J.-T. LU
KODAI MATH. J.
17 (1994), 290—298

# HYPERSURFACES OF A SPHERE WITH 3-TYPE QUADRIC REPRESENTATION

BY JI-TAN LU

**Abstract**

We study hypersurfaces of a sphere with 3-type quadric representation. Two theorems are obtained, and some eigenvalue inequalities are proved.

## 0. Introduction

Let $\Phi: M^n \to E^m$ be an isometric immersion of an $n$-dimensional compact Riemannian manifold into the Euclidean space, $\Delta$ and $\text{spec}(M^n)=\{0<\lambda_1<\lambda_2<\cdots \nearrow +\infty\}$ be the Laplacian and the spectrum of $M^n$, respectively. Then we have the decomposition $\Phi=\sum_{u\geq 0}\Phi_u$, $u\in N$, where $\Phi_u: M^n \to E^m$ is a differentiable mapping such that $\Delta\Phi_u=\lambda_u\Phi_u$, moreover $\Phi_0$ is a constant mapping (it is the center of mass of $M^n$). $M^n$ is said to be of finite type if the decomposition consists of only a finite number of non-zero terms, and of $k$-type if there are exactly $k$ non-zero $\Phi_u$'s$(\Phi_{u_1}, \cdots, \Phi_{u_k})$ in the decomposition. In the latter case, we also call the immersion $\Phi$ to be of $k$-type.

Finite type submanifolds of a hypersphere $S^{m-1}\subset R^m$ have been studied by many authors. For example, see [5], [2], [9], [3]. In [5] mass-symmetric 2-type hypersurfaces of $S^{m-1}$ were characterized. In [2] it was proved that a compact 2-type hypersurface of $S^{m-1}$ is mass-symmetric if and only if it has constant mean curvature. In [9] Nagatomo showed that many 2-type hypersurfaces of a hypersphere are mass-symmetric and that there is no compact hypersurface of constant mean curvature in a hypersphere which is of 3-type. In particular, Barros and Garay [3] proved that the Riemannian product of two plane circles of different and stuitable radii is the only 2-type surfaces in $S^3\subset R^4$.

On the other hand, let $\Psi: M^n \to S^{n+p}(1)$ be a minimal isometric immersion of an $n$-dimensional compact Riemannian manifold into the unit sphere, $SM(n+p+1)=\{P\in gl(n+p+1, R)\,|\,P^t=P\}$, and $f: S^{n+p}(1)\to SM(n+p+1)$ be the order 2 immersion of $S^{n+p}(1)$. We consider the associated isometric immersion $\Phi=f\circ\Psi: M^n \to SM(n+p+1)$, which is called the quadric representation of $M^n$. In [8], Ros characterized minimal submanifolds in $S^{n+p}(1)$ with 2-type quadric representation. Later, Lu [7] proved that the Clifford torus $M_{m,m}$ are the only

Received September 28, 1993.

full compact minimal hypersurfaces in $S^{2m+1}(1)$ with 2-type representation and that the Veronese surfaces in $S^4$ are the only full compact minimal surfaces in the unit sphere for which $\Phi$ is of 2-type. In this paper we study hypersurfaces of a sphere with 3-type quadric representation. Our main results are

THEOREM 1. *Let $M^n$ be a compact minimal hypersurface of a sphere with 3-type quadric representation. Then the length of the second fundamental form of $M^n$ in the sphere must be constant.*

On the basis of Theorem 1, we further prove

THEOREM 2. *There does not exist compact minimal surface in $S^3(1)$ with 3-type quadric representation.*

Theorem 2 is not valid for any dimensional compact minimal hypersurfaces in a sphere. For example, minimal Cartan hypersurface $SO(3)/Z_2\times Z_2$ in $S^4$ is showed to just have 3-type quadric representation.

Finally we also give some eigenvalue inequalities. The author wishes to thank professor W. H. Chen for many valuable comments and suggestions.

