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HOMOGENIZATION OF A REFLECTING BARRIER BROWNIAN MOTION IN A CONTINUUM PERCOLATION CLUSTER IN $\boldsymbol{R}^d$

Dedicated to Professor Hiroshi Tanaka on his 60th birthday

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1. Introduction and statement of result

Let $\mathfrak{M}$ be the set of all countable subsets η of $\boldsymbol{R}^d$ $(d\geqq 2)$ satisfying $N_K(\eta)<\infty$ for any compact subset K, where $N_A(\eta)$ is the number of points of η in $A\subset\boldsymbol{R}^d$. $\mathfrak{M}$ is equipped with $\mathfrak{B}(\mathfrak{M})$ the σ-field which is generated by $\{\eta\in\mathfrak{M}: N_A(\eta)=n\}$, $A\in\mathfrak{B}(\boldsymbol{R}^d)$, $n\in\boldsymbol{N}$. For $\eta\in\mathfrak{M}$, $r>0$ and two disjoint regions A_1 and A_2 in $\boldsymbol{R}^d$, we say that a continuous curve γ is an occupied (*resp.* a vacant) connection of A_1 and A_2 in a region A with respect to (η, r) if $\gamma\cap A_1\neq\emptyset$, $\gamma\cap A_2\neq\emptyset$, $\gamma\subset A$ and $\gamma\subset U_r(\eta)$ (*resp.* $\gamma\cap\overline{U_r(\eta)}=\emptyset$), where $U_r(\eta)$ stands for the r-neighborhood of η and $U_r(x)$ is the abbreviated form for $U_r(\{x\})$. A continuum percolation model is obtained if a distribution ν on the space $[0,\infty)\times\mathfrak{M}$ is given. In this paper we consider the case $\nu=\delta_r\otimes\mu_\lambda$, $r>0$, $\lambda>0$, where δ_r is the Dirac measure corresponding to the point r and μ_λ is a Poisson distribution on $\mathfrak{M}$ with intensity measure λdx, that is, for any disjoint system $\{A_1, A_2, \cdots, A_m\}\subset\mathfrak{B}(\boldsymbol{R}^d)$ such that $|A_i|=\int_{A_i}dx<\infty$, $i=1, 2, \ldots, m$, $N_{A_1}(\eta), \ldots, N_{A_m}(\eta)$ are independent random variables on the probability space $(\mathfrak{M}, \mathfrak{B}(\mathfrak{M}), \mu_\lambda)$ and

$$\mu_\lambda(N_{A_i}=n)=\frac{(\lambda|A_i|)^n}{n!}\exp(-\lambda|A_i|), \qquad i=1, 2, \ldots, m,\ n\in\boldsymbol{N}\cup\{0\}.$$

This percolation model is called the 'Poisson blob model'. (See Grimmett [9].) It should be viewed as a continuum analogue of the discrete site percolation model. Instead of sites being independently occupied we have a Poisson process on $\boldsymbol{R}^d$ with each Poisson point being the center of an occupied ball of radius r. Now we define two regions in $\boldsymbol{R}^d$,

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$$W(x, \eta, r)=\{y : \exists \text{ an occupied connection of } \{x\} \text{ and } \{y\} \text{ in } \boldsymbol{R}^d \text{ w.r.t. } (\eta, r)\},$$

$$W^*(x, \eta, r)=\{y : \exists \text{ a vacant connection of } \{x\} \text{ and } \{y\} \text{ in } \boldsymbol{R}^d \text{ w.r.t. } (\eta, r)\},$$

i.e., the occupied and vacant cluster of x, respectively. The critical values are defined as follows:

$$\lambda_c(r)=\inf\{\lambda>0 : \mu_\lambda(W(0, r) \text{ is unbounded})>0\},$$

$$\lambda_c^*(r)=\sup\{\lambda>0 : \mu_\lambda(W^*(0, r) \text{ is unbounded})>0\}.$$

It was shown that $\lambda_c(r)=\lambda_c^*(r)$, $r>0$ in the case where $d=2$ by Roy [16].

In this paper we study a homogenization problem of a reflecting barrier Brownian motion in an unbounded occupied (or vacant) cluster of the Poisson blob model.

Let A be a Lipschitz domain in $\boldsymbol{R}^d$, that is, the boundary ∂A of A can be locally represented as the graph of a Lipschitz function defined on some open ball of $\boldsymbol{R}^{d-1}$. We define a Dirichlet form $(\mathcal{E}_A, H^1(A))$ on $L^2(\bar{A}, dx)$ by

$$\mathcal{E}_A(u, v)=\frac{1}{2}\int_A dx \nabla u(x)\cdot\nabla v(x), \qquad u, v\in H^1(A),$$

$$H^1(A)=\{u\in L^2(A, dx) : |\nabla u|\in L^2(A, dx)\}.$$

It is known that $\{u|_A : u\in \boldsymbol{C}_0^\infty(\boldsymbol{R}^d)\}$ is dense in $H^1(A)$. (See Stein [17].) We introduce measurable subsets $\hat{\mathfrak{M}}$ and $\mathfrak{M}_x$, $x\in\boldsymbol{R}^d$ of $\mathfrak{M}$ defined by

$$\hat{\mathfrak{M}}=\{\eta\in\mathfrak{M}\setminus\Delta : W(0, \eta) \text{ is unbounded}\},$$

$$\mathfrak{M}_x=\{\eta\in\mathfrak{M}\setminus\Delta : W(x, \eta)\neq\emptyset\},$$

where

$$\Delta=\{\eta\in\mathfrak{M} : |x-y|=2r \text{ for some } x, y\in\eta\}.$$

In case $\eta\in\mathfrak{M}_x$, $W(x, \eta)$ is a Lipschitz domain and $(\mathcal{E}_{W(x,\eta)}, H^1(W(x, \eta)))$ is a regular Dirichlet form on $L^2(\overline{W(x, \eta)}, dx)$. Hence, by the theory of Dirichlet forms, we can construct the associated diffusion process starting from x, which is called a reflecting Brownian motion. (See Fukushima [7], [8], Bass and Hsu [1].) We denote by $(\Omega, \mathcal{F}, P_x^\eta, X(t))$ this diffusion process if $\eta\in\mathfrak{M}_x$; otherwise $X(t)=x$, $t\geqq 0$. The process $X(t)$ is conservative and has a transition density $p^\eta(t, x, y)$. For a probability measure μ on $\mathfrak{M}$ with $\mu(\hat{\mathfrak{M}})>0$ we put

$$\hat{\mu}(d\eta)=\frac{\mathbf{1}_{\hat{\mathfrak{M}}}(\eta)}{\mu(\hat{\mathfrak{M}})}\mu(d\eta).$$

Changing the role of $W(x, \eta)$ into $W^*(x, \eta)$, we define $\hat{\mathfrak{M}}^*$, $\mathfrak{M}_x^*$, $X^*(t)$, $p^n(t, x, y)^*$ and $\hat{\mu}^*$ by the same way as $\hat{\mathfrak{M}}$, $\mathfrak{M}_x$, $X(t)$, $p^n(t, x, y)$ and $\hat{\mu}$, respectively. For a probability measure μ on $\mathfrak{M}$ we put

$$P^\mu=\int_{\mathfrak{M}}\mu(d\eta)P_0^\eta.$$

Our main result is the following theorem.

