S. ROPPONGI
KODAI MATH. J.
18 (1995), 44—56

ON SINGULAR SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION

SUSUMU ROPPONGI

1. Introduction

Let Ω be a bounded domain in $\boldsymbol{R}^n$ $(n \geqq 2)$ with smooth boundary $\partial\Omega$. And let Σ be a C^∞-compact submanifold of Ω of dimension m $(0 \leqq m \leqq n-1)$. We take an arbitrary $\alpha(x) \in C^\infty(\Sigma)$ such that $\alpha(x) > 0$ on Σ and consider the following equation.

$$(1.1) \qquad \begin{cases} -\Delta u = u^p + \alpha\delta_\Sigma & \text{in } \mathcal{D}'(\Omega) \quad (p>1) \\ 0 \leqq u \in C^2(\Omega \setminus \Sigma), \end{cases}$$

where δ_Σ is the measure defined by

$$(1.2) \qquad \langle \delta_\Sigma, \eta \rangle = \int_\Sigma \eta(\sigma) d\sigma$$

for any $\eta \in C_0^\infty(\Omega)$.

What can one say about the existence of a solution of (1.1) and the local behaviour of its solution near Σ? We have the following.

THEOREM 1. *There exists a solution of* (1.1) *if and only if* $1 < p < (n-m)/(n-m-2)$ $(1 < p < \infty$ *if* $n-m \leqq 2)$. *And there exists a solution* u *of* (1.1) *satisfying*

$$(1.3) \qquad \begin{cases} C_1 d(x)^{-(n-m-2)} \leqq u(x) \leqq C_2 d(x)^{-(n-m-2)} & \textit{near } \Sigma \ (\textit{if } m \leqq n-3) \\ C_1 |\log d(x)| \leqq u(x) \leqq C_2 |\log d(x)| & \textit{near } \Sigma \ (\textit{if } m = n-2) \\ u(x) \in C^0(\Omega) \ (\textit{if } m = n-1), \end{cases}$$

where $d(x)$ *denotes the distance between* x *and* Σ. *Here* C_1 *and* C_2 *denote some positive constants.*

THEOREM 2. *Assume that* $p < n/(n-2)$ $(p < \infty$ *if* $n=2)$. *Then the same bounds as in* (1.3) *hold for any* u *satisfying* (1.1) *and, in addition,*

$$(1.4) \qquad u \in L^p_{\text{loc}}(\Omega) \quad \textit{if } m = n-1 .$$

Received May 2, 1992; revised February 21, 1994.

Remarks. 1. It is curious to the author that the exponent p such that (1.1) has a solution depends on the dimension of Σ. When $\Sigma=\{\text{a point}\}$, Lions [6] has proved the above results. Therefore we may assume that $m\geqq 1$ hereafter. When $p\in(1, (n+2)/(n-2))$ $(n\geqq 3)$, the following holds immediately from Gidas and Spruck [5, Theorem 3.1, pp. 540-541], since $-\Delta u=u^p$ in $\Omega\setminus\Sigma$.

$$u(x)\leqq C_2 d(x)^{-2/(p-1)} \quad \text{near } \Sigma \tag{1.5}$$

Since $p<(n-m)/(n-m-2)$, the upper bounds in (1.3) is more sharp than that in (1.5). When $p\geqq n/(n-2)$, we do not know the more sharp estimates than (1.5). Furthermore, when $p\geqq(n+2)/(n-2)$, we do not know the behaviour of the solution of (1.1) in general.

2. In the case $m\leqq n-2$, $u\in L^p_{\text{loc}}(\Omega)$ holds for any u satisfying (1.1) [see Lemma 2.1 in section 2]. We suspect that the assumption (1.4) is not necessary.

3. For the related papers we see Aviles [1], Brezis and Lions [3], Gidas and Spruck [5], Serrin [7] and the references in the above papers.

The other case where u^p is replaced by $-|u|^{p-1}u$ is discussed in, for example, Brezis and Veron [4], Vazquez and Veron [8], Veron [9], [10].

2. Asymptotics

Let Ω, Σ, $\alpha(x)$ be as before. Let $G(x, y)$ be the Green function of $-\Delta$ in Ω associated with the Dirichlet boundary condition. Then,

$$G(x, y)-S(x, y)=\begin{cases} K_n|x-y|^{2-n} & (\text{if } n\geqq 3) \\ -(1/2\pi)\log|x-y| & (\text{if } n=2), \end{cases}$$

where $S(x, y)\in C^\infty(\Omega\times\Omega)$ and

$$K_n=((n-2)|S^{n-1}|)^{-1} \quad (n\geqq 3).$$

Here $|S^{n-1}|$ denotes the surface area of the unit sphere of $\boldsymbol{R}^n$.

