsymbol{C} \setminus \{0\} \hookrightarrow \boldsymbol{C}, \quad \chi^q(x_1, \cdots, x_n) := x_1^{q_1} \cdot \cdots \cdot x_n^{q_n}.$$

To each cone $\sigma \in K$ of dimension k it is associated a tori $\Phi[\sigma]$ of dimension $n-k$, namely the factor group of $(\boldsymbol{C} \setminus \{0\})^n$ by the subgroup generated by

$$\{(t^{a_1}, \cdots, t^{a_n}) \mid t \in \boldsymbol{C} \setminus \{0\},\ (a_1, \cdots, a_n) \in \sigma\}.$$

The disjoint union of all the tori $\Phi[\sigma]$, for $\sigma \in K$, is the set of points of a toric variety, denoted by M_K.

2.4. Let $\sigma \in K$ and let $\bar{\sigma}$ be the conic polyhedron associated to σ, i.e. $\bar{\sigma}$ is the set of all cones $\sigma' \in K$ which are contained in σ. By $M_{\bar{\sigma}}$ we denote the corresponding toric variety. As a set of points, $M_{\bar{\sigma}}$ is the disjoint union of the tori $\Phi[\sigma']$, for $\sigma' \in \bar{\sigma}$. The variety $M_{\bar{\sigma}}$ is isomorphic to

$$C_k^n := \{u=(u_1, \cdots, u_n) \in C^n \mid u_{k+1} \cdot \cdots \cdot u_n \neq 0\}.$$

We recall the construction of this isomorphism.

The dual of the cone $\sigma \subseteq (R^n)^*$ is the cone $\sigma^* \subseteq R^n$ defined by

$$\sigma^* := \{x \in R^n \mid \forall a \in \sigma, \langle a, x \rangle \geqq 0\}.$$

The cone $\sigma^* \subseteq R^n$ contains a vector subspace of dimension $n-k$, namely

$$V_{\sigma^*} := \{x \in R^n \mid \forall a \in \sigma, \langle a, x \rangle = 0\}.$$

This vector subspace contains the vector space associated to the affine subvariety (of dimension $\leqq n-k$) generated by Δ^σ, hence

$$\Delta^\sigma \subseteq V_{\sigma^*} \Longleftrightarrow \Delta^\sigma \cap V_{\sigma^*} \neq \emptyset \Longleftrightarrow 0 \in \Delta^\sigma.$$

If $a_1, \cdots, a_k$ is a basis of the cone σ, then we can choose $A=\{m_1, \cdots, m_n\}$, a basis of $\sigma^* \subseteq R^n$, such that

$$\langle a_i, m_j \rangle = \delta_{ij}, \quad i=1, \cdots, k, \ j=1, \cdots, n.$$

Hence

$$\sigma^* = \left\{ \sum_{i=1}^n \lambda_i m_i \mid \lambda_i \in R, \lambda_1, \cdots, \lambda_k \geqq 0 \right\}$$

and V_{σ^*} is generated by $m_{k+1}, \cdots, m_n$.

For $q \in \sigma^* \cap Z^n$, the character χ^q will be extended to $\Phi[\sigma]$ as the zero function, if it does not factorize for the natural projection $(C \setminus \{0\})^n \rightarrow \Phi[\sigma]$. Otherwise, the extension will be the factorization through $\Phi[\sigma]$.

The isomorphism between $M_{\bar{\sigma}}$ and C_k^n is given by the following chart:

$$(1) \qquad \varphi_A : M_{\bar{\sigma}} \longrightarrow C_k^n, \quad \varphi_A(c) := (\chi_{\bar{\sigma}}^{m_1}(c), \cdots, \chi_{\bar{\sigma}}^{m_n}(c)).$$

In this formula, $\chi_{\bar{\sigma}}^{m_i}$ is obtained from the character χ^{m_i} by extending it to all the orbits $\Phi[\sigma']$, for $\sigma' \in \bar{\sigma}$.

By glueing all the varieties $M_{\bar{\sigma}}$ we obtain a compactification M_K of C^n. Note that for σ_+ we have that $M_{\overline{\sigma_+}}$ is the affine space C^n. M_K is a disjoint union of several tori $(C \setminus \{0\})^k$, for $k=0, \cdots, n$. There are only one torus $(C \setminus \{0\})^n$ and, by convention, $(C \setminus \{0\})^0 :=$ one point. Those tori of M_K which are not lying in $M_{\overline{\sigma_+}} = C^n$ will be called the *orbits at infinity* of M_K. We have:

$$M_K \setminus C^n = \text{a normal crossing divisor of } M_K$$

whose irreducible components are in one-to-one correspondence with the set of cones $\{\sigma \in K \mid \dim \sigma = 1, \sigma \notin \overline{\sigma_+}\}$. Note also that the orbits at infinity of M_K correspond to cones $\sigma \in K$ which do not belong to the conic polyhedron associated to σ_+. If $\dim \sigma = k$, then $M_{\bar{\sigma}}$ is isomorphic to C_k^n and the orbit $\Phi[\sigma]$ is described, using the chart (1), as

$$\{u \in C_k^n \mid u_1 = \cdots = u_k = 0\},$$

i.e. is isomorphic to $(C \setminus \{0\})^{n-k}$.