## 1. Preliminaries

Let $S^{n+p}(1)$ be an Euclidean sphere with radius 1 and $SM(n+p+1)=\{P\in gl(n+p+1, R)\mid P^t=P\}$ be the space of the real symmetric matrices of order $n+p+1$. We define on $SM(n+p+1)$ the metric $\langle P, Q\rangle=(1/2)\operatorname{tr} PQ$, for arbitrary $P, Q$ in $SM(n+p+1)$. We consider the mapping $f: S^{n+p}(1)\to SM(n+p+1)$ given by $f(x)=xx^t$, where $x$ is the position column vector of $S^{n+p}(1)$ in $R^{n+p+1}$, and $x^t$ is the transpose of $x$. Then $f$ is the order 2 immersion of the sphere, and the mass center of $f(S^{n+p}(1))$ is $I/(n+p+1)$, where $I$ is the identity matrix in $SM(n+p+1)$. We identify $x$ with $f(x)$. Then the normal space for the immersion $f$ at $x$ of $S^{n+p}(1)$ is given by

$$T_x^{\perp}(S^{n+p}(1))=\{Q\in SM(n+p+1)\mid Qx=\lambda x, \text{ for some real } \lambda\}. \tag{1.1}$$

We denote by $\bar{\nabla}$, $\tilde{\nabla}$ the Riemannian connection on $SM(n+p+1)$ and $S^{n+p}(1)$, respectively, and by $\tilde{\sigma}$, $\tilde{A}$ and $\tilde{H}$ the second fundamental form, the Weingarten endomorphism and the mean curvature vector of immersion $f$, respectively, the normal connection of $f$ is denoted by $\tilde{D}$. Then we have the following formulas

$$\tilde{D}\tilde{\sigma}=0, \tag{1.2}$$

$$\tilde{H}_x=\frac{2}{n+p}(I-(n+p+1)x), \tag{1.3}$$

$$\langle\tilde{\sigma}(X, Y), \tilde{\sigma}(V, W)\rangle=2\langle X, Y\rangle\langle V, W\rangle+\langle X, V\rangle\langle Y, W\rangle+\langle X, W\rangle\langle Y, V\rangle, \tag{1.4}$$

$$\tilde{A}_{\tilde{\sigma}(X,Y)}V=2\langle X, Y\rangle V+\langle X, V\rangle Y+\langle Y, V\rangle X, \tag{1.5}$$

(1.6) $\langle x, \tilde{\sigma}(X, Y)\rangle = -\langle X, Y\rangle$,

(1.7) $\langle I, \tilde{\sigma}(X, Y)\rangle = 0$,

where $X, Y, V, W$ are vector fields tangent to $S^{n+p}(1)$.

## 2. Compact minimal hypersurfaces in the sphere

Let $\Psi: M^n \to S^{n+1}(1)$ be a minimal isometric immersion of a hypersuface in $S^{n+1}(1)$. Let $e_1, \cdots, e_n, N$ be a local field of orthonormal frames of $S^{n+1}(1)$, such that restricted to $M^n$, $e_1, \cdots, e_n$ are tangent to $M^n$. We denote by $\nabla$, $D$ the Riemannian connection of $M^n$ and the normal connection of $\Psi$, and by $\sigma$, $A$, $H$ the second fundamental form, the Weingarten endomorphism and the mean curvature vector of $\Psi$, respectively. Considering the associated isometric immerison $\Phi = f \circ \Psi: M^n \to SM(n+2)$, we have the following formulas (see [8])

$$\Delta\Phi = -\sum_{i=1}^{n} \tilde{\sigma}(e_i, e_i), \tag{2.1}$$

$$\Delta^2\Phi = 2(n+1)\Delta\Phi - 2\sum_{i,j}\tilde{\sigma}(A_{\sigma(e_i, e_j)}e_i, e_j) + 2\sum_{i,j}\tilde{\sigma}(\sigma(e_i, e_j), \sigma(e_i, e_j)). \tag{2.2}$$