THEOREM 1. (i) *If* $\lambda > \lambda_c(r)$, *then the process* $\varepsilon X(t/\varepsilon^2)$ *on* $(\Omega, \mathcal{F}, P^{\widehat{\mu_\lambda}})$ *converges to* $\hat{\sigma}(\lambda)B(t)$ *in distribution on the space* $\boldsymbol{C}([0, \infty) \to \boldsymbol{R}^d)$ *equipped with the local uniform topology, where* $B(t)$ *is a d-dimensional Brownian motion and* $\hat{\sigma}(\lambda)$ *is a positive constant.*

(ii) *If* $\lambda < \lambda_c^*(r)$, *then the process* $\varepsilon X^*(t/\varepsilon^2)$ *on* $(\Omega, \mathcal{F}, P^{\hat{\mu}_\lambda^*})$ *converges to* $\hat{\sigma}^*(\lambda)B(\lambda)$ *in distribution on the space* $\boldsymbol{C}([0, \infty) \to \boldsymbol{R}^d)$ *equipped with the local uniform topology, where* $\hat{\sigma}^*(\lambda)$ *is a positive constant.*

Central limit theorems for stochastic processes in random sets were studied by many authors. De Masi, Ferrari, Goldstein and Wick [5] proved a central limit theorem for a random walk in the Bernoulli percolation cluster. Tanemura [19] generalized their result to a jump type Markov process in a continuum percolation cluster. Bhattacharya [2] and Ochi [14] studied homogenization problems of reflecting barrier Brownian motions in the case of periodic domains and Osada [15] studied the problems in the case of domains with stationary scatterers under certain geometric conditions.

We prove the uniqueness of unbounded cluster of the continuum percolation model (Proposition 3.1). The unbounded cluster, say $W(\eta)$, is regarded as a domain with stationary scatterers. If $W(\eta)$ satisfied the geometric conditions in [15], Theorem 1 would be derived from [15]. However, $W(\eta)$ does not satisfy the conditions. The conditions were essentially used in [15] to prove the tightness and the nondegeneracy of the limit. Then we can obtain the convergence of finite dimensional distributions in the same way as [15] (Section 3).

The tightness of the rescaled diffusion processes is proved in Section 2. Lyons and Zheng [13] obtained a decomposition of additive functionals of reversible Markov processes associated with Dirichlet forms. This decomposition is useful to check tightness. (See Takeda [18].) By virtue of the translation invariance of μ_λ we can apply their decomposition and show the tightness of the rescaled processes (Proposition 2.1). The technique used in Section 2 is thought to be applicable to more general situation.

We show the positivity of the diffusion constant $\hat{\sigma}(\lambda)$ in Section 4. The positivity of the diffusion constant $\hat{\sigma}^*(\lambda)$ can be proved by the same way. We introduce jump type Markov processes $X^n(t)$, $n \in \boldsymbol{N}$ in the continuum percolation cluster. The diffusion constant $\hat{\sigma}(\lambda)$ is bounded from below by the inferior limit of the diffusion constants $\hat{\sigma}_n(\lambda)$ of $X^n(t)$, $n \in \boldsymbol{N}$. In the case of the random walk in the Bernoulli percolation cluster, the positivity of the diffusion constant was shown in [5] by using an estimate of the effective conductivity associated with this process. Their technique was generalized to jump type Markov processes in continuum percolation clusters in [19]. Then the key part for proving the positivity of $\hat{\sigma}(\lambda)$ is to estimate the effective conductivities associated with $X^n(t)$ uniformly in n (Lemma 4.2). The proof of Lemma 4.2 is given in

Section 5 by using a result about the continuum percolation model given in [20].

2. Tightness

In this section we establish the tightness of the rescaled processes $\varepsilon X(t/\varepsilon^2)$, $\varepsilon>0$.

PROPOSITION 2.1. *Let μ be a translation invariant probability measure on $\mathfrak{M}$.*

(i) *If $\mu(\widehat{\mathfrak{M}})>0$, then $\{P^{\hat{\mu}}\circ(\varepsilon X(\cdot/\varepsilon^2))^{-1}:\varepsilon\in(0,1)\}$ is tight in the space $\boldsymbol{C}([0,\infty)\to\boldsymbol{R}^d)$ equipped with the local uniform topology.*

(ii) *If $\mu(\widehat{\mathfrak{M}^*})>0$, then $\{P^{\widehat{\mu^*}}\circ(\varepsilon X^*(\cdot/\varepsilon^2))^{-1}:\varepsilon\in(0,1)\}$ is tight in the space $\boldsymbol{C}([0,\infty)\to\boldsymbol{R}^d)$ equipped with the local uniform topology.*

We give a proof of (i). A proof of (ii) is obtained similarly. For any $\delta>0$, put

$$V_{h,T}=\{w\in\boldsymbol{C}([0,\infty)\to\boldsymbol{R}^d):\sup_{\substack{0\leq s,t\leq T\\|t-s|\leq h}}|w(t)-w(s)|>\delta\}.$$

Then it is sufficient to show that

$$\lim_{h\to 0}\sup_{\varepsilon\in(0,1)}P^{\mu}\circ\left(\varepsilon X\left(\frac{\cdot}{\varepsilon^2}\right)\right)^{-1}(V_{h,T})=0\,,\qquad\forall T>0\,.$$