LEMMA. *The closure of the orbit $\Phi[\sigma]$ is smooth and equals to the union of the orbits $\Phi[\sigma']$ with $\sigma \in \overline{\sigma'}$ (or, equivalently, with $\sigma \subseteqq \sigma'$).*

3. Orbits at infinity and fibers of f

3.1. We continue to use the notations introduced in Section 2. Let $m_\iota = (m_{\iota 1}, \cdots, m_{\iota n})$ and let us consider the matrices

$$M := \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \vdots & \vdots \\ m_{n1} & \cdots & m_{nn} \end{pmatrix} \quad \text{and} \quad W := M^{-1} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \vdots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix}.$$

If $w_\iota := (w_{\iota 1}, \cdots, w_{\iota n})$, then $\varphi_A^{-1}(u) = (u^{w_1}, \cdots, u^{w_n})$. The polynomial function f on C^n can be extended at C_k^n, via the diagram

$$\begin{tikzcd} & M_{\bar\sigma} \arrow[r, "\varphi_A"] & C_k^n \\ (C \setminus \{0\})^n \arrow[ur] \arrow[dr] & & \\ & C^n \arrow[r, "f"] & C \end{tikzcd},$$

to a Laurent polynomial function

$$f^W(u) = \sum_{\nu \in N^n} a_\nu u^{\nu W}.$$

3.2. On the other hand $0 \in \tilde{\Gamma}_-(f)$ and for all $a \in \sigma$ we have

$$0 = \langle a, 0 \rangle \geqq \min_{x \in \tilde{\Gamma}_-(f)} \langle a, x \rangle = \langle a, \omega \rangle.$$

Hence

$$\tilde{\Gamma}_-(f) \subseteqq \omega + \sigma^* := \{\omega + a \mid a \in \sigma^*\}$$

and for any $\nu \in \tilde{\Gamma}_-(f)$ we have a decomposition (which depends on the choice of ω)

$$\nu = \omega + \lambda_1(\nu) m_1 + \cdots + \lambda_n(\nu) m_n.$$

It follows that

$$\nu W = \omega W + (\lambda_1(\nu), \cdots, \lambda_n(\nu)) \quad \text{and} \quad f^W(u) = u^{\omega W} \cdot \sum_{\nu \in N^n} a_\nu \cdot u_1^{\lambda_1(\nu)} \cdot \cdots \cdot u_n^{\lambda_n(\nu)}$$

with $\lambda_1(\nu) \geqq 0, \cdots, \lambda_k(\nu) \geqq 0$. Note also that

$$\lambda_1(\nu) = \cdots = \lambda_k(\nu) = 0 \Longleftrightarrow \nu \in \Delta^\sigma.$$

3.3. We distinguish four types of orbits at infinity of M_K (compare with section 2.6 in [4]):

(A) orbits which correspond to cones $\sigma\subseteqq(\boldsymbol{R}^n)^*$ such that $0\notin\Delta^\sigma$;

(B) orbits which correspond to cones $\sigma\subseteqq(\boldsymbol{R}^n)^*$ such that $\Delta^\sigma=\{0\}$ (hence $\Delta^\sigma\cap\overline{\mathrm{supp}(f)}=\emptyset$);

(C) orbits which correspond to cones $\sigma\subseteqq(\boldsymbol{R}^n)^*$ such that $0\in\Delta^\sigma\neq\{0\}$ and such that $\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$ is not a bad face of $\overline{\mathrm{supp}(f)}$;

(D) orbits which correspond to cones $\sigma\subseteqq(\boldsymbol{R}^n)^*$ such that $0\in\Delta^\sigma$ and such that $\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$ is a bad face of $\overline{\mathrm{supp}(f)}$.

We shall denote by $\mathfrak{N}_A$, $\mathfrak{N}_B$, $\mathfrak{N}_C$ and $\mathfrak{N}_D$ the union of all the orbits of type (A), (B), (C) and respectively (D). Also, we shall denote by $\mathcal{A}\!f\!f(Y)$ the affine subvariety spaned by a subset $Y\subseteqq\boldsymbol{R}^n$. We have the following characterization of the orbits of type (D):

LEMMA. *Let $\Phi[\sigma]$ be an orbit at infinity. Equivalent are:*

(a) *$\Phi[\sigma]$ is of type* (D).

(b) $0\in\mathcal{A}\!f\!f(\Delta^\sigma\cap\overline{\mathrm{supp}(f)})$.

(c) *$\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$ is a bad face of $\overline{\mathrm{supp}(f)}$.*

Proof. The implications (a)$\Rightarrow$(c)$\Rightarrow$(b) follow directly from the definitions.

(b)$\Rightarrow$(c). Let $\Phi[\sigma]$ be an orbit at infinity such that $0\in\mathcal{A}\!f\!f(\Delta^\sigma\cap\overline{\mathrm{supp}(f)})$; then condition (i) from the definition of bad faces is fulfiled, for $\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$, and suppose that condition (ii) is not fulfield, i.e. for any hyperplane $H\subseteqq\boldsymbol{R}^n$ such that $H\cap\overline{\mathrm{supp}(f)}=\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$, we have that H does not contain any point from the interior of the positive octant of $\boldsymbol{R}^n$. It follows that $\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$ is contained in the intersection of $n-\dim\Delta^\sigma$ hyperplanes of coordinates of $\boldsymbol{R}^n$ and that $\sigma\subseteqq\sigma_+$. Hence $\Phi[\sigma]$ is not an orbit at infinity, a contradiction. Thus, we proved that (b)$\Leftrightarrow$(c).

(c)$\Rightarrow$(a). Suppose that $\Phi[\sigma]$ is not an orbit of type (D). It is easy to see that $\Phi[\sigma]$ can not be an orbit of type (C) or of type (B). Hence $\Phi[\sigma]$ is an orbit of type (A). Then $0\notin\Delta^\sigma$, and since Δ^σ is a face of $\tilde{\Gamma}_-(f)$=convex closure of $\{0\}\cup\mathrm{supp}(f)$, it follows that $0\notin\mathcal{A}\!f\!f(\Delta^\sigma)$, hence $0\notin\mathcal{A}\!f\!f(\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$, a contradiction. □

COROLLARY. *Let $\Phi[\sigma']$ be an orbit of type* (D) *and let $p^0\in\Phi[\sigma']$ be a point which is in the closure of an orbit at infinity, $\Phi[\sigma]$. Then $\Phi[\sigma]$ is an orbit of type* (D).

Proof. Since $p^0\in\Phi[\sigma']\cap\overline{\Phi[\sigma]}$, it follows, by Lemma 2.4, that $\Phi[\sigma']\subseteqq\overline{\Phi[\sigma]}$. Now use again Lemma 2.4 and next the above Lemma. □

3.4. PROPOSITION. *$\mathfrak{N}_A$ is compact and is a normal crossing divisor of M_K.*

Proof. It is enough to show that:

(α) the closure of an orbit of type (A) is a union of orbits of type (A);

(β) any orbit of type (A) is included in the closure of an orbit of type (A) of dimension $n-1$.