We denote by $S$ the square of the length of $\sigma$, then from (2.2) we have

$$\Delta^2\Phi = 2(n+1)\Delta\Phi + 2S\tilde{\sigma}(N, N) - 2\sum_{i=1}^{n} \tilde{\sigma}(Ae_i, Ae_i). \tag{2.3}$$

Let $(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y)$, for arbitrary vector feilds $X, Y$ tangent to $M^n$, and $\Delta A = -\sum_{k=1}^{n}\nabla_{e_k}(\nabla_{e_k}A) + \sum_{k=1}^{n}\nabla_{\nabla_{e_k}e_k}A$. We will prove the following Lemmas.

LEMMA 1. *Let $\Psi: M^n \to S^{n+1}(1)$ be a minimal isometric immersion of a hypersurface into the sphere, $\Phi = f \circ \Psi$. Then*

$$\begin{aligned}\Delta^3\Phi = {} & (2\Delta S + 4S^2 + 4|A^2|^2)\tilde{\sigma}(N, N) \\ & -4\,\mathrm{grad}\, S - 8(\mathrm{tr}\, A^3)N + 2(n+1)\Delta^2\Phi \\ & +12\tilde{\sigma}(N, A\,\mathrm{grad}\, S) + \frac{8}{3}\tilde{\sigma}(N, \mathrm{grad}(\mathrm{tr}\, A^3)) \\ & -4(S+1)\sum_{k=1}^{n}\tilde{\sigma}(Ae_k, Ae_k) - 4\sum_{k=1}^{n}\tilde{\sigma}(A^2e_k, A^2e_k) \\ & -4\sum_{k=1}^{n}\tilde{\sigma}((\Delta A)(e_k), Ae_k) + 4\sum_{i,k=1}^{n}\tilde{\sigma}((\nabla_{e_k}A)(e_i), (\nabla_{e_k}A)(e_i)).\end{aligned} \tag{2.4}$$

*Proof.* At first we compute the differential of $\Delta^2\Phi$. From (2.3) we obtain

(2.5) $$(\Delta^2\Phi)_*(e_k)=2(n+1)(\Delta\Phi)_*(e_k)+2e_k(S)\tilde{\sigma}(N, N)+2S\bar{\nabla}_{e_k}\tilde{\sigma}(N, N)-2\sum_{i=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(Ae_i, Ae_i).$$

We compute the last two terms respectively.

(2.6) $$\bar{\nabla}_{e_k}\tilde{\sigma}(N, N)=-\tilde{A}_{\tilde{\sigma}(N,N)}e_k+2\tilde{\sigma}(\tilde{\nabla}_{e_k}N, N)=-2e_k-2\tilde{\sigma}(Ae_k, N),$$

where we have used (1.5), (1.2) and $D_{e_k}N=0$. By the same way we have

(2.7) $$\begin{aligned}\sum_{i=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(Ae_i, Ae_i)&=-\sum_{i=1}^{n}\tilde{A}_{\tilde{\sigma}(Ae_i, Ae_i)}e_k+\sum_{i=1}^{n}\tilde{D}_{e_k}\tilde{\sigma}(Ae_i, Ae_i)\\&=-2Se_k-2\sum_{i=1}^{n}\langle Ae_i, e_k\rangle Ae_i\\&\quad+2\sum_{i=1}^{n}\tilde{\sigma}(\sigma(e_k, Ae_i), Ae_i)+2\sum_{i=1}^{n}\tilde{\sigma}(\nabla_{e_k}(Ae_i), Ae_i).\end{aligned}$$

Hence, from (2.5), (2.6) and (2.7) we have

(2.8) $$\begin{aligned}(\Delta^2\Phi)_*(e_k)=&-4S\tilde{\sigma}(Ae_k, N)+4A^2e_k\\&+2(n+1)(\Delta\Phi)_*(e_k)+2e_k(S)\tilde{\sigma}(N, N)\\&-4\sum_{i=1}^{n}\tilde{\sigma}(\sigma(e_k, Ae_i), Ae_i)-4\sum_{i=1}^{n}\tilde{\sigma}(\nabla_{e_k}(Ae_i), Ae_i).\end{aligned}$$