We put

$$g(x)=\int_\Sigma G(x, \sigma)\alpha(\sigma)d\sigma. \tag{2.1}$$

Then $0\leqq g\in C^\infty(\bar{\Omega}\setminus\Sigma)$ and g satisfies

$$\begin{cases} -\Delta g=\alpha\delta_\Sigma & \text{in } \mathscr{D}'(\Omega) \\ g=0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

By Propositions A.2 and A.3 in Appendix, we have

$$\begin{cases} C_1 d(x)^{-(n-m-2)}-D_1\leqq g(x)\leqq C_2 d(x)^{-(n-m-2)} & (\text{if } m\leqq n-3) \\ C_1|\log d(x)|-D_1\leqq g(x)\leqq C_2|\log d(x)|+D_2 & (\text{if } m=n-2) \\ g(x)\in C^0(\bar{\Omega}) \quad (\text{if } m=n-1) \end{cases} \tag{2.3}$$

$$\text{(2.4)}\qquad \begin{cases} g\in L^q_{\mathrm{loc}}(\Omega) & \text{for } q\in[1, (n-m)/(n-m-2)) \quad (\text{if } m\leqq n-3) \\ g\notin L^q_{\mathrm{loc}}(\Omega) & \text{for } q\in[(n-m)/(n-m-2), \infty) \quad (\text{if } m\leqq n-3) \end{cases}$$

$$\text{(2.5)}\qquad g\in L^q_{\mathrm{loc}}(\Omega) \quad \text{for } q\in[1, \infty) \ (\text{if } m=n-2).$$

Here C_1, C_2, D_1 and D_2 denote some positive constants.

LEMMA 2.1. *Assume that* $m\leqq n-2$ *and* u *satisfies*

$$\text{(2.6)}\qquad \begin{cases} -\Delta u=u^p & \text{in } \Omega\setminus\Sigma \quad (p>1) \\ 0\leqq u\in C^2(\Omega\setminus\Sigma). \end{cases}$$

Then $u\in L^p_{\mathrm{loc}}(\Omega)$.

Proof. We take a C^∞-convex function Φ on $[0, \infty)$ such that $\Phi(0)=1$, $\Phi(t)=0$ for $t\geqq 1$ and put

$$\xi_\varepsilon(x)=\begin{cases} \Phi(C_1^{-1}\varepsilon^{n-m-2}(g(x)+D_1)) & (\text{if } m\leqq n-3) \\ \Phi(C_1^{-1}|\log\varepsilon|^{-1}(g(x)+D_1)) & (\text{if } m=n-2). \end{cases}$$

Here C_1 and D_1 denote the same constants as in (2.3). Then, by (2.2) and (2.3), we can easily get

$$\text{(2.7)}\qquad \begin{array}{ll} \xi_\varepsilon(x)\in C^\infty(\Omega), \quad 0\leqq\xi_\varepsilon(x)\leqq 1 & \text{on } \Omega \\ \xi_\varepsilon(x)\to 1 & \text{as } \varepsilon\to 0 \ \text{ a.e. in } \Omega \\ \xi_\varepsilon(x)=0 & \text{if } d(x)<\varepsilon \\ \nabla\xi_\varepsilon(x)\to 0 & \text{as } \varepsilon\to 0, \text{ uniformly on any compact subsets of } \Omega\setminus\Sigma \\ \Delta\xi_\varepsilon(x)\geqq 0 & \text{a.e. in } \Omega. \end{array}$$

Let $\eta\in C_0^\infty(\Omega)$ such that $0\leqq\eta\leqq 1$, $\eta=1$ near Σ. Since $\eta\xi_\varepsilon\in C_0^\infty(\Omega\setminus\Sigma)$, we have

$$\text{(2.8)}\qquad \begin{aligned} \int_\Omega u^p\eta\xi_\varepsilon dx &= -\int_\Omega u\Delta(\eta\xi_\varepsilon)dx \\ &= -\int_\Omega u(\eta\Delta\xi_\varepsilon+2\nabla\eta\cdot\nabla\xi_\varepsilon+\xi_\varepsilon\Delta\eta)dx \\ &\leqq -\int_\Omega u(2\nabla\eta\cdot\nabla\xi_\varepsilon+\xi_\varepsilon\Delta\eta)dx. \end{aligned}$$

Notice that both $\nabla\eta$ and $\Delta\eta$ vanish near Σ. Thus, by using (2.7) and Fatou's Lemma with (2.8), we get

$$\int_\Omega u^p\eta dx\leqq -\int_\Omega u\Delta\eta dx<+\infty.$$

Since $u\in C^2(\Omega\setminus\Sigma)$, this implies $u\in L^p_{\mathrm{loc}}(\Omega)$. q. e. d.

Now we can get the lower bounds of u satisfying (1.1) and (1.4).

LEMMA 2.2. *Fix an arbitrary smooth domain Ω' satisfying $\Sigma\Subset\Omega'\Subset\Omega$. Then, for any u satisfying* (1.1) *and* (1.4),

$$u(x)\geqq g(x)-C \qquad x\in\bar{\Omega}'\setminus\Sigma \tag{2.9}$$

holds for a positive constant C.