Let $\Phi[\sigma]$ be an orbit of type (A). Then $0\notin\Delta^{\sigma}$, hence for all $\sigma'\supseteqq\sigma$ we have that $0\notin\Delta^{\sigma'}$, i.e. all such orbits $\Phi[\sigma']$ are of type (A). Now (α) follows, for $\Phi[\sigma]$, from Lemma 2.4.

To prove (β), it is enough to show that any orbit $\Phi[\sigma]$ of type (A) is in the closure of an orbit $\Phi[\sigma']$ of type (A), with $\dim\Phi[\sigma']=n-1$, or, equivalently, with $\dim\sigma'=1$. Let $a_1, \cdots, a_k$ be a basis of σ. Then the cones σ_i of dimension 1, defined by $\sigma_i=[0, \infty)\times a_i$, $i=1, \cdots, k$, have the properties

$$\Phi[\sigma]\subseteqq\Phi[\sigma_i] \quad\text{and}\quad 0\notin\Delta^{\sigma}=\bigcap_{j=1}^{k}\Delta^{\sigma_j}.$$

hence for some i, the orbit $\Phi[\sigma_i]$ is of type (A).

3.5. For $t\in\boldsymbol{C}$ we denote by X_t the closure of $f^{-1}(t)\subseteqq\boldsymbol{C}^n$ in M_K. X_t is a compact variety whose intersection with an orbit of type (A) does not depend on t. Indeed: for σ a cone of dimension k such that $\Phi[\sigma]$ is an orbit of type (A), the equation $f-t=0$ is written, in $\boldsymbol{C}_k^n$, like

$$f^W(u)-t=u^{\omega W}\cdot\sum_{\nu\in\boldsymbol{N}^n}a_\nu\cdot u_1^{\lambda_1(\nu)}\cdot\cdots\cdot u_n^{\lambda_n(\nu)}-t\cdot u^{0\cdot W}=0. \tag{2}$$

Note also that

$$0\in\tilde{\Gamma}_-(f)\subseteqq\omega+\sigma^*,\quad 0\notin\Delta^{\sigma}\ \text{and}\ \omega\in\Delta^{\sigma}.$$

Since $0\in V_{\sigma^*}$ and $\omega\notin V_{\sigma^*}$, it follows that in the decomposition of $0\in\boldsymbol{R}^n$

$$0=\omega+\lambda_1(0)\cdot m_1+\cdots+\lambda_n(0)\cdot m_n$$

we have that at least one of $\lambda_1(0), \cdots, \lambda_k(0)$ is not equal to 0, hence is >0. And now, the equation (2), after dividing by $u^{\omega W}$, gives us in $\Phi[\sigma]$

$$\sum_{\nu\in\Delta^{\sigma}}a_\nu\cdot u_{k+1}^{\lambda_{k+1}(\nu)}\cdot\cdots\cdot u_n^{\lambda_n(\nu)}=0,$$

i.e. is independent of t.

In fact, by the Newton nondegeneracy condition, we have that the intersection between X_t and an orbit of type (A) is transverse and that X_t is smooth in a neighbourhood of this orbit, see [4].

3.6. Let σ be a cone such that the corresponding orbit is of type (B), (C) or (D). Then we take $\omega=0$ as $\omega\in\Delta^{\sigma}$. Hence the decomposition of $0\in\boldsymbol{R}^n$ will be

$$0=0+\lambda_1(0)\cdot m_1+\cdots+\lambda_n(0)\cdot m_n$$

with all $\lambda_i(0)=0$. Note that $\tilde{\Gamma}_-(f)\subseteqq\sigma^*$, hence f can be extended holomorphicaly to C_k^n, hence to $M_K\backslash\mathcal{N}_A$. The equation $f-t=0$ is written in C_k^n like

$$f^W(u)-t=\sum_{\nu\in N^n} a_\nu\cdot u_1^{\lambda_1(\nu)}\cdot\cdots\cdot u_n^{\lambda_n(\nu)}-t=0$$

and for $u_1=\cdots=u_k=0$ gives us

$$f^W(0,\cdots,0,u_{k+1},\cdots,u_n)-t=\sum_{\nu\in\Delta^\sigma} a_\nu\cdot u_{k+1}^{\lambda_{k+1}(\nu)}\cdot\cdots\cdot u_n^{\lambda_n(\nu)}-t=0. \tag{3}$$

3.7. Let $\Phi[\sigma]$ be an orbit of type (B). Then, if $t\neq 0$, we have $X_t\cap\Phi[\sigma]=\emptyset$, while for $t=0$ we get $\Phi[\sigma]\subseteqq X_0$. This follows from the equation (3), since $\Delta^\sigma=\{0\}$ and $f(0)=0$.

3.8. Let $\sigma\subseteqq(\boldsymbol{R}^n)^*$ be a cone of dimension k such that the corresponding orbit, $\Phi[\sigma]$, is of type (C). Hence $\Delta^\sigma\neq\{0\}$ and Lemma 3.3 shows us that condition (i) from the definition of bad faces is not fulfiled; thus $\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$ is contained in a hyperplane which does not contains the origin. It follows that f_{Δ^σ} can be considered as a weighted homogeneous polynomial, for suitable integer weights for the coordinates. With a good choice of $m_{k+1},\cdots,m_n$, we obtain that $\lambda_{k+1}(\nu)\geqq 0,\cdots,\lambda_n(\nu)\geqq 0$, for all $\nu\in\Delta^\sigma$, and that

$$f^W(0,\cdots,0,u_{k+1},\cdots,u_n)=\sum_{\nu\in\Delta^\sigma} a_\nu\cdot u_{k+1}^{\lambda_{k+1}(\nu)}\cdot\cdots\cdot u_n^{\lambda_n(\nu)}$$

is a weighted homogeneous polynomial with integer weights for the coordinates $u_{k+1},\cdots,u_n$. Hence, in

$$\Phi[\sigma]=\{u\in C_k^n\mid u_1=\cdots=u_k=0\}$$

the hypersurfaces $(f^W)^{-1}(t)\cap\Phi[\sigma]=X_t\cap\Phi[\sigma]\subseteqq\Phi[\sigma]$ are diffeomorphic, when t varies in $C\backslash\{0\}$, since we can construct a trivialization of the global Milnor fibration

$$(f^W)^{-1}(C\backslash\{0\})\cap\Phi[\sigma]\ni(0,\cdots,0,u_{k+1},\cdots,u_n)$$
$$\longmapsto f^W(0,\cdots,u_{k+1},\cdots,u_n)\in C\backslash\{0\}$$

which respect the action of $C\backslash\{0\}$ on $\Phi[\sigma]$, associated to the integer weights of the coordinates $u_{k+1},\cdots,u_n$.