Let $x$ be an arbitrary point in $M^n$, we may assume that $\nabla_{e_j}e_i=0$ at $x$. We compute $\Delta^3\Phi$ at $x$ as follows

(2.9) $$\begin{aligned}\Delta^3\Phi(x)=&-\sum_{i=1}^{n}\bar{\nabla}_{e_k}(\Delta^2\Phi)_*(e_k)\\=&2(n+1)\Delta^2\Phi(x)+2\Delta(S)\tilde{\sigma}(N, N)\\&+4S\sum_{k=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(Ae_k, N)-4\sum_{k=1}^{n}\bar{\nabla}_{e_k}(A^2e_k)\\&+4\sum_{k=1}^{n}e_k(S)\tilde{\sigma}(Ae_k, N)-2\sum_{k=1}^{n}e_k(S)\bar{\nabla}_{e_k}\tilde{\sigma}(N, N)\\&+4\sum_{k,i=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(\sigma(e_k, Ae_i), Ae_i)+4\sum_{k=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(\nabla_{e_k}(Ae_i), Ae_i).\end{aligned}$$

It is obvious that

$$\sum_{k=1}^{n}e_k(S)\tilde{\sigma}(Ae_k, N)=\tilde{\sigma}(A\,\mathrm{grad}\,S, N).$$

By a direct computation, using Codazzi equation, $H=0$ and (1.2), (1.5) we obtain the following relations

$$\sum_{k=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(Ae_k, N)=S\tilde{\sigma}(N, N)-\sum_{k=1}^{n}\tilde{\sigma}(Ae_k, Ae_k),$$

$$\sum_{k=1}^{n}\bar{\nabla}_{e_k}(A^2e_k)=\sum_{k=1}^{n}\tilde{\sigma}(Ae_k, Ae_k)+(\mathrm{tr}\, A^3)N+\frac{1}{2}\mathrm{grad}\, S,$$

$$\sum_{k=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(\nabla_{e_k}(Ae_i), Ae_i)=\frac{1}{3}\tilde{\sigma}(N, \mathrm{grad}(\mathrm{tr}\, A^3))-\sum_{k=1}^{n}\tilde{\sigma}((\Delta A)(e_i), Ae_i)$$
$$-\frac{3}{2}\mathrm{grad}\, S+\frac{1}{2}\tilde{\sigma}(N, A\,\mathrm{grad}\, S)+\sum_{k,i=1}^{n}\tilde{\sigma}((\nabla_{e_k}A)(e_i), (\nabla_{e_k}A)(e_i)),$$

$$\sum_{k=1}^{n}\bar{\nabla}_{e_k}\tilde{\sigma}(\sigma(e_k, Ae_i), Ae_i)=\frac{1}{2}\tilde{\sigma}(N, A\,\mathrm{grad}\, S)$$
$$+\frac{1}{3}\tilde{\sigma}(N, \mathrm{grad}(\mathrm{tr}\, A^3))-\sum_{k=1}^{n}\tilde{\sigma}(A^2e_k, A^2e_k)-\mathrm{tr}(A^3)N+|A^2|^2\tilde{\sigma}(N, N).$$

From (2.6), (2.9) and the above relations we have (2.4).

LEMMA 2. *Let $\Psi$: $M^n \to S^{n+1}(1)$ be a minimal isometric immersion of a hypersurface in the sphere, $\Phi=f\circ\Psi$. Then we have the following relations*

$$\langle\Phi, \Phi\rangle=\frac{1}{2}, \tag{2.10}$$

$$\langle\Phi, \Delta\Phi\rangle=n, \tag{2.11}$$

$$\langle\Phi, \Delta^2\Phi\rangle=2n(n+1), \tag{2.12}$$

$$\langle\Phi, \Delta^3\Phi\rangle=4n(n+1)^2+S, \tag{2.13}$$

$$\langle\Delta\Phi, \Delta^2\Phi\rangle=4n(n+1)^2+S. \tag{2.14}$$

*Proof*. The above relations can be obtainted by a long but direct computation using (2.1), (2.2), (2.4), (1.4) and (1.6).