Using the translation invariance of μ, we obtain

$$\begin{aligned}P^{\mu}\circ\left(\varepsilon X\left(\frac{\cdot}{\varepsilon^2}\right)\right)^{-1}(V_{h,T})&=\int_{\mathfrak{M}}\mu(d\eta)P^{\eta}_0\circ\left(\varepsilon X\left(\frac{\cdot}{\varepsilon^2}\right)\right)^{-1}(V_{h,T})\\&=\frac{1}{|U_l(0)|}\int_{U_l(0)}dy\int_{\mathfrak{M}}\mu(d\eta)P^{\tau_{-y}\eta}_0\circ\left(\varepsilon X\left(\frac{\cdot}{\varepsilon^2}\right)\right)^{-1}(V_{h,T})\\&=\int_{\mathfrak{M}}\mu(d\eta)\frac{1}{|U_l(0)|}\int_{U_T(\eta)\cap U_l(0)}dyP^{\eta}_y\circ\left(\varepsilon X\left(\frac{\cdot}{\varepsilon^2}\right)\right)^{-1}(V_{h,T}),\end{aligned}$$

for any $l>0$, where $\tau_x\eta=\{x+y:y\in\eta\}$. Put $Q^{\eta,\varepsilon}_x=P^{\eta}_{x/\varepsilon}\circ(\varepsilon X(\cdot/\varepsilon^2))^{-1}$. Then

$$P^{\mu}\circ\left(\varepsilon X\left(\frac{\cdot}{\varepsilon^2}\right)\right)^{-1}(V_{h,T})=\int_{\mathfrak{M}}\mu(d\eta)\frac{1}{|U_1(0)|}\int_{\varepsilon U_T(\eta)\cap U_1(0)}dxQ^{\eta,\varepsilon}_x(V_{h,T}),$$

where $\alpha A=\{\alpha x:x\in A\}$ for $A\subset\boldsymbol{R}^d$, $\alpha>0$. Hence, Proposition 2.1 follows from the following lemma. We show the lemma by the same procedure as in Theorem 3.1 in Takeda [18].

LEMMA 2.1. *For any $\eta\in\mathfrak{M}$*

$$\lim_{h\to 0}\sup_{\varepsilon\in(0,1)}\int_{\varepsilon U_T(\eta)\cap U_1(0)}dxQ^{\eta,\varepsilon}_x(V_{h,T})=0\,,\qquad\forall T>0\,.$$

Proof. From the definition of $X(t)$ this lemma is trivial if $\eta\in\Delta$. So we assume $\eta\in\mathfrak{M}\setminus\Delta$. Suppose that $x\in\varepsilon U_r(\eta)$. Then $(w(t), Q_x^{\eta,\varepsilon})$ is a diffusion process associated with the Dirichlet form $(\mathcal{E}_{\varepsilon U_r(\eta)}, H^1(\varepsilon U_r(\eta)))$ starting from x. For any $\varepsilon\in(0, 1)$ there exists an increasing sequence $\{a_n\}_{n\in N}$ such that $n+2<a_n\leqq n+3$ and $A_n^\varepsilon[\eta]=\varepsilon U_r(\eta)\cap U_{a_n}(0)$ is a Lipschitz domain for any $n\in\boldsymbol{N}$ with $A_n^\varepsilon[\eta]\neq\emptyset$. We denote the diffusion process starting from x associated with the Dirichlet form $(\mathcal{E}_{A_n^\varepsilon[\eta]}, H^1(A_n^\varepsilon[\eta]))$ by $(w(t), Q_x^{\eta,\varepsilon,n})$. Then,

$$\begin{aligned}(2.1)\qquad &\int_{\varepsilon U_r(\eta)\cap U_1(0)} dx Q_x^{\eta,\varepsilon}(V_{h,T})\\ &\leqq\int_{\varepsilon U_r(\eta)\cap U_1(0)} dx\{Q_x^{\eta,\varepsilon,n}(\sup_{t\in[0,T]}|w(t)-w(0)|>n)+Q_x^{\eta,\varepsilon,n}(V_{h,T})\}\\ &\leqq|A_n^\varepsilon[\eta]|\{Q^{\eta,\varepsilon,n}(\sup_{t\in[0,T]}|w(t)-w(0)|>n)+Q^{\eta,\varepsilon,n}(V_{h,T})\},\end{aligned}$$

where

$$Q^{\eta,\varepsilon,n}=\frac{1}{|A_n^\varepsilon[\eta]|}\int_{A_n^\varepsilon[\eta]} dx Q_x^{\eta,\varepsilon,n}.$$

From the decomposition obtained by Lyons-Zheng [13] we claim

$$(2.2)\qquad w(t)-w(s)=\frac{1}{2}(M(t)-M(s))+\frac{1}{2}(\tilde{M}(T-t)-\tilde{M}(T-s)),\quad Q^{\eta,\varepsilon,n}\text{-a.s.}$$

where $M(t)$ and $\tilde{M}(t)$ are d-dimensional Brownian motions with respect to $Q^{\eta,\varepsilon,n}$. Denoting by P_W a Wiener measure on $\boldsymbol{C}([0,\infty)\to\boldsymbol{R}^d)$, by Doob's inequality we obtain

$$\begin{aligned}(2.3)\qquad &Q^{\eta,\varepsilon,n}(\sup_{t\in[0,T]}|w(t)-w(0)|>n)\\ &\leqq Q^{\eta,\varepsilon,n}(\sup_{t\in[0,T]}|M(t)|>n)\\ &\quad+Q^{\eta,\varepsilon,n}(\sup_{t\in[0,T]}|\tilde{M}(T-t)-\tilde{M}(T)|>n)\\ &\leqq 2\int_{C([0,\infty)\to\boldsymbol{R}^d)} P_W(dw)\exp(|w(T)|)\exp(-n),\end{aligned}$$

$$\begin{aligned}(2.4)\qquad &Q^{\eta,\varepsilon,n}(V_{h,T})\leqq Q^{\eta,\varepsilon,n}(\sup_{\substack{0\leqq s,t\leqq T\\|t-s|\leqq h}}|M(t)-M(s)|>\delta)\\ &\quad+Q^{\eta,\varepsilon,n}(\sup_{\substack{0\leqq s,t\leqq T\\|t-s|\leqq h}}|\tilde{M}(t)-\tilde{M}(s)|>\delta)\\ &\leqq\frac{2T}{h}P_W\Big(\sup_{0\leqq t\leqq 2h}|w(t)|>\frac{\delta}{2}\Big)\end{aligned}$$

$$\leqq \frac{2T}{h}\left(\frac{2}{\delta}\right)^4 \int_{C([0,\infty)\to \boldsymbol{R}^d)} P_W(dw)|w(2h)|^4$$

$$\leqq \frac{3\cdot 2^7 dT}{\delta^4} h\,.$$

Setting $n=[-\log h]$ and combining (2.1), (2.3) and (2.4), we conclude that

$$\int_{\varepsilon U_T(\eta)\cap U_1(0)} dx Q_x^{\eta,\varepsilon}(V_{h,T})$$

$$\leqq |U_1(0)|\left\{2\int_{C([0,\infty)\to \boldsymbol{R}^d)} P_W(dw)\exp(|w(T)|)+\frac{3\cdot 2^7 dT}{\delta^4}\right\}(3-\log h)^d h,$$

where $[\alpha]$ denotes the integer part of a nonnegative constant α. Therefore, this completes the proof of Lemma 2.1. ■

We denote the expectation with respect to P_x^η and P^μ by E_x^η and E^μ, respectively. We show a lemma which ensures the existence of the moment $E^{\hat{\mu}}[|X(t)|^p]$ (*resp.* $E^{\hat{\mu}*}[|X^*(t)|^p]$) for any $p\in \boldsymbol{N}$ and any translation invariant probability measure μ on $\mathfrak{M}$ with $\mu(\widehat{\mathfrak{M}})>0$ (*resp.* $\mu(\widehat{\mathfrak{M}^*})>0$).