The case of orbits of type (D) is discussed in the next Section.

4. Main result

4.1. Let $\sigma\subseteqq(\boldsymbol{R}^n)^*$ be a cone such that $\Phi[\sigma]$ is of type (D). Then $\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$ is a bad face of $\overline{\mathrm{supp}(f)}$. The following Proposition describes a reason for which the elements of Σ_Δ, for Δ a bad face of $\overline{\mathrm{supp}(f)}$, should be "special" values of f.

PROPOSITION. *Let f be as in Theorem* (1.3) *and let $t_0 \in \boldsymbol{C}$. Then there exists $\Delta \in \mathcal{B}$ such that $t_0 \in \Sigma_\Delta$ if and only if there exists an orbit $\Phi[\sigma]$ of type* (D) *such that either X_{t_0} is not smooth in a point situated in $\Phi[\sigma]$, or X_{t_0} is smooth in any point of $\Phi[\sigma]$ but does not intersect transversally the orbit $\Phi[\sigma]$. Moreover, if one of these conditions holds, we have $\Delta^\sigma \cap \overline{\mathrm{supp}(f)} = \Delta$.*

Proof. Suppose that there exists a bad face $\Delta \in \mathcal{B}$ such that $t_0 \in \Sigma_\Delta$. This means that there exists a point $z^0 \in (\boldsymbol{C} \setminus \{0\})^n$ such that $\mathrm{grad} f_\Delta(z^0) = 0$ and $f_\Delta(z^0) = t_0$. Let us denote by Δ' the convex closure of $\{0\} \cup \Delta$. Then Δ' is a face of $\tilde{\Gamma}_-(f)$. Condition (ii) for the bad face Δ and construction of K imply that there exists a cone $\sigma \in K \setminus \{\overline{\sigma_+}\}$ such that $\Delta^\sigma = \Delta'$. For such a cone σ we have $\Delta^\sigma \cap \overline{\mathrm{supp}(f)} = \Delta$ and $0 \in \mathcal{A}\!f\!f(\Delta^\sigma \cap \overline{\mathrm{supp}(f)})$; hence $\Phi[\sigma]$ is of type (D). Let $k = \dim \sigma$ and let $u^0 = (u_1^0, \cdots, u_n^0) \in \boldsymbol{C}_k^n$ be the point which corresponds to z^0 in chart (1). Then $u_1^0 \neq 0, \cdots, u_k^0 \neq 0$ and the function

$$h : \boldsymbol{C}_k^n \longrightarrow \boldsymbol{C}, \quad h(u) := \sum_{\nu \in \Delta} a_\nu u_1^{\lambda_1(\nu)} \cdot \cdots \cdot u_n^{\lambda_n(\nu)},$$

which corresponds to f_Δ in chart (1), does not depend on $u_1, \cdots, u_k$, by (3.2). We denote $p^0 := (0, \cdots, 0, u_{k+1}^0, \cdots, u_n^0)$ and let $g = f^W$ be the function which corresponds to f. Then $p^0 \in \Phi[\sigma]$, $g(p^0) = h(p^0) = h(u^0) = t_0$, $\mathrm{grad}\, h(p^0) = \mathrm{grad}\, h(u^0) = 0$ and

$$\frac{\partial g}{\partial u_{k+1}}(p^0) = \cdots = \frac{\partial g}{\partial u_n}(p^0) = 0.$$

Hence $(X_{t_0} \cap \Phi[\sigma], \Phi[\sigma])$ is singular at p^0 and according to the values of the partial derivatives

$$\frac{\partial g}{\partial u_1}(p^0), \cdots, \frac{\partial g}{\partial u_k}(p^0),$$

we have one of the following two situations:

- either X_{t_0} is not smooth in $p^0 \in \Phi[\sigma]$, or
- X_{t_0} is smooth in $p^0 \in \Phi[\sigma]$ but X_{t_0} does not intersect transversally the orbit $\Phi[\sigma]$ in the point p^0. Equivalently: X_{t_0} is smooth in p^0 but $(X_{t_0} \cap \Phi[\sigma], \Phi[\sigma])$ is singular at p^0.

Conversely, let us suppose that there exists an orbit $\Phi[\sigma]$, of type (D), such that X_{t_0} is not smooth in a point $p^0 \in \Phi[\sigma]$, or X_{t_0} is smooth in $p^0 \in \Phi[\sigma]$ but X_{t_0} does not intersect transversally the orbit $\Phi[\sigma]$ in the point $p^0 := (0, \cdots, 0, p_{k+1}^0, \cdots, p_n^0)$. It follows, in both cases, that $g(p^0) = t_0$ and

$$\frac{\partial g}{\partial u_{k+1}}(p^0) = \cdots = \frac{\partial g}{\partial u_n}(p^0) = 0,$$

hence $(X_{t_0}\cap\Phi[\sigma], \Phi[\sigma])$ is singular at p^0. Since $h(u_1, \cdots, u_n)=g(0, \cdots, 0, u_{k+1}, \cdots, u_n)$, it follows that in the point

$$v^0:=(1, \cdots, 1, p^0_{k+1}, \cdots, p^0_n)$$

we have $h(v^0)=t_0$ and $\operatorname{grad} h(v^0)=0$. Next, the point v^0 corresponds to a point $z^0\in(\boldsymbol{C}\setminus\{0\})^n$ and then we apply Lemma 3.3. □

We shall consider the following

4.2. HYPOTHESIS. *The singular points of $X_{t_0}\subseteq M_K$ situated in $\mathfrak{N}_D$ are isolated. Also, for any orbit $\Phi[\sigma]$ of type* (D), *the singular points of $X_{t_0}\cap\overline{\Phi[\sigma]}\subseteq\overline{\Phi[\sigma]}$ are isolated. We denote by $\mu(X_{t_0}, p^0)$ and by $\mu(X_{t_0}\cap\overline{\Phi[\sigma]}, p^0)$ the corresponding Milnor numbers in a point $p^0\in X_{t_0}\cap\mathfrak{N}_D$ such that $p^0\in\overline{\Phi[\sigma]}$.*

We shall see later conditions on $\overline{\operatorname{supp}(f)}$ which imply that f satisfies this Hypothesis.