*Note*. In fact, (2.10), (2.11), (2.12), (2.14) hold for any co-dimention $p$. (see [8], Lemma 2.2).

## 3. Proof of the Theorems 1 and 2

*Proof of Theorem* 1. Let $\Psi: M^n\to S^{n+1}(1)$ be a minimal isometric immersion, $\Phi=f\circ\Psi$, if $\Phi$ is of 3-type ($\{u_1, u_2, u_3\}$). Then we have

$$\Phi=\Phi_0+\Phi_{u_1}+\Phi_{u_2}+\Phi_{u_3},$$

and

$$\Delta\Phi=\lambda_{u_1}\Phi_{u_1}+\lambda_{u_2}\Phi_{u_2}+\lambda_{u_3}\Phi_{u_3},$$

$$\Delta^2\Phi=\lambda_{u_1}^2\Phi_{u_1}+\lambda_{u_2}^2\Phi_{u_2}+\lambda_{u_3}^2\Phi_{u_3},$$

$$\Delta^3\Phi=\lambda_{u_1}^3\Phi_{u_1}+\lambda_{u_2}^3\Phi_{u_2}+\lambda_{u_3}^3\Phi_{u_3}.$$

Hence

$$\Delta^3\Phi=a\Delta^2\Phi+b\Delta\Phi+c\Phi-c\Phi_0, \tag{3.1}$$

where

$$a=\sum_{i=1}^{3}\lambda_{u_i},\ b=-\sum_{1\leq i<j\leq 3}\lambda_{u_i}\lambda_{u_j},\ c=\prod_{i=1}^{3}\lambda_{u_i}.$$

From (1.1), (2.1) and (2.2) we know that $I$, $\Phi$, $\Delta\Phi$ and $\Delta^2\Phi$ are all normal to $S^{n+1}(1)$. Hence, for any vector field $X$ tangent to $M^n$, we use (2.4) and (3.1) to obtain

$$\langle\Delta^3\Phi, X\rangle=-4\langle \operatorname{grad} S, X\rangle=-c\langle X, \Phi_0\rangle,$$

but

$$X\langle\Phi, \Phi_0\rangle=\langle X, \Phi_0\rangle,$$

and

$$\langle \operatorname{grad} S, X\rangle=X(S).$$

Therefore

$$X(4S-c\langle\Phi, \Phi_0\rangle)=0.$$

This means

$$4S-c\langle\Phi, \Phi_0\rangle=constant. \tag{3.2}$$

On the other hand, by using (2.10), (2.11), (2.12) and (3.1) we have

$$\langle\Delta^3\Phi, \Phi\rangle=2n(n+1)a+nb+\frac{1}{2}c-c\langle\Phi_0, \Phi\rangle.$$

Combining with (2.14), we have

$$S+c\langle\Phi, \Phi_0\rangle=2n(n+1)a+nb+\frac{1}{2}c-4(n+1)^2n \tag{3.3}$$

Hence, from (3.3) and (3.2) we obtain $S=constant$, and $\langle\Phi, \Phi_0\rangle=constant$. This finishes the proof of Theorem 1.