LEMMA 2.2. *Let μ be a translation invariant probability measure on $\mathfrak{M}$. For any $t\in(0,\infty)$ and any $l\in(0,\infty)$*

(i) $$P^\mu(\sup_{t\in[0,T]}|X(t)|\geqq l)\leqq 2P_W(\sup_{t\in[0,T]}|w(t)|\geqq l)\,,$$

(ii) $$P^\mu(\sup_{t\in[0,T]}|X^*(t)|\geqq l)\leqq 2P_W(\sup_{t\in[0,T]}|w(t)|\geqq l)\,,$$

where P_W is a Wiener measure on $\boldsymbol{C}([0,\infty)\to\boldsymbol{R}^d)$.

Proof. Let $\eta\notin\Delta$. There exists an increasing sequence $\{b_n\}_{n\in\boldsymbol{N}}$ such that $(n+1)l<b_n\leqq(n+2)l$ and $A(n,\eta)=U_r(\eta)\cap U_{b_n}(0)$ is a Lipschitz domain for any $n\in\boldsymbol{N}$ with $A(n,\eta)\neq\emptyset$. We denote the diffusion process starting from $x\in A(n,\eta)$ associated with the Dirichlet form $(\mathcal{E}_{A(n,\eta)}, H^1(A(n,\eta)))$ by $(X(t), P_x^{\eta,n})$ and put $P^{\eta,n}=(1/|A(n,\eta)|)\int_{A(n,\eta)} dx P_x^{\eta,n}$.

$$P^\mu(\sup_{t\in[0,T]}|X(t)|\geqq l)=\int_{\mathfrak{M}}\mu(d\eta)P_0^\eta(\sup_{t\in[0,T]}|X(t)|\geqq l)$$

$$=\int_{\mathfrak{M}}\mu(d\eta)\frac{1}{|U_{nl}(0)|}\int_{U_{nl}(0)} dx P_0^{\tau_{-x}\eta}(\sup_{t\in[0,T]}|X(t)|\geqq l)$$

$$=\int_{\mathfrak{M}}\mu(d\eta)\frac{1}{|U_{nl}(0)|}\int_{U_{nl}(0)} dx P_x^\eta(\sup_{t\in[0,T]}|X(t)-X(0)|\geqq l)$$

$$\leqq \int_{\mathfrak{M}\setminus\Delta} \mu(d\eta) \frac{1}{|U_{nl}(0)|} \int_{A(n,\eta)} dx P_x^{\eta,n}(\sup_{t\in[0,T]} |X(t)-X(0)| \geqq l)$$

$$\leqq \left(\frac{n+2}{n}\right)^d \int_{\mathfrak{M}\setminus\Delta} \mu(d\eta) P^{\eta,n}(\sup_{t\in[0,T]} |X(t)-X(0)| \geqq l)$$

From the decomposition obtained by Lyons-Zheng [13] we claim

$$X(t)-X(s)=\frac{1}{2}(N(t)-N(s))+\frac{1}{2}(\tilde{N}(T-t)-\tilde{N}(T-s)), \quad P^{\eta,n}\text{-a.s.}$$

where $N(t)$ and $\tilde{N}(t)$ are d-dimensional Brownian motions with respect to $P^{\eta,n}$. For this reason, we get

$$\begin{aligned} &P^{\eta,n}(\sup_{t\in[0,T]} |X(t)-X(0)| \geqq l) \\ &\leqq P^{\eta,n}(\sup_{t\in[0,T]} |N(t)| \geqq l) + P^{\eta,n}(\sup_{t\in[0,T]} |\tilde{N}(T-t)-\tilde{N}(T)| \geqq l) \\ &= 2P_W(\sup_{t\in[0,T]} |w(t)| \geqq l), \end{aligned}$$

and conclude that

$$P^{\mu}(\sup_{t\in[0,T]} |X(t)| \geqq l) \leqq 2\left(\frac{n+2}{n}\right)^d P_W(\sup_{t\in[0,T]} |w(t)| \geqq l), \qquad \forall n \in \boldsymbol{N}.$$

This completes the proof of (i) of Lemma 2.2. We can prove (ii) by the same way. ■

3. Convergence of finite dimensional distributions

The purpose of this section is the following theorem.

THEOREM 3.1. (i) *If* $\mu_\lambda(\widehat{\mathfrak{M}})>0$, *then the process* $\varepsilon X(t/\varepsilon^2)$ *on* $(\Omega, \mathcal{F}, P^{\hat{\mu}_\lambda})$ *converges to* $\hat{\sigma}(\lambda)B(t)$ *in the sense of finite dimensional distributions, where* $\hat{\sigma}(\lambda)$ *is a nonnegative constant determined by*

$$\hat{\sigma}(\lambda)^2 = \lim_{t\to\infty} \frac{1}{t} E^{\hat{\mu}_\lambda}[X_1(t)^2] .$$

(ii) *If* $\mu_\lambda(\widehat{\mathfrak{M}^*})>0$, *then the process* $\varepsilon X^*(t/\varepsilon^2)$ *on* $(\Omega, \mathcal{F}, P^{\hat{\mu}^*_\lambda})$ *converges to* $\hat{\sigma}^*(\lambda)B(t)$ *in the sense of finite dimensional distributions, where* $\hat{\sigma}^*(\lambda)$ *is a nonnegative constant determined by*

$$\hat{\sigma}^*(\lambda)^2 = \lim_{t\to\infty} \frac{1}{t} E^{\hat{\mu}^*_\lambda}[X^*_1(t)^2] .$$

Let $W(x, \eta)$ and $W^*(x, \eta)$ be the occupied cluster and the vacant cluster containing x for η, which is defined in Section 1. We denote the collections

of all unbounded clusters for η by $\mathcal{W}(\eta)$ and $\mathcal{W}^*(\eta)$:

$$\mathcal{W}(\eta)=\{\mathcal{W}(x, \eta): W(x, \eta) \text{ is unbounded, } x\in \boldsymbol{R}^d\},$$

$$\mathcal{W}^*(\eta)=\{W^*(x, \eta): W^*(x, \eta) \text{ is unbounded, } x\in \boldsymbol{R}^d\}.$$

We first study the number of elements of $\mathcal{W}(\eta)$ and $\mathcal{W}^*(\eta)$. Burton and Keane [3] proved the uniqueness of the infinite cluster of a site percolation model on $\boldsymbol{Z}^d$ under a translation invariant finite energy probability measure. Using their technique, we show the uniqueness of the unbounded cluster under a Poisson distribution μ_λ.