Our main result is the following

4.3. THEOREM. *Suppose that f is a polynomial Newton nondegenerate, not convenient and $f(0)=0$. Let K be a conic polyhedron constructed as in* (2.2). *For a bad face $\Delta\in\mathcal{B}$, let $t_0\in\Sigma_\Delta$ be such that $t_0\notin\{0\}\cup\Sigma_f$ and let us suppose that Hypothesis* 4.2 *is satisfied for t_0. Then $t_0\in B_f$ and for $t\notin B_f$, we have*

$$(4)\qquad \begin{aligned}&\chi(f^{-1}(t_0))-\chi(f^{-1}(t))\\ &=(-1)^{n-1}\cdot\sum_{p^0\in X_{t_0}\cap\mathfrak{N}_D}\Big[\mu(X_{t_0}, p^0)+\sum_{\Phi[\sigma]\subseteq\mathfrak{N}_D,\ \overline{\Phi[\sigma]}\ni p^0}\mu(X_{t_0}\cap\overline{\Phi[\sigma]}, p^0)\Big].\end{aligned}$$

Note that, by Hypothesis 4.2, we have $\mu(X_{t_0}, p^0)=\mu(X_{t_0}\cap\overline{\Phi[\sigma]}, p^0)=0$ for all, but a finite numbers of points $p^0\in X_{t_0}\cap\mathfrak{N}_D$.

4.4. *Proof.* Let $p^0\in X_{t_0}\cap\mathfrak{N}_D$ be such that $\mu(X_{t_0}\cap\overline{\Phi[\sigma]}, p^0)\neq0$ for at least one cone σ, and let $\varepsilon(p^0)>0$ be a Milnor radius for the Milnor fibrations of $X_{t_0}\cap\overline{\Phi[\sigma]}$, for any σ with $p^0\in\overline{\Phi[\sigma]}$; this includes also the case when $\sigma=\{0\}$. Let $\boldsymbol{B}(p^0)$ be the open Milnor ball centered at p^0, of radius $\varepsilon(p^0)$. We denote by $\boldsymbol{B}$ the union of all such Milnor balls.

We shall denote $Y:=\mathfrak{N}_B\cup\mathfrak{N}_C\cup\mathfrak{N}_D$. For any $t\in\boldsymbol{C}$ we have:

$$\begin{aligned}&\chi(f^{-1}(t))=\chi(X_t\setminus(\mathfrak{N}_A\cup Y))\\ &=\chi(X_t\setminus(\mathfrak{N}_A\cup\boldsymbol{B}))-\chi((X_t\cap Y)\setminus\boldsymbol{B})+\chi((X_t\setminus Y)\cap\boldsymbol{B}).\end{aligned}$$

The function f can be extended holomorphically to a function $f_K: M_K\setminus\mathfrak{N}_A\to\boldsymbol{C}$ and $f_K^{-1}(t)=X_t\setminus\mathfrak{N}_A$. Let $\Lambda\subseteq\boldsymbol{C}$ be a small disc centered at t_0, such that

$$\Lambda\cap\Big(\{0\}\cup\Sigma_f\cup\bigcup_{\Delta\in\mathcal{B}}\Sigma_\Delta\Big)=\{t_0\}.$$

Theorem 4.3 follows from the following Lemmas, in which is described the variation of

$$\chi(X_t\backslash(\mathcal{N}_A\cup\boldsymbol{B})),\quad \chi((X_t\cap Y)\backslash\boldsymbol{B}) \text{ and } \chi((X_t\backslash Y)\cap\boldsymbol{B})$$

when $t\in\Lambda$. □

4.5. LEMMA. *The restriction of f_K,*

$$f_K:(M_K\backslash(\mathcal{N}_A\cup\boldsymbol{B}))\cap f^{-1}(\Lambda)\longrightarrow\Lambda, \tag{5}$$

is a locally trivial fibration. In particular, $\chi(X_t\backslash(\mathcal{N}_A\cup\boldsymbol{B}))$ does not depend on $t\in\Lambda$.

Proof. To see this, we apply the Ehresmann fibration theorem, outside a small neighbourhood of $\mathcal{N}_A$; and in a neighbourhood of $\mathcal{N}_A$, we can construct a vector field which produce a trivialisation of (5). The vector field can be constructed since outside $\boldsymbol{B}$, X_t is smooth and intersects transversally any orbit $\Phi[\sigma]$, for any $t\in\Lambda$. Namely, there exists a vector field $\boldsymbol{w}$, defined in a small neighbourhood U of $\mathcal{N}_A\cap X_{t_0}$ such that if t is sufficiently close to t_0, then in any point $p\in X_t\cap U$ we have $\boldsymbol{w}(p)\pitchfork T_pX_t$, and if $p\in\Phi[\sigma]\cap U\subseteq\mathcal{N}_A$, then $\boldsymbol{w}(p)\in T_p\Phi[\sigma]$; the construction of $\boldsymbol{w}$ is done using the prolongation of a tangent vector to a vector field, the compactness of $\mathcal{N}_A\cap X_{t_0}$ and a partition of unity. □