*Proof of Theorem* 2. When $n=2$, i.e, $M^2$ is a compact minimal surface in $S^3(1)$. For the Gauss curvature $K$ of $M^2$, we have $K=1-(1/2)S$. If $\Phi$ is of 3-type, from Theorem 1, we know that $K$ is constant. But Bryant [1] had proved that there is no minimal surface of constant negative Guassian curvature in $S^n$, so $K\geqq 0$. If $K=0$, then $S=2$. From the well-known result of Chern and others [6], we know that $M^2$ must be the Clifford torus $M_{1,1}$. But we know that for $M_{1,1}$ $\Phi$ is of 2-type. This is a contradiction in consideration of $\Phi$ being of 3-type. If $K>0$, Calabi [4] told us $K$ must be 1, thus, $S=0$. This means that $M^2$ is the geodesic sphere and therefore $\Phi$ is of 1-type. This is also a contradiction to that $\Phi$ is of 3-type. Theorem 2 is thereby proved.

## 4. Eigenvalue inequalities

Let $\Psi$: $M^n \to S^{n+p}(1)$ be a minimal isometric immersion of a compact $n$-dimentional Riemannian manifold into the sphere. Then $\Phi = f \circ \Psi$ is an isometric immersion of $M^n$ into $SM(n+p+1, R)$. Let $\Phi = \sum_{u \geqq 0} \Phi_u$ be the spectral decomposition. Then we have

$$(4.1) \qquad \int_{M^n} \langle \Phi_u, \Phi_v \rangle *1 = 0 \qquad \text{for all} \quad u, v \in N, \ u \neq v,$$

where *1 is the volume element of $M^n$.

We put

$$\int_{M^n} \langle \Phi_u, \Phi_u \rangle *1 = a_u \qquad \text{for all} \quad u \in N,$$

and

$$\Omega_k = \int_{M^n} \langle \Delta \Phi, \Phi \rangle *1 - \lambda_k \int_{M^n} \langle \Phi - \Phi_0, \Phi \rangle *1,$$

then

$$(4.2) \qquad \int_{M^n} \langle \Delta^3 \Phi, \Phi \rangle *1 = \sum_{u \geqq 1} \lambda_u^3 a_u .$$

From the above relations, we obtain

$$(4.3) \qquad \Omega_1 = \sum_{u > 1} (\lambda_u - \lambda_1) a_u \geqq 0$$

the equality in (4.3) holds if and only if $\Phi$ is of order 1.

THEOREM 3. *Let $\Psi$: $M^n \to S^{n+p}(1)$ be an minimal isometric immersion, and $\Phi = f \circ \Psi$. Then*

$$\lambda_1 \leqq \frac{n}{(1/2) - |\Phi_0|^2},$$

*the equality holds if and only if $\Phi$ is of order* 1.

The theorem is obtained from (4.3), (2.10) and (2.11).

THEOREM 4. *Let $\Psi$: $M^n \to S^{n+p}(1)$ be a minimal isometric immersion, and $\Phi = f \circ \Psi$. If $\Phi$ is of 3-type, then we have*

$$(4.4) \qquad 2n(n+1) \sum_{i=1}^{3} \lambda_{u_i} - n \sum_{i<j} \lambda_{u_i} \lambda_{u_j} + \frac{n+p}{2(n+p+1)} \prod_{i=1}^{3} \lambda_{u_i} - 4n(n+1)^2 \geqq \frac{\int_{M^n} S*1}{\mathrm{Vol}(M^n)},$$

*the equality holds if and only if the centres of $M^n$ and $S^{n+p}(1)$ in $SM(n+p+1)$ are the same.*

*Proof.* If $\Phi$ is of 3-type, we have

$$\Delta^3\Phi=\Big(\sum_{i=1}^{3}\lambda_{u_i}\Big)\Delta^2\Phi-\Big(\sum_{i<j}\lambda_{u_i}\lambda_{u_j}\Big)\Delta\Phi+\Big(\prod_{i=1}^{3}\lambda_{u_i}\Big)(\Phi-\Phi_0). \tag{4.5}$$

