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DYNKIN GRAPHS AND TRIANGLE SINGULARITIES

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In Arnold's classification list of singularities (Arnold [1].) we find interesting singularities to be studied. Though we find singularities of any dimension in Arnold's list, we consider singularities of dimension two in particular. Among them there is a class called exceptional singularities or triangle singularities. This class consists of fourteen singularities. It is known that they are closely related to K3 surfaces with the structure of elliptic surfaces. (Looijenga [4].) Here we would like to consider the following nine singularities of these fourteen ones:

$$E_{12},\ Z_{11},\ Q_{10},$$
$$E_{13},\ Z_{12},\ Q_{11},$$
$$E_{14},\ Z_{13},\ Q_{12}.$$

(The remaining five triangle singularities are W_{12}, W_{13}, S_{11}, S_{12} and U_{12}.) We assume that the ground field is the complex field $\mathbf{C}$.

Recall here that a connected Dynkin graph of type A, D or E corresponds to a surface singularity called a rational double point. Let Ξ be a class of surface singularities. By $PC(\Xi)$ we denote the set of Dynkin graphs Γ with several components such that there exists a small deformation fiber Y of a singularity belonging to Ξ satisfying the following conditions:

1. Y has only rational double points as singularities.
2. The combination of rational double points on Y corresponds exactly to Γ. (The type of each component of Γ corresponds to the type of the singularity on Y and the number of components of each type corresponds to the number of singularities of each type on Y.)

Note that by definition every graph in $PC(\Xi)$ has only components of type A, D or E. We would like to study $PC(\Xi)$ for $\Xi = E_{12},\ Z_{11},\ \dots,\ Q_{12}$

THEOREM. *Let Ξ be one of the above nine classes of singularities. The following two conditions are equivalent.*

(A) $\Gamma \in PC(\Xi)$.

(B) *The Dynkin graph Γ has only components of type A, D or E, and can be made from the essential basic graph depending on Ξ by a combination of two of elementary transformations and tie transformations.*

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The respective essential basic graph
corresponding to the above nine singularities

$$E_8, \quad E_7, \quad E_6$$
$$E_8+BC_1, \; E_7+BC_1, \; E_6+BC_1$$
$$E_8+G_2, \quad E_7+G_2, \quad E_6+G_2$$

In the above list of essential basic graphs the plus sign + denotes the disjoint union of graphs. In the above condition (B) an elementary transformation and a tie transformation are operations by which we can make a new Dynkin graph from a given Dynkin graph. We give the definition of them below. In the condition (B) four kinds of combinations – "elementary" twice, "tie" twice, "elementary" after "tie", and "tie" after "elementary" – are all permitted.

DEFINITION. (An elementary transformation) The following procedure is called an elementary transformation of a Dynkin graph.

(1) Replace each connected component by the corresponding extended Dynkin graph.
(2) Choose in arbitrary manner at least one vertex from each component (of the extended Dynkin graph) and then remove these vertices together with edges issuing from them.

We can find the definition of the extended Dynkin graph in any book on Lie algebras. (Bourbaki [2].) They can be made by adding one vertex and one or two edges to each connected component of the Dynkin graph. The position of the added vertex and edges depends on the type of the component. Also we can find the definition of the coefficients of the maximal root in any book on Lie algebras.

DEFINITION. (A tie transformation) Assume that applying the following procedure to a Dynkin graph Γ, we have obtained the Dynkin graph $\overline{\Gamma}$. Then we call the following procedure a tie transformation of a Dynkin graph.