4.6. LEMMA. *$\chi((X_t\cap Y)\backslash\boldsymbol{B})$ does not depend on $t\in\Lambda$.*

Proof. By the additivity of the Euler characteristic, it is enough to show that

$$\chi((X_t\cap\Phi[\sigma])\backslash\boldsymbol{B})$$

does not depend on $t\in\Lambda$, for any orbit $\Phi[\sigma]$ of type (B), (C) or (D). When $\Phi[\sigma]$ is of type (B) or (C), this follows from Corollary 3.3 and from (3.7) and (3.8). When $\Phi[\sigma]$ is of type (D), it can be proved, similarly to the proof of Lemma 4.5, that the restriction

$$f_K:(\overline{\Phi[\sigma]}\backslash(\mathcal{N}_A\cup\boldsymbol{B}))\cap f^{-1}(\Lambda)\longrightarrow\Lambda$$

is a locally trivial fibration. □

4.7. LEMMA. (a) *The pair $(Y\cap B(p^0), B(p^0))$ is isomorphic to $(\{x_1\cdot\cdots\cdot x_k=0\}, \boldsymbol{C}^n)$, for some $1\leq k\leq n-1$.*

(b) *For $t\in\Lambda\backslash\{t_0\}$ we have*

$$\chi((X_{t_0}\backslash Y)\cap \boldsymbol{B})-\chi((X_t\backslash Y)\cap \boldsymbol{B})$$
$$=(-1)^{n-1}\cdot \sum_{p^0\in X_{t_0}\cap \mathcal{N}_D}\Big[\mu(X_{t_0}, p^0)+\sum_{\Phi[\sigma]\subseteq \mathcal{N}_D,\ \overline{\Phi[\sigma]}\ni p^0}\mu(X_{t_0}\cap\overline{\Phi[\sigma]}, p^0)\Big].$$

Proof. (a) From (2.4), $\mathcal{N}_A\cup Y$ is a normal crossing divisor. By Corollary 3.3, we can choose $\varepsilon(p^0)$ sufficiently small such that $\mathcal{N}_A\cap \boldsymbol{B}(p^0)=\mathcal{N}_B\cap \boldsymbol{B}(p^0)=\mathcal{N}_C\cap \boldsymbol{B}(p^0)=\emptyset$. Hence

$$Y\cap \boldsymbol{B}(p^0)=\mathcal{N}_D\cap \boldsymbol{B}(p^0)\subseteq \overline{\mathcal{N}_D}\cap \boldsymbol{B}(p^0)\subseteq (\mathcal{N}_A\cup Y)\cap \boldsymbol{B}(p^0)=\mathcal{N}_D\cap \boldsymbol{B}(p^0),$$

and this proves part (a).

(b) For any point $p^0\in X_{t_0}\cap \mathcal{N}_D$ and for any $t\in \Lambda\backslash\{t_0\}$ we have:

$$\chi((X_{t_0}\backslash Y)\cap \boldsymbol{B}(p^0))-\chi((X_t\backslash Y)\cap \boldsymbol{B}(p^0))$$
$$=(\chi(X_{t_0}\cap \boldsymbol{B}(p^0))-\chi(X_t\cap \boldsymbol{B}(p^0)))-(\chi(X_{t_0}\cap Y\cap \boldsymbol{B}(p^0))-\chi(X_t\cap Y\cap \boldsymbol{B}(p^0))).$$

From [6] we have that $\chi(X_{t_0}\cap \boldsymbol{B}(p^0))-\chi(X_t\cap \boldsymbol{B}(p^0))=(-1)^{n-1}\cdot\mu(X_{t_0}, p^0)$ and from [5] we know that $X_t\cap Y\cap \boldsymbol{B}(p^0)$ has the homotopy type of a bouquet of $(n-2)$-sphere. The number of spheres in this bouquet is equal to

$$\sum_{\Phi[\sigma]\subseteq \mathcal{N}_D,\ \overline{\Phi[\sigma]}\ni p^0}\mu(X_{t_0}\cap\overline{\Phi[\sigma]}, p^0);$$

this formula is a consequence of the following result, proved, for example, in [14], where it is considered the more general case of an isolated singularity defined on an arrangement of hyperplanes. □

4.8. THEOREM. *Let $\mathcal{A}=\{H_1, \cdots, H_k\}$ be an arrangement of hyperplanes in $\boldsymbol{C}^n$ such that $0\in H_1\cap\cdots\cap H_k$ and let $\mathcal{L}(\mathcal{A})$ denote the intersection poset of $\mathcal{A}$. Let $f:(H_1\cup\cdots\cup H_k, 0)\to \boldsymbol{C}$ be a germ of a holomorphic function in the origin with the property that the restriction of f to any $X\in\mathcal{L}(\mathcal{A})$, $X\neq \boldsymbol{C}^n$, $X\neq\{0\}$, has an isolated critical point in 0. Let's denote by F the Milnor fiber of f. For $X\in\mathcal{L}(\mathcal{A})$, $X\neq \boldsymbol{C}^n$, $X\neq\{0\}$, let $\mu(f|_X)$ denote the Milnor number of the restriction of f to X; for $\{0\}\in\mathcal{L}(\mathcal{A})$ we put $\mu(f|_{\{0\}})=1$. Let $\mu:\mathcal{L}(\mathcal{A})\to \boldsymbol{Z}$ be the Möbius function of $\mathcal{L}(\mathcal{A})$. Then we have:*

$$\dim H_i(F)=\begin{cases} 0, & \text{for } i\neq n-2\\ \sum_{X\in\mathcal{L}(\mathcal{A}),\, X\neq \boldsymbol{C}^n}|\mu(X)|\cdot\mu(f|_X), & \text{for } i=n-2.\end{cases}$$

As a general reference on arrangements, we refer to [9].

5. Examples and remarks

5.1. Theorem (4.3) is similar to the results in [3], [10] and [12], since in all these cases the "singularities at infinity" are isolated.

5.2. The following lemma will help us to prove that Hypothesis 4.2 is satisfied, if for all $\Delta\in\mathscr{B}$ we have $\dim\Delta=n-1$.