PROPOSITION 3.1. (i) *If* $\mu_\lambda(\hat{\mathfrak{M}})>0$, *then* $\#\mathcal{W}(\eta)=1$ *for* μ_λ*-almost all* η.
(ii) *If* $\mu_\lambda(\hat{\mathfrak{M}}^*)>0$, *then* $\#\mathcal{W}^*(\eta)=1$ *for* μ_λ*-almost all* η.

Proof. First, note that $\#\mathcal{W}(\eta)$ is constant, μ_λ-a.s. from the ergodicity of μ_λ under translation. Suppose that $\#\mathcal{W}(\eta)=n$, μ_λ-a.s. η, $2\leqq n<\infty$. There exists $l_0>0$ such that

$$\mu_\lambda(W\cap U_{l_0}(0)\neq\emptyset,\ \forall W\in\mathcal{W})>0\,. \tag{3.1}$$

We define measurable subsets $\Lambda_1(l, \alpha)$, $\Lambda_2(l, \alpha)$, l, $\alpha>0$, of $\mathfrak{M}$ by

$$\Lambda_1(l, \alpha)=\{\eta\in\mathfrak{M}: W\cap U_{l+\alpha}(0)\neq\emptyset,\ \forall W\in\mathcal{W}(\eta\setminus U_l(0))\},$$

$$\Lambda_2(l, \alpha)=\{\eta\in\mathfrak{M}: U_r(\eta\cap U_l(0))\supset U_{l+\alpha}(0)\}.$$

Since any cluster W is open set, from (3.1) we can choose $\alpha_0\in[0, r)$ such that $\mu_\lambda(\Lambda_1(l_0, \alpha_0))>0$. It is easy to see that $\mu_\lambda(\Lambda_2(l_0, \alpha))>0$, $\alpha\in[0, r)$. From a property of a Poisson distribution we have $\mu_\lambda(\Lambda_1(l_0, \alpha_0)\cap\Lambda_2(l_0, \alpha_0))>0$. On the other hand, if $\eta\in\Lambda_1(l_0, \alpha_0)\cap\Lambda_2(l_0, \alpha_0)$, then $\#\mathcal{W}(\eta)=1$. Then, $\mu_\lambda(\#\mathcal{W}=1)>0$. It is a contradiction. Hence,

$$\mu_\lambda(2\leqq\#\mathcal{W}<\infty)=0\,. \tag{3.2}$$

Suppose that $\#\mathcal{W}(\eta)=\infty$, μ_λ-a.s. η. We introduce the following notion. Let $l>0$. A point $x\in l\boldsymbol{Z}^d$ is called an l-encounter site for $\eta\in\mathfrak{M}$, if $W(x, \eta)$ is unbounded and $W(x, \eta)\setminus\overline{U_{l/2}(x)}$ has exactly three unbounded connected components. Denote by $\mathcal{N}_l(\eta)$ the number of unbounded connected components of the open set $W(0, \eta)\setminus\overline{U_l(0)}$. We define a measurable subset $\Lambda_3(l, \alpha)$, l, $\alpha>0$, of $\mathfrak{M}$ by

$$\Lambda_3(l, \alpha)=\{\eta\in\mathfrak{M}: \#\{W\in\mathcal{W}(\eta\setminus U_l(0)): W\cap U_{l+\alpha}(0)\neq\emptyset\}\geqq 3\}.$$

By the same argument as the above we see that there exist $l_1>0$ and $\alpha_1>0$ such that $\mu_\lambda(\Lambda_2(l_1, \alpha_1)\cap\Lambda_3(l_1, \alpha_1))>0$. If $\eta\in\Lambda_2(l_1, \alpha_1)\cap\Lambda_3(l_1, \alpha_1)$, then $\mathcal{N}_{l_1+r}(\eta)\geqq 3$. Then,

$$\mu_\lambda(\mathcal{N}_{l_1+r}\geqq 3)>0\,.$$

Since $\mathcal{N}_{\cdot}(\eta)$ is a right continuous increasing function and satisfies for μ_λ-a.s. η $\mathcal{N}_l(\eta)-\mathcal{N}_{l-}(\eta)\leqq 1$ for all $l>0$, there exists $l_2>0$ such that

$$\mu_\lambda(\mathcal{N}_{l_2/2}=3)>0\,.$$

Then, we have

$$\mu_\lambda(\{\eta : 0 \text{ is an } l_2\text{-encounter site for } \eta\})=\varepsilon>0\,.$$

It follows from the ergodic theorem that for almost all η

(3.3) the number of l_2-encounter sites for η in $V_k=O(k^d)$, $k\to\infty$,

where $V_k=[-(k+1/2)l_2, (k+1/2)l_2]^d$, $k\in \boldsymbol{N}$. Put $\mathcal{W}(\eta)=\{W_1(\eta), W_2(\eta), \ldots\}$ and

$$\mathcal{Y}_i(\eta)=\{Y : Y \text{ is an unbounded connected component of } W_i(\eta)\setminus V_k\}\,.$$