(1) Add one vertex and a few edges to each component of Γ and make it into the extended Dynkin graph of the corresponding type. Moreover attach the corresponding coefficient of the maximal root to each vertex.
(2) Choose in an arbitrary manner subsets A, B of the set of the vertices of the extended graph $\widetilde{\Gamma}$ satisfying the following conditions:
$\langle a \rangle$ $A \cap B = \emptyset$.
$\langle b \rangle$ Let V be the set of vertices of an arbitrarily chosen component $\widetilde{\Gamma}'$ of $\widetilde{\Gamma}$. Let ℓ be the number of elements in $V \cap A$ and $n_1, n_2, \ldots, n_\ell$ be the numbers attached to $V \cap A$. Furthermore let N be the sum of numbers attached to $V \cap B$. (If $V \cap B = \emptyset$, then $N = 0$.) Then the greatest common divisor of the $\ell + 1$ numbers $N, n_1, n_2, \ldots, n_\ell$ is 1.
(3) Erase all attached integers and remove vertices belonging to A together with edges issuing from them.
(4) Draw another new vertex ◯ corresponding to a root α with $\alpha^2 = 2$. Connect this new vertex ◯ and each vertex in B by an edge.

Remark. Often the resulting graph $\overline{\Gamma}$ after the above procedure (1) – (4) is not a Dynkin graph. We consider only the cases where the resulting graph $\overline{\Gamma}$ is a Dynkin graph and then we call the above procedure a tie transformation. Under this restriction the number $\#(B)$ of elements in the set B satisfies $0 \leq \#(B) \leq 3$. $\ell = \#(V \cap A) \geq 1$.

Here we give some explanation on Dynkin graphs and root systems of type BC. A root system R is a finite subset of a Euclidean space satisfying axioms on symmetry. Usually we assume moreover the following axiom $(*)$ of the reduced condition:

$$\text{If } \alpha \in R, \text{ then } 2\alpha \notin R \qquad (*)$$

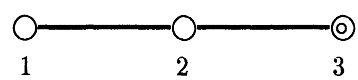
Under these axioms we obtain irreducible root systems of type A, B, C, D, E, F and G as in any book on Lie algebras. However, under the absence of the axiom $(*)$ we have further a series of irreducible root systems, which are called of type BC_k ($k = 1, 2, 3, \ldots$). (Bourbaki [2].) It is easy to generalize the concept of Dynkin graphs to root systems of type BC. (Urabe [6].) The Dynkin graph of type BC_1 is the following: $\otimes$

We explain the meaning of this BC_1 graph. Recall first the meaning of Dynkin graphs. Let R be an irreducible root system and $\Delta \subset R$ be the root basis. We can assume that the longest root $\alpha \in R$ satisfies $\alpha^2 = 2$ after normalizing the inner product of the ambient Euclidean space. The Dynkin graph Γ of R is the graph drawn by the following rules: (1) The vertices of Γ have one-to-one correspondence with the set Δ (the root basis). (2) Two vertices in Γ corresponding to two elements α, $\beta \in \Delta$ are connected by an edge in Γ if and only if the inner product $(\alpha, \beta) \neq 0$.

If R is of type A, D or E, then R consists of only roots α with $\alpha^2 = 2$, and every $\alpha \in \Delta$ satisfies $\alpha^2 = 2$. Therefore in these cases every vertex in the Dynkin graph can be denoted by a small white circle $\bigcirc$.

If R is of type BC_1, then Δ consists of a unique root δ with $\delta^2 = 1/2$ and $R = \{-2\delta, -\delta, \delta, 2\delta\}$. The vertex in the Dynkin graph corresponding to a root δ with $\delta^2 = 1/2$ is denoted by $\otimes$. The BC_1 graph is the graph consisting of a unique vertex of this kind. In this case the maximal root η is equal to 2δ, and thus the extended Dynkin graph, i.e., the graph corresponding to $\Delta^+ = \Delta \cup \{-\eta\}$ is the following: 1 $\bigcirc$━━━━$\otimes$ 2 (The edge is bold. The numbers are the coefficients of the maximal root.)