LEMMA. *Let $f:\boldsymbol{C}^n\to\boldsymbol{C}$ be a Newton nondegenerate polynomial function such that $f(0)=0$. Suppose that the hypersurface $f^{-1}(t)\subseteqq\boldsymbol{C}^n$ has non-isolated singularities. Then either $t\neq0$ and there exists a bad face Δ of $\overline{\mathrm{supp}(f)}$ such that $t\in\Sigma_\Delta$, or $t=0$.*

Proof. Suppose that for some $t\in\boldsymbol{C}\setminus\{0\}$ the hypersurface $f^{-1}(t)$ has non-isolated singularities. Then there exists a Laurent series $p(s)\in f^{-1}(t)$, for $s>0$ sufficiently small, such that $\mathrm{grad}\, f(p(s))\equiv0$ and such that $\|p(s)\|\to\infty$ as $s\to0$. Suppose that

$$p(s)=(\alpha_{11}s^{a_1}+\alpha_{12}s^{a_1+1}+\cdots,\ \cdots,\ \alpha_{k1}s^{a_k}+\alpha_{k2}s^{a_k+1}+\cdots,\ 0,\ \cdots,\ 0)$$

with $\alpha_{11}\cdot\alpha_{21}\cdot\cdots\cdot\alpha_{k1}\neq0$, $a_1\leqq a_2\leqq\cdots\leqq a_k$, $a_1<0$, and let $H\subseteqq\boldsymbol{R}^n$ be the hyperplane of equation $a_1x_1+\cdots+a_kx_k=0$. We shall prove that $\Delta=H\cap\overline{\mathrm{supp}(f)}$ is a bad face of $\overline{\mathrm{supp}(f)}$.

Suppose that $k=n$ and let $l:\boldsymbol{R}^n\to\boldsymbol{R}$ be the function defined by $l(x)=a_1x_1+a_2x_2+\cdots+a_nx_n$; hence $l^{-1}(0)=H$. Let Δ^0 be the face of $\tilde{\Gamma}_-(f)$ where the restriction of l to $\tilde{\Gamma}_-(f)$ takes its minimal value, and let d^0 be this minimal value. We have $d^0\leqq0$ since $0\in\tilde{\Gamma}_-(f)$.

If Δ^0 is also a face of $\overline{\mathrm{supp}(f)}$, then the condition $\mathrm{grad}\, f(p(s))\equiv0$ gives us that $\mathrm{grad}\, f_{\Delta^0}(\alpha_{11},\cdots,\alpha_{n1})=0$, which is in contradiction with the nondegeneracy condition on the face Δ^0.

It remains that Δ^0 is not a face of $\overline{\mathrm{supp}(f)}$, hence $0\in\Delta^0$ and $d^0=0$. Consider now the restriction of l to $\overline{\mathrm{supp}(f)}$ and let d be the minimal value of this restriction. Since $\overline{\mathrm{supp}(f)}\subseteqq\tilde{\Gamma}_-(f)$, we have $d\geqq d^0=0$.

If $d>0$, then $f(p(s))\to0$ as $s\to0$, and this contradicts the hypothesis that $f(p(s))\equiv t\neq0$.

Hence we have that $d=0$. This implies that $a_n>0$, hence condition (ii) from the definition of a bad face is satisfied for $\Delta=H\cap\overline{\mathrm{supp}(f)}=\Delta^0\cap\overline{\mathrm{supp}(f)}$. Suppose now that condition (i) is not fulfilled. Then, as above, the nondegeneracy condition fails on the face Δ, a contradiction. Hence the condition (i) is also satisfied for the face Δ, i.e. Δ is a bad face of $\overline{\mathrm{supp}(f)}$. And now it is easy to see that $t=f(p(s))=f_\Delta(\alpha_1,\cdots,\alpha_n)$.

It remains to consider the case when $k<n$. By taking the restriction of f to $\boldsymbol{C}^k\times\{0\}$, we reduce the case $k<n$ to the situation $k=n$. We have only to remark that, for $t\neq0$, the condition $p(s)\in f^{-1}(t)$ implies that $\overline{\mathrm{supp}(f)}\cap(\boldsymbol{R}^k\times\{0\})\neq\emptyset$. $\square$

5.3. PROPOSITION. *Let f be a polynomial as in Theorem 1.3 and suppose, moreover, that for all $\Delta\in\mathscr{B}$ we have $\dim\Delta=n-1$. Then Hypothesis 4.2 is satisfied for all*

$$t\in\Big(\bigcup_{\Delta\in\mathscr{B}}\Sigma_{\Delta}\Big)\setminus\{0\}\setminus\Sigma_{f}.$$

In particular, we have

$$\Sigma_{f}\cup\bigcup_{\Delta\in\mathscr{B}}(\Sigma_{\Delta}\setminus\{0\})\subseteqq B_{f}\subseteqq\{0\}\cup\Sigma_{f}\cup\bigcup_{\Delta\in\mathscr{B}}\Sigma_{\Delta}.$$

Proof. Suppose that there exists $\Delta\in\mathscr{B}$ and $t_0\in\Sigma_{\Delta}\setminus\{0\}\setminus\Sigma_f$ such that Hypothesis 4.2 is not fulfilled for t_0. Then there exists $p^0\in X_{t_0}\cap\mathcal{N}_D$ and an orbit $\Phi[\sigma]$ of type (D), with $p^0\in\overline{\Phi[\sigma]}$, such that either p^0 is one of the non isolated singular points of $X_{t_0}\subseteqq M_K$ contained in $\overline{\Phi[\sigma]}$, or $X_{t_0}\cap\overline{\Phi[\sigma]}\subseteqq\overline{\Phi[\sigma]}$ has a non isolated singularity at p^0. In both cases, using chart (1), we obtain that the restriction g_σ of f^W (=the function which corresponds to f) to $\overline{\Phi[\sigma]}$ has a non isolated singularity in p^0, hence, by Lemma 5.2, there exists bad faces for the Newton polyhedron of g_σ. But this Newton polyhedron corresponds to $\Delta=\Delta^\sigma\cap\overline{\mathrm{supp}(f)}$=bad face of $\overline{\mathrm{supp}(f)}$; hence $\overline{\mathrm{supp}(f)}$ should have bad faces of codimension at least 2, included in Δ, which is in contradiction with our hypothesis. □