Moreover $p<(n-m)/(n-m-2)$ holds if (1.1) *has a solution and if $m\leqq n-3$.*

Proof. By (1.1), (1.4), (2.2), (2.3), (2.4), (2.5) and Lemma 2.1,

$$0\leqq -\Delta(u-g)\in L^1_{\mathrm{loc}}(\Omega), \qquad u-g\in L^1_{\mathrm{loc}}(\Omega), \tag{2.10}$$

and $u-g\in C^2(\Omega\setminus\Sigma)$. Thus we get (2.9) by the maximum principle.

Assume that $p\geqq(n-m)/(n-m-2)$ holds. Since $0\leqq g(x)\leqq u(x)+C\in L^p_{\mathrm{loc}}(\Omega)$, $g\in L^q_{\mathrm{loc}}(\Omega)$ holds with $q=(n-m)/(n-m-2)$. But this contradicts (2.4). Therefore $p<(n-m)/(n-m-2)$ holds if $m\leqq n-3$. q. e. d.

Next we consider the upper bounds of u satisfying (1.1) and (1.4). We recall (2.10). Therefore, by using L^1-elliptic regularity theory (see, for example, Benilan, Brezis and Crandall [2, Appendix, pp. 547-555]), we get the following.

LEMMA 2.3. *For any u satisfying* (1.1) *and* (1.4), *$u\in L^q_{\mathrm{loc}}(\Omega)$ holds for any $q\in[1, n/(n-2))$ ($q\in[1, \infty)$ if $n=2$).*

LEMMA 2.4. *Assume that $n=2$ and $m=1$. Then, for any u satisfying* (1.1) *and* (1.4), *$u\in C^0(\Omega)$ holds.*

Proof. We fix an arbitrary $q>p$. By (1.1), (1.4), (2.2) and Lemma 2.3, $-\Delta(u-g)=u^p\in L^{q/p}_{\mathrm{loc}}(\Omega)$. Thus, by using the Sobolev embedding, $u-g\in W^{2,q/p}_{\mathrm{loc}}(\Omega)\subset C^0(\Omega)$. Since $g\in C^0(\bar{\Omega})$, $u\in C^0(\Omega)$ holds. q. e. d.

We introduce the following function $h(x)$ for the case $n\geqq 3$.

$$h(x)=\int_\Sigma |x-\sigma|^{-\tau}d\sigma \quad x\in\bar{\Omega}\setminus\Sigma \qquad (n\geqq 3), \tag{2.11}$$

where

$$\tau=\begin{cases} m-2+(n-m-2)p & (\text{if } m\leqq n-3)\\ m-1/2 & (\text{if } m=n-2)\\ m-3/2 & (\text{if } m=n-1). \end{cases}$$

Then we have the following.

LEMMA 2.5. *Let $h(x)$ be as in* (2.11). *Assume that $n\geqq 3$ and that $p<(n-m)/(n-m-2)$ if $m\leqq n-3$. Then*

$$(2.12)\quad \begin{cases} h(x)\in C^0(\bar{\Omega}) & (\text{if } m\geqq n-2, \text{ or if } m=n-3 \text{ and } p\in(1,2)) \\ h(x)\leqq C|\log d(x)|+D & (\text{if } m=n-3 \text{ and } p=2) \\ h(x)\leqq Cd(x)^{2-(n-m-2)p} & (\text{if } m\leqq n-4, \text{ or if } m=n-3 \text{ and } p\in(2,3)), \end{cases}$$

$$(2.13)\qquad -\Delta h(x)\geqq \begin{cases} Cd(x)^{-(n-m-2)p} & (\text{if } m\leqq n-3) \\ Cd(x)^{-3/2} & (\text{if } m=n-2) \\ Cd(x)^{-1/2} & (\text{if } m=n-1), \end{cases}$$

and

$$(2.14)\qquad 0\leqq g(x)^p\leqq -C\Delta h(x)$$

hold for $x\in\bar{\Omega}\setminus\Sigma$. Here C and D denote some positive constants.

Proof. We can immediately get (2.12) from (2.11) and Proposition A.2 in Appendix. Differentiating (2.11), we see

$$(2.15)\qquad -\Delta h(x)=\tau(n-2-\tau)\int_{\Sigma}|x-\sigma|^{-(\tau+2)}d\sigma \qquad x\in\bar{\Omega}\setminus\Sigma\,,$$

where

$$\tau(n-2-\tau)=\begin{cases} ((n-m-2)p+m-2)(n-m-p(n-m-2)) & (\text{if } m\leqq n-3) \\ \frac{1}{2}\left(n-\frac{5}{2}\right) & (\text{if } m=n-2, \text{ or if } m=n-1). \end{cases}$$

We recall that $m\geqq 1$ and $p>1$. Thus $\tau(n-2-\tau)>0$ holds. And we can easily get (2.13) from (2.15) and Proposition A.2. Furthermore (2.14) easily follows from (2.3) and (2.13). q.e.d.

Now we have the following.

LEMMA 2.6. *Assume that $n\geqq 3$ and $p<n/(n-2)$. Fix an arbitrary smooth domain Ω' satisfying $\Sigma\Subset\Omega'\Subset\Omega$. Then, for any u satisfying* (1.1) *and* (1.4),

$$(2.16)\qquad u(x)\leqq g(x)+Ch(x)+C \qquad x\in\bar{\Omega}'\setminus\Sigma$$

holds for some positive constant C.