If R is of type G_2, then δ consists of two elements α with $\alpha^2 = 2$ and γ with $\gamma^2 = 2/3$. We denote the vertex corresponding to a root γ with $\gamma^2 = 2/3$ by $\circledcirc$. Our Dynkin graph of type G_2 is the following; $\bigcirc$———$\circledcirc$ and our extended Dynkin graph of type G_2 is the following (The numbers are the coefficients of the maximal root.):



Note that as a result of an elementary or a tie transformation, a graph consisting of a unique vertex corresponding to γ with $\gamma^2 = 2/3$ can appear. We call the graph $\circledcirc$ the Dynkin graph of type G_1. This corresponds to the root system $R = \{-\gamma, \gamma\}$ with $\gamma^2 = 2/3$. The extended Dynkin graph of type G_1 is the following: 1 $\circledcirc$━━━━$\circledcirc$ 1. (The edge is bold. The numbers are the coefficients of the maximal root.)

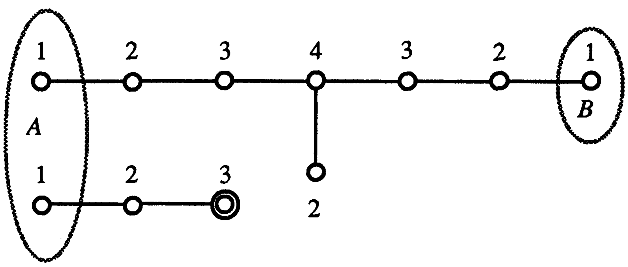
We can explain why we do not use the standard expression $\bigcirc \Rrightarrow \bigcirc$ of the G_2 graph. (Bourbaki [2].) If we use the standard expression, we cannot define the concept of the G_1 graph.

Now, for an irreducible root system R of the remaining types, i.e., of type B_k, C_k, F_4 or BC_l with $l \geqq 2$, the root basis Δ contains a root $\beta \in \Delta$ with $\beta^2 = 1$. However, in the case of our nine triangle singularities such a root β with $\beta^2 = 1$ never appears. Therefore in our case the type of a connected Dynkin graph is either A_k with $k \geqq 1$, D_l with $l \geqq 4$, E_6, E_7, E_8, G_2, G_1 or BC_1.

Note that since we have assumed that the Dynkin graph Γ in our Theorem has only components of type A, D or E, any Dynkin graph with a component of type G_2, G_1 or BC_1 made by two transformations has no meaning, and is to be thrown out.

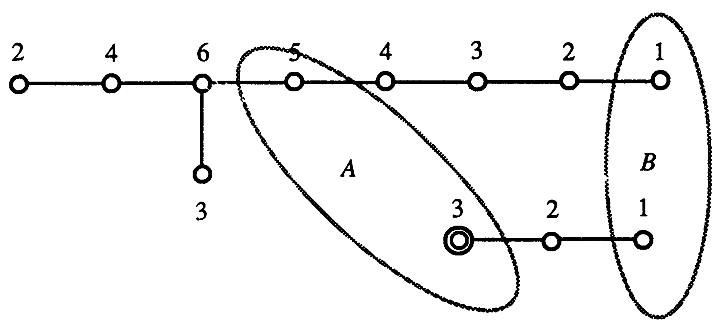
Example. We show $A_7 + A_4 \in PC(Z_{13})$ and $D_8 + A_2 \in PC(Z_{13})$.

For Z_{13} the corresponding essential basic graph is $E_7 + G_2$. We can start from $E_7 + G_2$. As an example, we apply a tie transformation to this graph. After the first step of the transformation the following graph is obtained and we can choose the subsets A and B as follows.