Any l_2-encounter site x for η with $x\in W_i(\eta)\cap V_k$ determines a partition $\boldsymbol{P}(x)=\{\boldsymbol{P}_1(x), \boldsymbol{P}_2(x), \boldsymbol{P}_3(x)\}$ of $\mathcal{Y}_i(\eta)$. If partitions $\boldsymbol{P}(x)=\{\boldsymbol{P}_1(x), \boldsymbol{P}_2(x), \boldsymbol{P}_3(x)\}$ and $\boldsymbol{P}(y)=\{\boldsymbol{P}_1(y), \boldsymbol{P}_2(y), \boldsymbol{P}_3(y)\}$ of $\mathcal{Y}_i(\eta)$ are determined by l_2-encounter sites $x, y\in W_i(\eta)\cap V_k$, $x\neq y$, then $\boldsymbol{P}(x)\neq\boldsymbol{P}(y)$ and there is an ordering of each such that $\boldsymbol{P}_1(y)\supset\boldsymbol{P}_1(x)\cup\boldsymbol{P}_2(x)$. Using Lemma 2 in [3], if $\#\mathcal{P}_i\neq 0$, then

$$\#\mathcal{P}_i(\eta)\leqq\#\mathcal{Y}_i(\eta)-2\,,$$

where $\mathcal{P}_i(\eta)$ is the collection of all partitions of $\mathcal{Y}_i(\eta)$ determined by l_2-encounter sites for η in $W_i(\eta)\cap V_k$. Since $|Y\cap(V_{k+(h/l_2)}\setminus V_k)|\geqq|U_h(0)|$ for any $Y\in\mathcal{Y}_i(\eta)$ with $\#\mathcal{P}_i\neq 0$, we have

$$\sum_{i\,:\,\#\mathcal{P}_i\neq 0}\#\mathcal{Y}_i(\eta)=O(k^{d-1})\,,\qquad k\to\infty\,,$$

and so

(3.4) the number of l_2-encounter sites for η in $V_k=O(k^{d-1})$, $k\to\infty$.

This contradicts (3.3). Hence, we have

(3.5) $$\mu_\lambda(\#\mathcal{W}=\infty)=0\,.$$

This completes the proof of (i). We can prove (ii) in the same way. ■

We introduce the environment process $\eta(t)$ (*resp.* $\eta^*(t)$) seen from $X(t)$ (*resp.* $X^*(t)$) defined by $\eta(t)=\tau_{-X(t)}\eta$ (*resp.* $\eta^*(t)=\tau_{-X^*(t)}\eta$). We denote by $O(\eta)$ (*resp.* $O^*(\eta)$) the union of all elements of $\mathcal{W}(\eta)$ (*resp.* $\mathcal{W}^*(\eta)$). From Proposition 3.1, if $\mu_\lambda(\widehat{\mathfrak{M}})>0$ (*resp.* $\mu_\lambda(\widehat{\mathfrak{M}^*})>0$), then $O(\eta)$ (*resp.* $O^*(\eta)$) is a connected open set for μ_λ-almost all η. Then we can show the following by the same procedure as Theorem 2.1 in [15].

LEMMA 3.2. (i) *Let* $\mu_\lambda(\widehat{\mathfrak{M}})>0$. *Then the process* $(\eta(t), P^{\hat{\mu}_\lambda})$ *is an ergodic reversible Markov process with transition density*

$$\boldsymbol{P}(t,\ \eta,\ \Lambda)=\begin{cases} P_0^{\eta}(X(t)\in\Lambda_\eta), & \textit{if } \eta\in\mathfrak{M}_0\,, \\ \delta_\eta(\Lambda), & \textit{otherwise,} \end{cases}$$

$t>0$, $\eta\in\mathfrak{M}$, $\Lambda\subset\mathfrak{M}$, *where* $\Lambda_\eta=\{x\in\boldsymbol{R}^d:\tau_{-x}\eta\in\Lambda\}$.

(ii) *Let* $\mu_\lambda(\widehat{\mathfrak{M}^*})>0$. *Then the process* $(\eta^*(t),\ P^{\hat{\mu}_\lambda^*})$ *is an ergodic reversible Markov process with transition density*

$$\boldsymbol{P}^*(t,\ \eta,\ \Lambda)=\begin{cases} P_0^{\eta}(X^*(t)\in\Lambda_\eta), & \textit{if } \eta\in\mathfrak{M}_0^*\,, \\ \delta_\eta(\Lambda), & \textit{otherwise,} \end{cases}$$

$t>0$, $\eta\in\mathfrak{M}$, $\Lambda\subset\mathfrak{M}$.

Proof of Theorem 3.1. We introduce the measurable subset $\mathfrak{A}$ of $\boldsymbol{C}(\boldsymbol{R}\to\widehat{\mathfrak{M}})$ defined by

$$\mathfrak{A}=\{\zeta\in\boldsymbol{C}(\boldsymbol{R}\to\widehat{\mathfrak{M}}):\exists\eta\in\widehat{\mathfrak{M}}\setminus\Xi \text{ s.t. } \zeta(t)\in\Gamma(\eta),\ \forall t\geqq 0\},$$

where Ξ and $\Gamma(\eta)$ are measurable subsets of $\mathfrak{M}$ given by

$$\Xi=\{\eta\in\mathfrak{M}:\eta=\tau_u\eta \text{ for some } u\in\boldsymbol{R}^d\setminus\{0\}\},$$

$$\Gamma(\eta)=\{\tau_{-x}\eta:x\in\overline{W(0,\ \eta)}\},\qquad \eta\in\mathfrak{M}\,.$$

We define a functional F_t on $\mathfrak{A}$ by

$$F_t(\zeta(\cdot))=x(t)-x(0),\qquad t\in[0,\ \infty),$$

where $x(\cdot)\in\boldsymbol{C}([0,\ \infty)\to\boldsymbol{R}^d)$ such that $\tau_{x(0)-x(t)}\zeta(0)=\zeta(t)$. The functional F_t is additive and antisymmetric, that is,

$$F_{t+s}(\zeta(\cdot))=F_t(\zeta(\cdot))+F_s(\zeta(t+\cdot)),\qquad F_t(\zeta(\cdot))=-F_t(\zeta(t-\cdot)).$$

From the definition of the process $\eta(t)$ we have

$$X(s)-X(0)=F_s(\eta(\cdot)),\qquad P^{\hat{\mu}_\lambda}\text{-a.s.}$$

Then, $(X(t),\ P^{\hat{\mu}_\lambda})$ is a square integrable antisymmetric additive functional of the ergodic reversible Markov process $(\eta(t),\ P^{\hat{\mu}_\lambda})$. Applying Theorem 2.1 in [5], we obtain that the process $\varepsilon X(t/\varepsilon^2)$ converges to $DB(t)$ in the sense of finite dimensional distribution. (See Section 3 in [15].) Here D is a symmetric and nonnegative definite $d\times d$ matrix determined by

$$(D^2)_{ij}=\lim_{t\to\infty}\frac{1}{t}E^{\hat{\mu}_\lambda}[X_i(t)X_j(t)]\,,$$

however, by the rotation invariance of $X(t)$ the matrix D must be a constant multiple of the unit matrix. We have thus proved (i) of Theorem 3.1. We can prove (ii) in the same way. ■