Proof. We fix an arbitrary $q\in(p, n/(n-2))$. Then, by Lemma 2.3, $u\in L^q(\Omega')$. We put

$$(2.17)\qquad u=u_0+g\,,$$

where u_0 satisfies

(2.18)
$$\begin{cases} -\Delta u_0 = u^p & \text{in } \Omega' \\ u_0 = u & \text{on } \partial\Omega'. \end{cases}$$

Since $0 \leqq u \in C^2(\bar{\Omega}' \setminus \Sigma)$ and $u^p \in L^{q(0)}(\Omega')$ with $q(0) = q/p > 1$, (2.18) has a unique non-negative solution $u_0 \in W^{2,q(0)}(\Omega')$. By the Sobolev embedding,

$$u_0 \in W^{2,q(0)}(\Omega') \subset L^{r(0)}(\Omega'),$$

where

$$r(0) = nq(0)/(n - 2q(0)) > n/(n-2).$$

Let u_1 be the solution of

(2.19)
$$\begin{cases} -\Delta u_1 = (u_0)^p & \text{in } \Omega' \\ u_1 = u & \text{on } \partial\Omega'. \end{cases}$$

Since $u_0 \in L^{r(0)}(\Omega')$ holds with $r(0) > n/(n-2) > p$, the same argument as above implies

$$u_1 \in W^{2,q(1)}(\Omega') \subset L^{r(1)}(\Omega'),$$

where

$$q(1) = r(0)/p > 1,$$

$$r(1) = nq(1)/(n - 2q(1)) > n/(n-2).$$

Furthermore, by (2.14), (2.17), (2.18) and (2.19),

$$\begin{aligned} -\Delta u_0 = u^p &\leqq (u_0 + g)^p \\ &\leqq 2^{p-1}((u_0)^p + g^p) \leqq -C\Delta u_1 - C\Delta h \qquad \text{a.e. in } \Omega' \end{aligned}$$

and

$$u_0 - Cu_1 - Ch = -(C-1)u - Ch \leqq 0 \qquad \text{on } \partial\Omega'$$

hold, where $C > 1$ denotes some positive constant. Thus, by the maximum principle,

$$0 \leqq u_0(x) \leqq C_1 u_1(x) + C_1 h(x) \qquad \text{a.e. in } \Omega'$$

hold for some positive constant $C_1 > 1$.

By (2.3) and (2.12), $h(x) \leqq \text{const.}\ g(x)$ holds for $x \in \Omega' \setminus \Sigma$. Observing this fact and (2.14), we have

(2.20)
$$0 \leqq u_0(x) \leqq C_j u_j(x) + C_j h(x) \qquad \text{a.e. in } \Omega'$$

for $j \geqq 1$, where C_j denotes some positive constant and a sequence of functions $\{u_j\}_{j \geq 1}$ is defined by letting u_{j+1} be a unique solution of

$$\begin{cases} -\Delta u_{j+1} = (u_j)^p & \text{in } \Omega' \\ u_{j+1} = u & \text{on } \partial\Omega' \end{cases}$$

inductively for $j\geqq 0$. Furthermore,

$$u_j\in W^{2,q(j)}(\Omega')\subset L^{r(j)}(\Omega')$$

hold for $j\geqq 0$, where $q(0)=q/p$ and

$$q(j)=r(j-1)/p\ , \qquad r(j)=nq(j)/(n-2q(j))$$

for $j\geqq 1$. Since $q>p$, we can easily see

$$\begin{aligned} 1/q(j)-2/n &= 2/n(p-1)-(2/n(p-1)-1/q)p^j \\ &< 2/n(p-1)-(2/n(p-1)-1/p)p^j \\ &= (2-(n-(n-2)p)p^{j-1})/n(p-1) \end{aligned}$$

for $j\geqq 0$. We recall that $p<n/(n-2)$. Thus, $2q(k)>n$ holds for some positive integer k. By the Sobolev embedding,

$$u_k\in W^{2,q(k)}(\Omega')\subset C^0(\bar{\Omega}')\,. \tag{2.21}$$

By (2.17), (2.20) and (2.21), we get (2.16). q. e. d.

Now we are in a position to prove Theorem 2. From (2.3), Lemmas 2.2, 2.4, 2.5 and 2.6, we can immediately get (1.3) for the case $m\leqq n-2$ or the case $n=2$ and $m=1$. Therefore we only treat the case $m=n-1$ and $n\geqq 3$. We take an arbitrary u satisfying (1.1) and (1.4). Then, by (2.2), Lemmas 2.5 and 2.6,

$$-\Delta(u-g)=u^p\in L^\infty_{\mathrm{loc}}(\Omega)\,, \qquad u-g\in C^2(\Omega\setminus\Sigma)\,.$$

Thus $u-g\in C^0(\Omega)$ holds. Since $g\in C^0(\bar{\Omega})$ by (2.3), $u\in C^0(\Omega)$. Now we get the desired Theorem 2.