5.4. *Remark.* In fact, the same argument as in the proof of Proposition 5.3 gives us that for any bad face $\Delta\in\mathscr{B}$ of dimension $\dim\Delta=n-1$, we have that

$$\Sigma_{\Delta}\setminus\{0\}\setminus\Big(\bigcup_{\delta\subseteqq\Delta,\,\delta\in\mathscr{B}}\Sigma_{\delta}\Big)\subseteqq B_{f}.$$

5.5. *Example.* Let $f:\boldsymbol{C}^3\to\boldsymbol{C}$ be the polynomial function defined by $f(x,y,z)=x^2y^2z^2-2xyz+x^2+y^2$. It is easy to see that there exists three bad faces for $\overline{\mathrm{supp}(f)}$, namely Δ_1, with $f_{\Delta_1}=x^2y^2z^2-2xyz+x^2$, Δ_2, with $f_{\Delta_2}=x^2y^2z^2-2xyz+y^2$ and $\Delta=\Delta_1\cap\Delta_2$, with $f_{\Delta}=x^2y^2z^2-2xyz$. Next, we have: $\Sigma_f=\{0\}$, $\Sigma_{\Delta_1}=\Sigma_{\Delta_2}=\emptyset$ and $\Sigma_{\Delta}=\{-1\}$; hence $B_f\subseteqq\{0,-1\}$. We shall see that Theorem 4.3 gives us that $-1\in B_f$.

The vectors $v_\infty:=(-1,-1,1)$, $v_1:=(0,1,-1)$, $v_2:=(1,0,-1)$ and $v_0:=(0,0,1)$ are orthogonal to the faces of dimension 2 of $\tilde{\Gamma}_-(f)$ and generates the cones of dimension 1 in $\tilde{\Gamma}_-(f)^*$. We can obtain a unimodular simplicial subdivision of $\tilde{\Gamma}_-(f)$ by subdividing the cone generated in $(\boldsymbol{R}^3)^*$ by the vectors $\{v_0,v_1,v_2\}$, into the cones generated by the vectors $\{v_0,e_1,e_2\}$, $\{e_1,e_2,e_3\}$, $\{e_1,e_3,v_2\}$, $\{e_2,e_3,v_1\}$ and $\{v_1,v_2,e_3\}$, where

$$e_1:=(1,0,0),\quad e_2:=(0,1,0)\ \text{and}\ e_3:=(1,1,-1)=-v_\infty.$$

Let $\sigma\subseteqq(\boldsymbol{R}^3)^*$ be the cone generated by the vectors v_1, v_2 and e_3. Then the dual cone of $\sigma\subseteqq(\boldsymbol{R}^3)^*$ is the cone $\sigma^*\subseteqq\boldsymbol{R}^3$ generated by $m_1=(-1,0,-1)$, $m_2=(0,-1,-1)$ and $m_3=(1,1,1)$. Hence

$$M=\begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad W=M^{-1}=\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

By the isomorphism between $M_{\bar{\sigma}}$ and $\boldsymbol{C}_3^3$ given by the chart (1), the function f gives us

$$g:\boldsymbol{C}_3^3 \longrightarrow \boldsymbol{C}, \quad g(u_1, u_2, u_3)=u_3^2-2u_3+u_1^2u_3^2+u_2^2u_3^2.$$

It is easy to see that in the point $p^0=(0, 0, 1)\in X_{-1}$, the restrictions of g to the subspaces $\{u_1=u_2=0\}$, $\{u_1=0\}$, $\{u_2=0\}$ and the function g have an isolated singularity of type A_1. It follows, by Theorem 4.3, that $-1\in B_f$ and for $t\in \boldsymbol{C}\setminus B_f$ we have

$$\chi(f^{-1}(-1))-\chi(f^{-1}(t))=4.$$

5.6. Finally, we remark that when f is a convenient Newton nondegenerate polynomial, then, cf. [1], f is a *tame* polynomial, hence $B_f=\Sigma_f$.

References

[1] S. A. Broughton, Milnor numbers and the topology of polynomial hypersurfaces, Invent. Math., **92** (1988), 217-241.

[2] L. Fourrier, Topologie d'un polynôme de deux variables complexes au voisinage de l'infini, Preprint **39** (1994), Univ. Toulouse III.

[3] Há, Huy Vuí and Lê, Dung Tráng, Sur la topologie des polynômes complexes, Acta Math. Vietnam., **9** (1984), 21-32.

[4] A. G. Hovanski, Newton polyhedrons (resolution of singularities), Current problems in mathematics **22**, 207-239, Itogi Nauki i Tekhniki, Viniti AN SSSR, Moscow 1983.

[5] Lê, Dũng Tráng, Le concept de singularité isolée de fonction analytique, Complex Analytic Singularities, editors, T. Suwa and P. Wagreich, Adv. Stud. Pure, Math., 8, 1986, 215-227.

[6] J. Milnor, Singular Points of Complex Hypersurfaces, Ann. of Math. Stud. **61**, Princeton Univ. Press, 1968.

[7] A. Némethi and A. Zaharia, On the bifurcation set of a polynomial function and Newton boundary, Publ. Res. Inst. Math. Sci., **26** (1990), 681-689.

[8] M. Oka, Non-degenerate complete intersection singularity, manuscript.

[9] P. Orlik and H. Terao, Arrangements of hyperplanes, Grundelehren Math. Wiss., 300, Springer-Verlag, 1992.

[10] A. Parusiński, On the bifurcation set of a complex polynomial with isolated singularities at infinity, Compositio Math., **97** (1995), 369-384.

[11] F. Pham, Vanishing homologies and the n-variables saddlepoint method, Proc. Sympos. Pure Math., vol. 40, part 2, Amer. Math. Soc., 1983, 319-333.

[12] D. Siersma and M. Tibăr, Singularities at infinity and their vanishing cycles, Duke Math. J., **80** (1995), 771-783.

[13] R. Thom, Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc., **75** (1969), 240-284.

[14] M. IOACHIM ZAHARIA, The homology of the Milnor fiber of a function germ defined on an arrangement of hyperplanes, submitted to Saitama Math. J..

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