4. Positivity of the diffusion constants

Combining Proposition 2.1 and Theorem 3.1, we see that the rescaled processes $\varepsilon X(t/\varepsilon^2)$ and $\varepsilon X^*(t/\varepsilon^2)$ converge to $\hat{\sigma}(\lambda)B(t)$ and $\hat{\sigma}^*(\lambda)B(t)$, respectively. Then, the only assertion left to be proved is positivity of the diffusion constants $\hat{\sigma}(\lambda)$ and $\hat{\sigma}^*(\lambda)$. Let $\sigma(\lambda)$ and $\sigma^*(\lambda)$ be nonnegative constants determined by

$$\sigma(\lambda)^2=\lim_{t\to\infty}\frac{1}{t}E^{\mu_\lambda}[X_1(t)^2]\,,\qquad \sigma^*(\lambda)^2=\lim_{t\to\infty}\frac{1}{t}E^{\mu_\lambda}[X_1^*(t)^2]\,.$$

It is easy to see that $\hat{\sigma}(\lambda)^2=(\sigma(\lambda)^2/\mu_\lambda(\widehat{\mathfrak{M}}))$ and $\hat{\sigma}^*(\lambda)^2=(\sigma^*(\lambda)^2/\mu_\lambda(\widehat{\mathfrak{M}^*}))$. Then, it is enough to show the following proposition.

PROPOSITION 4.1. (i) *If* $\lambda>\lambda_c(r)$, *then* $\sigma(\lambda)>0$.
(ii) *If* $\lambda<\lambda_c^*(r)$, *then* $\sigma^*(\lambda)>0$.

For a subset A of $\boldsymbol{R}^d$ and $\beta>0$, we define a measurable kernel $q^\beta[A]$ on $\boldsymbol{R}^d\times\mathscr{B}(\boldsymbol{R}^d)$ by

$$q^\beta[A](x,\,dy)=\frac{\beta^{d+2}d}{\int_{U_1(0)}|z|^2dz}\mathbf{1}_{[0,1/\beta)}(|x-y|)\mathbf{1}_A(x)\mathbf{1}_A(y)dy\,.$$

We abbreviate $q^n[U_r(\eta)]$ to q_η^n for $\eta\in\mathfrak{M}$, $n\in\boldsymbol{N}$. Denote by $(\Omega,\,\mathscr{F},\,P_x^\eta,\,X^n(t))$ a right continuous Markov process starting from x with generator

$$L_\eta^n u(x)=\int_{\boldsymbol{R}^d}q_\eta^n(x,\,dy)(u(y)-u(x))\,,\qquad u\in\boldsymbol{C}_\infty(\boldsymbol{R}^d)\,,$$

where $\boldsymbol{C}_\infty(\boldsymbol{R}^d)$ is the space of continuous functions u on $\boldsymbol{R}^d$ such that $u(x)\to 0$ as $|x|\to\infty$. Clearly L_η^n can be regarded as a bounded linear operator on $L^2(\boldsymbol{R}^d,\,dx)$.

LEMMA 4.1. *Let* μ *be a translation invariant probability measure on* $\mathfrak{M}$. *For any* $t\in(0,\,\infty)$

$$\lim_{n\to\infty}E^\mu[X_1^n(t)^2]=E^\mu[X_1(t)^2]\,.$$

Proof. Let $\eta\in\mathfrak{M}\setminus\Delta$ and $W(x,\,\eta)\neq\emptyset$. Since $\mathscr{E}_{W(x,\eta)}$ is a Dirichlet form, by a standard result there is a nonpositive selfadjoint operator L_η with $\mathscr{D}((-L_\eta)^{1/2})=H^1(W(x,\,\eta))$ such that

$$\mathscr{E}_{W(x,\eta)}(u,\,v)=((-L_\eta)^{1/2}u,\,(-L_\eta)^{1/2}v)_{L^2(W(x,\eta),dx)}\,.$$

Put

$$\mathscr{A}=\left\{u|_{W(x,\eta)}:u\in\boldsymbol{C}_0^\infty(\boldsymbol{R}^d),\ \frac{\partial u}{\partial \boldsymbol{n}}=0\ \text{a.e.}\ dS\ \text{on}\ \partial W(x,\,\eta)\right\},$$

where S is the surface measure for $\partial W(x, \eta)$ and $\boldsymbol{n}$ is the interior normal. We can show that $\mathcal{A}$ is a core for L_η. (See [7].) On the other hand it is easy to see that for any $u \in \mathcal{A}$

$$\lim_{n\to\infty} L_\eta^n u(y) = \Delta u(y) = L_\eta u(y), \qquad \forall y \in W(x, \eta),$$

$$\sup_{n\in N} \sup_{y\in W(x,\eta)} L_\eta^n u(y) < \infty,$$

and so

$$L_\eta^n u \longrightarrow L_\eta u, \qquad \text{as } n\to\infty \text{ in } L^2(W(x, \eta), dx).$$

Applying Theorem 5.2 in Trotter [21], we obtain that for any $u \in L^2(W(x, \eta), dx)$ and $t \in (0, \infty)$

$$T_\eta^n(t)u \longrightarrow T_\eta(t)u \qquad \text{as } n\to\infty \text{ in } L^2(W(x, \eta), dx),$$

where $T_\eta^n(t)$ and $T_\eta(t)$ are semigroups on $L^2(W(x, \eta), dx)$ associated with L_η^n and L^η, respectively. Hence, for any $u, v \in L^2(\boldsymbol{R}^d, dx)$

$$\lim_{n\to\infty} \int_{W(x,\eta)} dy E_y^\eta[u(X^n(0))v(X^n(t))] = \int_{W(x,\eta)} dy E_y^\eta[u(X(0))v(X(t))].$$

Noting that $X^n(t) = X(t) = y$, $t \in [0, \infty)$, P_y^η-a.s. if $y \notin U_r(\eta)$, we obtain

$$\lim_{n\to\infty} \int_{U_1(0)} dy E_y^\eta[u(X^n(0))v(X^n(t))] = \int_{U_1(0)} dy E_y^\eta[u(X(0))v(X(t))]. \tag{4.1}$$

From the translation invariance of μ we have

$$\begin{aligned} E^\mu[u(X^n(t))] &= \int_{\mathfrak{M}} \mu(d\eta) E_0^\eta[u(X^n(t))] \\ &= \frac{1}{|U_1(0)|} \int_{U_1(0)} dx \int_{\mathfrak{M}} \mu(d\eta) E_x^{\tau_x\eta}[u(X^n(t) - X^n(0))] \\ &= \int_{\mathfrak{M}} \mu(d\eta) \frac{1}{|U_1(0)|} \int_{U_1(0)} dx E_x^\eta[u(X^n(t) - X^n(0))]. \end{aligned} \tag{4.2}$$