3. Existence of a solution

Let u be a solution of (1.1) satisfying (1.3). We take an arbitrary $\lambda>0$ and put $v=\lambda^{-1/(p-1)}u$, $\beta=\lambda^{-p/(p-1)}\alpha$. Then v satisfies

$$\begin{cases} -\Delta v=\lambda(v^p+\beta\delta_\Sigma) & \text{in } \mathcal{D}'(\Omega) \\ 0\leqq v\in C^2(\Omega\setminus\Sigma) & (p>1,\ \lambda>0) \end{cases} \tag{3.1}$$

and the same bounds as in (1.3). Therefore we treat (3.1) hereafter. We put

$$\bar{g}(x)=\int_\Sigma G(x,\sigma)\beta(\sigma)d\sigma\,. \tag{3.2}$$

Then $0\leqq\bar{g}\in C^\infty(\bar{\Omega}\setminus\Sigma)$ and $\bar{g}$ satisfies

$$\begin{cases} -\Delta \bar{g} = \beta \delta_\Sigma & \text{in } \mathcal{D}'(\Omega) \\ \bar{g} = 0 & \text{on } \partial\Omega \end{cases}$$

and the same properties as in (2.3), (2.4) and (2.5).

At first we construct a supersolution of (3.1).

LEMMA 3.1. *Let $n \geqq 3$ and $h(x)$ be as in* (2.11). *Then there exist $\lambda > 0$ and $\bar{v}$ satisfying*

$$0 \leqq \bar{v}(x) = \lambda(\bar{g}(x) + h(x)) \in C^\infty(\bar{\Omega} \setminus \Sigma), \tag{3.3}$$

$$-\Delta \bar{v} \geqq \lambda(\bar{v}^p + \beta \delta_\Sigma) \quad \text{in } \mathcal{D}'(\Omega). \tag{3.4}$$

Furthermore $\bar{v}$ satisfies the same bounds as in (1.3).

Proof. We only treat the case $m \leqq n-4$, since the other cases can be treated similarly. We put

$$\bar{v}(x) = A(\bar{g}(x) + h(x)) \qquad x \in \bar{\Omega} \setminus \Sigma, \tag{3.5}$$

where $A > 0$ is some constant which will be defined later. Then $\bar{v}$ satisfies (3.3) and

$$-\Delta \bar{v} = A(\beta \delta_\Sigma - \Delta h) \qquad \text{in } \mathcal{D}'(\Omega). \tag{3.6}$$

By Lemma 2.2, we may assume that $p < (n-m)/(n-m-2)$ holds. Thus, by (2.3), (2.12), (2.13) and (3.5),

$$\begin{aligned} \bar{v}^p(x) &\leqq A^p 2^{p-1}(\bar{g}(x)^p + h(x)^p) \\ &\leqq A^p 2^{p-1}(C_2 d(x)^{-(n-m-2)p} + C d(x)^{(2-(n-m-2)p)p}) \\ &= A^p 2^{p-1} d(x)^{-(n-m-2)p}(C_2 + C d(x)^{(n-m-(n-m-2)p)p}) \\ &\leqq C_3 A^p d(x)^{-(n-m-2)p} \leqq B A^p(-\Delta h(x)) \end{aligned} \tag{3.7}$$

hold for $x \in \bar{\Omega} \setminus \Sigma$, where C, C_2, C_3 and B denote some positive constant independent of A. By (3.6) and (3.7) we have

$$-\Delta \bar{v} \geqq B A^{1-p} \bar{v}^p + A \beta \delta_\Sigma \qquad \text{in } \mathcal{D}'(\Omega).$$

Therefore we get (3.4) if we choose $\lambda = A = B^{1/p}$. q.e.d.

LEMMA 3.2. *Assume that $n=2$ and $m=1$. Then there exist $\lambda > 0$ and $\bar{v}$ satisfying*

$$\begin{cases} -\Delta \bar{v} \geqq \lambda(\bar{v}^p + \beta \delta_\Sigma) & \text{in } \mathcal{D}'(\Omega) \\ 0 \leqq \bar{v} \in C^0(\bar{\Omega}). \end{cases} \tag{3.8}$$

Proof. We put

(3.9) $$w(x)=\int_{\Sigma}(\log(R|x-\sigma|^{-1}))^{1/2}d\sigma \qquad x\in\bar{\Omega}\setminus\Sigma .$$

Here R denotes the diameter of Ω. By Proposition A.3 in Appendix, $0\leqq w(x)\in C^0(\bar{\Omega})$. Differentiating (3.9), we have

$$-\Delta w(x)=4^{-1}\int_{\Sigma}|x-\sigma|^{-2}(\log(R|x-\sigma|^{-1}))^{-3/2}d\sigma$$

for $x\in\bar{\Omega}\setminus\Sigma$. Since $0<\log t\leqq 3t^{1/3}$ hold for any $t>1$,

$$0<\log(R|x-\sigma|^{-1})\leqq 3R^{1/3}|x-\sigma|^{-1/3}$$

hold for any $x\in\bar{\Omega}\setminus\Sigma$ and any $\sigma\in\Sigma$. Thus, by Proposition A.2 in Appendix, we have

(3.10) $$-\Delta w(x)\geqq C_1\int_{\Sigma}|x-\sigma|^{-3/2}d\sigma\geqq C_2 d(x)^{-1/2}\geqq C_3>0$$

for $x\in\bar{\Omega}\setminus\Sigma$. Here C_1, C_2 and C_3 denote some positive constants.