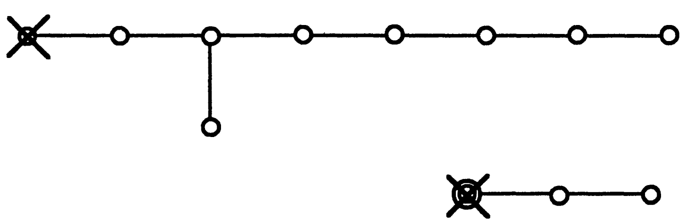
Obviously the condition $\langle a \rangle$ $A \cap B = \emptyset$ is satisfied. For the component E_7, $\ell = \#(V \cap A) = 1$ and $n_1 = 1$, $N = 1$. Thus $G.C.D.(n_1, N) = 1$. For the component G_2, $\ell = 1$, $n_1 = 1$, $N = 0$ and $G.C.D.(n_1, N) = 1$. The condition $\langle b \rangle$ is also satisfied. In the next step all vertices in A are erased, and drawing a new vertex ◯, we connect it and the vertex in B by an edge. One knows that the resulting graph is $E_8 + G_2$.

We can apply a transformation once more starting from $E_8 + G_2$. First we apply a tie transformation.



The above choice of A and B satisfies the conditions and we get the graph $A_7 + A_4$ as the result. By our Theorem $A_7 + A_4 \in PC(Z_{13})$.

If we apply an elementary transformation to $E_8 + G_2$, and if we erase two vertices as follows, we obtain the graph $D_8 + A_2$.



By our Theorem one can conclude $D_8 + A_2 \in PC(Z_{13})$.

Below we sketch the verification of our Theorem briefly.

First we apply the results in Looijenga [4]. In [4] Looijenga shows that our singularity is closely related K3 surfaces and that by the theory of periods for K3 surfaces we can reduce our problem into a problem on the lattice theory. Let Λ_N be the even unimodular lattice with signature $(16 + N,\ N)$. It is unique up to isomorphisms if $N \geqq 1$. A certain lattice P is defined corresponding to each Ξ of the nine triangle singularities. We can consider the quotient module Λ_N/P when a primitive embedding $P \subset \Lambda_N$ is given. A bilinear form on Λ_N/P with values in rational numbers can be defined. By Looijenga one knows that for a Dynkin graph Γ with only components of type A, D or E, $\Gamma \in PC(\Xi)$ if and only if the associated root lattice $Q(\Gamma)$ has a lattice embedding into Λ_3/P satisfying certain conditions. (Note that the suffix of Λ is $N = 3$ here.)

Second we translate Looijenga's conditions on the lattice theory into a simpler condition. We say that an embedding $Q(\Gamma) \subset \Lambda_N/P$ is *full* if the root system of $Q(\Gamma)$ and the root system of the *primitive hull* of $Q(\Gamma)$ in Λ_N/P coincide. (For a submodule M of a module L, the set $\widetilde{M} = \{ x \in L \mid \text{For some non-zero integer } m, mx \in M\}$ is called the primitive hull of M in L. Obviously it is a submodule containing M.) We consider here root systems including roots δ with $\delta^2 = 1/2$ and roots γ with $\gamma^2 = 2/3$. One knows that a lattice embedding $Q(\Gamma) \subset \Lambda_3/P$ satisfies Looijenga's conditions if and only if it is full.

Let $\overline{PC}(\Xi)$ denote the set of Dynkin graphs satisfying the condition (B) in our Theorem.

The inclusion relation $\overline{PC}(\Xi) \subset PC(\Xi)$ is an immediate consequence of our general theory of elementary transformation and tie transformations. Indeed, let Γ' be a Dynkin graph obtained from a Dynkin graph Γ by an elementary transformation or a tie transformation. We can show that if there exists a full embedding $Q(\Gamma) \subset \Lambda_N/P$, then there exists a full embedding $Q(\Gamma') \subset \Lambda_{N+1}/P$. Note here that the suffix of Λ increases by one. Besides, for some primitive embedding $P \subset \Lambda_1$ the corresponding essential basic graph Γ_0 has a full embedding $Q(\Gamma_0) \subset \Lambda_1/P$. Thus we can conclude $\overline{PC}(\Xi) \subset PC(\Xi)$.