Then

$$\int_{M^n}\langle\Delta^3\Phi,\ \Phi\rangle*1=\Big(\sum_{i=1}^{3}\lambda_{u_i}\Big)\int_{M^n}\langle\Delta^2\Phi,\ \Phi\rangle*1 \tag{4.6}$$
$$-\Big(\sum_{i<j}\lambda_{u_i}\lambda_{u_j}\Big)\int_{M^n}\langle\Delta\Phi,\ \Phi\rangle*1+\Big(\prod_{i=1}^{3}\lambda_{u_i}\Big)\Big(\int_{M^n}\langle\Phi,\ \Phi\rangle*1-\int_{M^n}\langle\Phi,\ \Phi_0\rangle*1\Big).$$

Using (2.14), (2.12), (2.11), (2.10) and (4.2), we get

$$2n(n+1)\Big(\sum_{i=1}^{3}\lambda_{u_i}\Big)-n\Big(\sum_{i<j}\lambda_{u_i}\lambda_{u_j}\Big) \tag{4.7}$$
$$+\Big(\frac{1}{2}-|\Phi_0|^2\Big)\Big(\prod_{i=1}^{3}\lambda_{u_i}\Big)-4n(n+1)^2=\frac{\int_{M^n}S*1}{\mathrm{Vol}(M^n)},$$

where $\mathrm{Vol}(M^n)$ is the volume of $M^n$.

We recall that $\Phi_0$ is a constant mapping and $\Phi_0=\int_{M^n}\Phi*1/\mathrm{Vol}(M^n)$. So,

$$\mathrm{tr}\,\Phi_0=\mathrm{tr}\Big(\frac{\int_{M^n}\Phi*1}{\mathrm{Vol}(M^n)}\Big)=\frac{\int_{M^n}\mathrm{tr}\,\Phi*1}{\mathrm{Vol}(M^n)}=1, \tag{4.8}$$

where we use the fact that $\mathrm{tr}\,\Phi=1$.

Let $\mu_1, \cdots, \mu_{n+p+1}$ be the eigenvalues of the matrix $\Phi_0$, then $\sum_{i=1}^{n+p+1}\mu_i=1$, and

$$|\Phi_0|^2=\frac{1}{2}\sum_{i=1}^{n+p+1}\mu_i^2\geqq\frac{1}{2(n+p+1)}\Big(\sum_{i=1}^{n+p+1}\mu_i\Big)^2=\frac{1}{2(n+p+1)}, \tag{4.9}$$

the equality holds if and only if $\mu_1=\cdots=\mu_{n+p+1}$. This means $\Phi_0=I/(n+p+1)$. Combining (4.9) and (4.7) we have (4.4).

## REFERENCES

[1] R.L. BRYANT, Minimal surfaces of constant curvature in $S^n$, Trans. Amer. Math. Soc., **290** (1985), 259-271.

[2] M. BARROS, B.Y. CHEN AND O.J. GARAY, Spherical finite type hypersurfaces, Algebras Groups Geom., **4** (1987), 58-72.

[3] M. BARROS AND O.J. GARAY, 2-type surfaces in $S^3$, Geom. Dedicata, **14** (1987), 329-336.

[4] E. CALABI, Minimal immersions of surfaces in Euclidean spheres, J. Differential Geom., **1** (1967), 111-125.

[5] B.Y. CHEN, Total Mean Curvature and Submanifolds of Finite Type, World Scientific, 1984.

[6] S.S. CHERN, M. DO CARMO AND S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length, Shing-Shen Chern Selected Papers, Springer-verlag, 1978, 393-409.

[7] J.T. LU, Spectral geometry of minimal submanifolds in $S^n(1)$, J. Tsinghua Univ., **33** (1993) S4, 108-113.

[8] A. ROS, Eigenvalue inequalities for minimal submanifolds and P-manifolds, Math. Z., **187** (1984), 393-404.

[9] Y. NAGATOMO, Finite type hypersurfaces of a sphere, Tokyo J. Math., **14** (1991), 85-92.

[10] I. DIMITRIC, Quadric representation of a submanifold, Proc. Amer. Math. Soc., **114** (1992), 201-210.

DEPARTMENT OF APPLIED MATHEMATICS
TSINGHUA UNIVERSITY (100084)
BEIJING, CHINA