We put

(3.11) $$\bar{v}(x)=A(\bar{g}(x)+w(x)) \qquad x\in\bar{\Omega}\setminus\Sigma ,$$

where $A>0$ is some positive constant which will be defined later. Since $0\leqq\bar{g}\in C^0(\bar{\Omega})$, $0\leqq\bar{v}\in C^0(\bar{\Omega})$ and $\bar{v}(x)^p\leqq A^pC$ $(x\in\bar{\Omega})$ hold for some positive constant C independent of A. By (3.10) and (3.11), we have

$$\begin{aligned}-\Delta\bar{v}&=A(\beta\delta_{\Sigma}-\Delta w)\\ &\geqq AC_3+A\beta\delta_{\Sigma}\\ &\geqq A^{1-p}C_3C^{-1}\bar{v}^p+A\beta\delta_{\Sigma} \qquad \text{in } \mathcal{D}'(\Omega).\end{aligned}$$

Therefore we get (3.8) if we choose $\lambda=A=(C_3/C)^{1/p}$. q. e. d.

Now we are in a position to prove Theorem 1. Let $\bar{v}$ be as in Lemma 3.1 (resp. Lemma 3.2) for the case $n\geqq 3$ (resp. $n=2$ and $m=1$). We define a sequence of functions $\{v_j\}_{j\geqq 0}$ by $v_0=0$ and by letting v_{j+1} be a unique solution of

$$\begin{cases}-\Delta v_{j+1}=\lambda((v_j)^p+\beta\delta_{\Sigma}) & \text{in } \mathcal{D}'(\Omega)\\ \quad v_{j+1}=0 & \text{on } \partial\Omega ,\end{cases}$$

inductively. It is easy to see by induction that

$$0\leqq v_j(x)\leqq v_{j+1}(x)\leqq\bar{v}(x) \qquad \text{a.e. in } \Omega$$

for $j\geqq 0$. Thus, $v_j(x)\to v(x)$ $(j\to\infty)$ a.e. in Ω, which is a solution of (3.1) and satisfies

(3.12) $$\lambda\bar{g}(x)=v_1(x)\leqq v(x)\leqq\bar{v}(x) \qquad \text{a.e. in } \Omega .$$

From (2.3), (3.12), Lemmas 3.1 and 3.2, we can easily get Theorem 1.

4. Appendix

Let Ω, Σ be as in Introduction. At first we consider the following integral.

(A.1) $$I(x)=\int_{\Sigma}|x-\sigma|^{-s}d\sigma\,,\qquad x\in\bar{\Omega}\setminus\Sigma\quad(s\in(0,\infty)).$$

Then we have the following.

LEMMA A.1. *We fix an arbitrary* $a\in\Sigma$. *Then there exists a small* $\varepsilon>0$ *such that*

(A.2) $$C_1d(x)^{m-s}\leqq I(x)\leqq C_2d(x)^{m-s}+D_2\qquad (\textit{if } s\neq m)$$

(A.3) $$-C_1(\log d(x))-D_1\leqq I(x)$$
$$\leqq -C_2(\log d(x))+D_2\qquad (\textit{if } s=m)$$

hold for any $x\in B\ (\varepsilon\,;\,a)$. *Here* C_1, C_2, D_1 *and* D_2 *are some positive constants independent of* x, ε *and* $B(\varepsilon\,;\,a)$ *denotes the ball of radius* ε *with the center* a.

Proof. When $m=n-2$, the first inequality in (A.3) is proved in Vazquez and Veron [8, Lemma 2.3, pp. 129-130]. Therefore we use the same notations as in [8].

We fix $a\in\Sigma$ and set $B_\eta=B(\eta\,;\,a)$, $\Sigma_\eta=\Sigma\cap B_\eta$ for $\eta>0$. And we put

(A.4) $$I(x)=I_1(x)+I_2(x),$$

where

$$I_1(x)=\int_{\Sigma_\eta}|x-\sigma|^{-s}d\sigma$$

$$I_2(x)=\int_{\Sigma\setminus\Sigma_\eta}|x-\sigma|^{-s}d\sigma\,.$$

There exists a local diffeomorphism from an open subset $G\subset\boldsymbol{R}^n$ onto B_η such that $\Psi(0)=a$ and $\Psi(\omega)=\Sigma_\eta$ if $\omega=G\cap\boldsymbol{R}^m$. And the restriction $\bar{\Psi}$ of Ψ to ω is a parametrization of Σ_η. If $y=(\bar{y},\rho)\in\boldsymbol{R}^m\times\boldsymbol{R}^{n-m}$, $\bar{\Psi}(\bar{y})=\Psi(\bar{y},0)$. Thus we have