Let μ be the Milnor number of one of the nine triangle singularities under consideration. This number is equal to the suffix of the corresponding symbol of the singularity. (For E_{12}, Z_{12} and Q_{12} $\mu = 12$.) Let Γ be a Dynkin graph with r vertices. We can show that if $\Gamma \in PC(\Xi)$, then $r \leqq \mu - 2$. Besides, if $r \leqq \mu - 5$, then conditions $\Gamma \in PC(\Xi)$ and $\Gamma \in \overline{PC}(\Xi)$ are equivalent. The last assertion follows from Meyer's theorem "Any indefinite rational quadratic form represents zero, if the number of variables is greater than or equal to five."

In order to show the opposite inclusion relation $\overline{PC}(\Xi) \supset PC(\Xi)$ it suffices to show $(S_\mu \cap \overline{PC}(\Xi)) \cup (S_\mu - PC(\Xi)) = S_\mu$, where S_μ denotes the set of Dynkin graphs Γ with only components of type A, D or E whose number r of vertices satisfies $\mu - 4 \leq r \leq \mu - 2$. To tell the truth, we could not succeed in finding any effective method to show this equality except case-by-case checking. This is a weak point of our theory. I regret this fact and hope that somebody can improve it. The theory of monodromy groups of elliptic surfaces may be effective for the improvement. Anyway, by the elementary lattice theory, the surface theory in the algebraic geometry, and the p-adic lattice theory due to Nikulin (Nikulin [5]), we can accomplish the checking.

Details of the verification will appear elsewhere.

Before concluding this article we would like to refer to the remaining five hypersurface triangle singularities W_{12}, W_{13}, S_{11}, S_{12} and U_{12}. Recall here in particular that the number of transformations in the condition (B) in our Theorem for the nine singularities is two. For the remaining five singularities W_{12}, W_{13}, S_{11}, S_{12} and U_{12}, if we try to formulate the corresponding theorem including a description in which the number of transformations is two, the formulation becomes very complicated and it is not worth mentioning. This is because the property of the *Milnor lattice* for W_{12}, W_{13}, S_{11}, S_{12} and U_{12} is very different from that of the nine singularities in this article.

We consider one of fourteen hypersurface triangle singularities and let F be the corresponding Milnor fiber. The pair $(L, -(\ , \ ))$ of the second homology group $L = H_2(F, \mathbf{Z})$ of F and (-1) times the intersection form $(\ , \ )$ is called the Milnor lattice. Let $H = \mathbf{Z}u + \mathbf{Z}v$ denote the hyperbolic plane, i.e., a lattice of rank 2 with $u^2 = v^2 = 0$ and $(u, v) = 1$.

For any of fourteen cases L has the following decomposition

$$L \cong M \oplus H \oplus H \tag{$**$}$$

where M is a positive definite lattice, and the symbol $\oplus$ denotes the orthogonal direct sum. For the nine singularities considered in this article for every decomposition $(**)$ the co-root system $R^\vee$ of M, i.e., the set

$$R^\vee = \{\, x \in M \mid x^2 = 2,\ 4 \text{ or } 6. \text{ For every } y \in M\ 2(x, y)/x^2 \text{ is an integer.}\}$$

spans M over $\mathbf{Q}$. However, for W_{12}, W_{13}, S_{11}, S_{12} and U_{12} for every decomposition $(**)$ the co-root system never spans M.

Instead of a theorem with the number of transformations two, we can formulate a theorem in which the number of transformations is one. In this case the basic graph used at the start of transformations is not Dynkin but a so-called Gabriélov graph. (Gabriélov [3].) For W_{12}, W_{13}, S_{11}, S_{12} and U_{12} under this formulation we can get theorems worth mentioning. Besides, even for the nine singularities considered in this article also theorems under the formulation with the number of transformations one are worth mentioning.

These results will appear elsewhere.

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