(A.5) $$I_1(x)=\int_{\omega}J(\bar{y})|x-\bar{\Psi}(\bar{y})|^{-s}d\bar{y}\,,$$

where

$$J(\bar{y})=\left|\det\left\langle\frac{\partial\bar{\Psi}}{\partial\bar{y}_i},\frac{\partial\bar{\Psi}}{\partial\bar{y}_j}\right\rangle\right|^{1/2}.$$

As $\bar{\Psi}$ is a parametrization of Σ_η, we may assume that

(A.6) Ψ and Ψ^{-1} are uniformly Lipschitz continuous

(A.7) $$0<C_1\leqq J(\bar{y})\leqq C_2$$

(A.8) $$\{y\in \boldsymbol{R}^n;\ |y|\leqq b\}\subset G\subset\{y\in\boldsymbol{R}^n;\ |y|\leqq c\}.$$

Here C_1, C_2, b and c denote some positive constants.

We take $\varepsilon>0$ such that $0<\varepsilon<\eta/2$. And we take an arbitrary $x\in B_\varepsilon$ and write $x=\Psi(\bar{z},\rho)$, $\bar{z}\in\boldsymbol{R}^m$, $\rho\in\boldsymbol{R}^{n-m}$. If ε is small enough,

(A.9) $$|\bar{z}|^2+|\rho|^2<(b/2)^2$$

holds. By (A.8) and (A.9), we have the following.

(A.10) $$\{\bar{y}\in\boldsymbol{R}^m;\ |\bar{y}|<b/2\}\subset\{\bar{y}\in\boldsymbol{R}^m;\ |\bar{y}-\bar{z}|<b\}$$

(A.11) $$\{\bar{y}\in\boldsymbol{R}^m;\ |\bar{y}-\bar{z}|<c\}\subset\{\bar{y}\in\boldsymbol{R}^m;\ |\bar{y}|<2c\}$$

Summing up (A.5), (A.6), (A.7), (A.10) and (A.11),

(A.12) $$C_3I_3(x)\leqq I_1(x)\leqq C_4I_4(x)$$

hold, where C_3 and C_4 are some positive constants and

$$I_3(x)=\int_0^{b/2}r^{m-1}(r^2+|\rho|^2)^{-s/2}dr\,,$$

$$I_4(x)=\int_0^{2c}r^{m-1}(r^2+|\rho|^2)^{-s/2}dr\,.$$

Since $|\rho|<b/2<2c$ hold from (A.8) and (A.9),

(A.13) $$I_4(x)\leqq\int_0^{|\rho|}r^{m-1}|\rho|^{-s}dr+\int_{|\rho|}^{2c}r^{m-s-1}dr$$
$$\leqq\begin{cases} C|\rho|^{m-s}+D & (\text{if } s\neq m)\\ C\log(2c/|\rho|)+D & (\text{if } s=m)\,,\end{cases}$$

(A.14) $$I_3(x)\geqq\int_0^{|\rho|}r^{m-1}(r^2+|\rho|^2)^{-s/2}dr$$
$$\geqq 2^{-s/2}|\rho|^{-s}\int_0^{|\rho|}r^{m-1}dr=m^{-1}2^{-s/2}|\rho|^{m-s}\qquad (s\in(0,\infty))$$

and

(A.15) $$I_3(x)\geqq\int_{|\rho|}^{b/2}r^{m-1}(r^2+|\rho|^2)^{-s/2}dr$$
$$\geqq 2^{-s/2}\int_{|\rho|}^{b/2}r^{m-s-1}dr=2^{-m/2}\log(b/2|\rho|)\qquad(\text{if } s=m)$$

hold for some positive constants C and D. By the way, from (A.6), (A.16) $$C_5 d(x) \leqq |\rho| \leqq C_6 d(x)$$

hold for some positive constants C_5 and C_6.

By (A.12), (A.13), (A.14), (A.15) and (A.16), we can see that $I_1(x)$ satisfies the same bounds as in (A.2) and (A.3) for any $x \in B(\varepsilon ; a)$. On the other hand,

$$0 < I_2(x) \leqq (2/\eta)^s |\Sigma|$$

hold for any $x \in B(\varepsilon ; a)$, since $0 < \varepsilon < \eta/2$ and

$$|x-\sigma| \geqq |\sigma - a| - |x-a| \geqq \eta - \varepsilon > \eta/2$$

hold for any $x \in B(\varepsilon ; a)$ and any $\sigma \in \Sigma \setminus \Sigma_\eta$. Therefore we get the desired results. q.e.d.

Now we have the following.

PROPOSITION A.2. *Let $I(x)$ be as in* (A.1).

i) *If $s > m$,*

$$C_1 d(x)^{m-s} \leqq I(x) \leqq C_2 d(x)^{m-s} \qquad x \in \bar{\Omega} \setminus \Sigma ,$$

$$I \in L^q(\Omega) \qquad \text{for any } q \in (0, (n-m)/(s-m)) ,$$

$$I \notin L^q(\Omega) \qquad \text{for any } q \in [(n-m)/(s-m), \infty) .$$

ii) *If $s = m$,*

$$C_1 |\log d(x)| - D_1 \leqq I(x) \leqq C_2 |\log d(x)| + D_2 \qquad x \in \bar{\Omega} \setminus \Sigma ,$$

$$I \in L^q(\Omega) \qquad \text{for any } q \in (0, \infty) .$$

iii) *If $s < m$, $I \in C^0(\bar{\Omega})$.*

Here C_1, C_2, D_1 and D_2 denote some positive constants.

Proof. At first we treat the case $s < m$. Fix an arbitrary $a \in \Sigma$. Then, by (A.2), $0 < I(x) \leqq C$ holds for any $x \in B(\varepsilon ; a)$. Here C is a positive constant independent of x and ε. Thus, by using Fatou's Lemma, $I(x) \to I(a)$ as $x \to a$, Since $a \in \Sigma$ is arbitrary, we get iii).

Next we treat the case $s \geqq m$. Using the compactness of Σ, (A.2) and (A.3) remain valid in some neighbourhood of Σ. Since $I(x) \in C^\infty(\bar{\Omega} \setminus \Sigma)$, (A.2) and (A.3) remain valid in $\bar{\Omega}$.

Notice the following formula in Weyl [11].

(A.17) $$\int_{d(x)<\varepsilon} 1 dx = |B_1^{n-m}| |\Sigma| \varepsilon^{n-m} + o(\varepsilon^{n-m+1}) \qquad \text{as } \varepsilon \to 0 .$$

Here $|B_1^{n-m}|$ denotes the volume of the unit ball in $\boldsymbol{R}^{n-m}$.

From (A.2), (A.3) and (A.17), we can easily get i) and ii). q.e.d.

Next we consider the following integral.

(A.18) $$K(x)=\int_{\Sigma}(\log(R|x-\sigma|^{-1}))^{s}d\sigma \qquad x\in\bar{\Omega}\setminus\Sigma \quad (s\in(0, \infty)),$$

where R denotes the diameter of Ω.
Since $|x-\sigma|<R$ for $\sigma\in\Sigma$, $x\in\bar{\Omega}$ and $0\leqq\log t\leqq 2st^{1/(2s)}$ for $t\geqq 1$, we get

(A.19) $$0\leqq K(x)\leqq(2s)^{s}R^{1/2}\int_{\Sigma}|x-\sigma|^{-1/2}d\sigma .$$

We fix an arbitrary $a\in\Sigma$. Then, by (A.1), (A.2) and (A.19),

$$0\leqq K(x)\leqq C_2 d(x)^{m-1/2}+D_2\leqq C$$

hold for any $x\in B(\varepsilon; a)$. Here C denotes some positive constant. Therefore the same argument as in the proof of Proposition A.2 yields the following.

PROPOSITION A.3. *Let* $K(x)$ *be as in* (A.18). *Then,* $K(x)\in C^{0}(\bar{\Omega})$ *holds for any* $s\in(0, \infty)$.

REFERENCES

[1] P. AVILES, On isolated singularities in some nonlinear partial differential equations, Indiana Univ. Math. J., **32** (1983), 773-791.

[2] P. BENILAN, H. BREZIS AND M.G. CRANDALL, A semilinear equation in $L^1(\boldsymbol{R}^N)$, Ann. Scuola Norm. Sup. Pisa, **2** (1975), 523-555.

[3] H. BREZIS AND P.L. LIONS, A note on isolated singularities for linear elliptic equations, Mathematical Analysis and Applications, Part A, Advances in Mathematics Supplementary Studies, **74**, Academic Press, 1980, 263-266.

[4] H. BREZIS AND L. VERON, Removable singularities for some nonlinear elliptic equations, Arch. Ratiotal Mech. Anal., **75** (1980), 1-6.

[5] B. GIDAS AND J. SPRUCK, Global and local behaviour of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math., **34** (1981), 525-598.

[6] P.L. LIONS, Isolated singularities in semilinear problems, J. Differential Equations, **38** (1980), 441-450.

[7] J. SERRIN, Isolated singularities of solutions of quasilinear equations, Acta Math., **113** (1965), 219-240.

[8] J.L. VAZQUEZ AND L. VERON, Singularities of elliptic equations with an exponential nonlinearity, Math. Ann., **296** (1984), 119-135.

[9] L. VERON, Singular solutions of some nonlinear elliptic equations, Nonlinear Anal., **5** (1981), 225-242.

[10] L. VERON, Singularités éliminable d'équations elliptics nonlinéaris, J. Differential Equations, **41** (1981), 87-95.

[11] H. WEYL, On the volume of tubes, Amer. J. Math., **61** (1939), 461-472.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
TOKYO INSTITUTE OF TECHNOLOGY
O-OKAYAMA, MEGUROKU,
TOKYO 152
JAPAN