Similarly,

$$E^\mu[u(X(t))] = \int_{\mathfrak{M}} \mu(d\eta) \frac{1}{|U_1(0)|} \int_{U_1(0)} dx E_x^\eta[u(X(t) - X(0))]. \tag{4.3}$$

Hence, from (4.1), (4.2) and (4.3) we arrive at

$$\lim_{n\to\infty} E^\mu[u(X^n(t))] = E^\mu[u(X(t))], \tag{4.4}$$

for any $u \in \boldsymbol{C}_0(\boldsymbol{R}^d)$ and $t \in (0, \infty)$. Then for the proof of the lemma it is sufficient to show that

$$\lim_{M\to\infty} \sup_{n\in N} E^\mu[X_1^n(t)^2 : X_1^n(t)^2 > M] = 0.$$

This equality is derived from the estimate

$$\sup_{n\in N} E^{\mu}[|X_1^n(t)|^3]<\infty\,,$$

which can be shown by the same argument as in the proof of Lemma 2.2. ■

We give the definition of an electrical network and an effective conductivity in the context of an electrical network, which is a generalization of that in Doyle and Snell [6]. Let (q, m) be a pair of a measurable kernel q on $\boldsymbol{R}^d\times\mathscr{B}(\boldsymbol{R}^d)$ and a Radon measure m on $\boldsymbol{R}^d$ satisfying

$$q(x, dy)m(dx)=q(y, dx)m(dy)\,.$$

We call the pair (q, m) an electrical network. Throughout this paper, we consider the case where $q(\cdot, A)$ is bounded for any bounded set A. For $k\in\boldsymbol{N}$, the effective conductivity in (q, m) is defined by

$$\mathscr{I}_k(q, m)=\min_u\left\{\frac{1}{2}\int_{[-k-1/2,\,k+1/2)^d}m(dx)\int_{[-k-1/2,\,k+1/2)^d}q(x, dy)(u(y)-u(x))^2\right\},$$

where u ranges all measurable functions u on $[-k-1/2, k+1/2)^d$ satisfying $u=0$ on $[-k-1/2, -k+1/2)\times[-k-1/2, k+1/2)^{d-1}$ and $u=1$ on $[k-1/2, k+1/2)\times[-k-1/2, k+1/2)^{d-1}$.

We state the following result whose proof is given in Section 5.

LEMMA 4.2. *Let $\lambda>\lambda_c(r)$. Then there exists a positive constant $c(\lambda)$ such that*

$$\varliminf_{k\to\infty}(2k)^{2-d}\mathscr{I}_k(q_\eta^n, dx)\geqq c(\lambda)\,,\qquad \mu_\lambda\text{-}a.s.\ \forall n\in\boldsymbol{N}.$$

Now, we are in a position to prove Proposition 4.1.

Proof of Proposition 4.1. In [19] we have shown that $X^n(t)$ is an anti-symmetric additive functional of a reversible Markov process. By the same calculation as in the proof of Theorem 2.2 in [5], we obtain that

$$E^{\mu_\lambda}[X_1^n(t)^2]=t\int_{\mathfrak{M}}\mu_\lambda(d\eta)\int_{\boldsymbol{R}^d}q_\eta^n(0, du)u_1^2$$
$$-2\int_0^t ds\int_0^s dv\int_{\mathfrak{M}}\mu_\lambda(d\eta)G(\eta)S_vG(\eta)\,,$$

where S_t is the semigroup on $L^2(\mathfrak{M}, \mu_\lambda)$ associated with the reversible process and $G(\eta)=\int_{\boldsymbol{R}^d}q_\eta^n(0, du)u_1$. Since S_t is symmetric, we see that

$$\lim_{t\to\infty}\frac{1}{t}E^{\mu_\lambda}[X_1^n(t)^2]\leqq\frac{1}{t}E^{\mu_\lambda}[X_1^n(t)^2]\,,\qquad\forall t\in(0, \infty)\,. \tag{4.5}$$

On the other hand, we proved in [19] that

$$\lim_{t\to\infty}\frac{1}{t}E^{\mu_\lambda}[X_1^n(t)^2]\geqq\int_{\mathfrak{M}}\mu_\lambda(d\eta)\varliminf_{k\to\infty}2(2k)^{2-d}\mathcal{G}_k(q_\eta^n,\ dx). \tag{4.6}$$

Then, from Lemmas 4.1 and 4.2 we conclude that

$$\begin{aligned}\sigma(\lambda)^2&=\lim_{t\to\infty}\frac{1}{t}E^{\mu_\lambda}[X_1(t)^2]\\&=\lim_{t\to\infty}\frac{1}{t}\lim_{n\to\infty}E^{\mu_\lambda}[X_1^n(t)^2]\\&\geqq\varliminf_{n\to\infty}\lim_{t\to\infty}\frac{1}{t}E^{\mu_\lambda}[X_1^n(t)^2]\\&\geqq\varliminf_{n\to\infty}\int_{\mathfrak{M}}\mu_\lambda(d\eta)\varliminf_{k\to\infty}2(2k)^{2-d}\mathcal{G}_k(q_\eta^n,\ dx)\\&\geqq 2c(\lambda).\end{aligned}$$

This completes the proof of (i). A proof of (ii) is obtained similarly. ■

5. Proof of Lemma 4.2

In this section we give the proof of Lemma 4.2. First we introduce a site percolation model on $\boldsymbol{Z}^d$. For each $x=(x_1, x_2, \dots, x_d)$ and $y=(y_1, y_2, \dots, y_d)$ we write $\langle x, y\rangle$ and say that x is adjacent to y if $\sum_{j=1}^d |x_j-y_j|=1$. The element $x\in\boldsymbol{Z}^d$ is called a site. Put $\boldsymbol{X}=\{0, 1\}^{\boldsymbol{Z}^d}$ and equip $\boldsymbol{X}$ with the σ-field generated by $\{\xi\in\boldsymbol{X}:\xi(x)=0\}$, $x\in\boldsymbol{Z}^d$. For $\xi\in\boldsymbol{X}$, a site $x\in\boldsymbol{Z}^d$ is called occupied (*resp.* vacant), if $\xi(x)=1$ (*resp.* $\xi(x)=0$). We write $x\leftrightarrow y$ if there is an occupied path from x to y; there is a sequence $x^0=x, x^1, \dots, x^n=y$ of occupied sites so that $\langle x^{m-1}, x^m\rangle$, $1\leqq m\leqq n$. For $A_1, A_2\subset\boldsymbol{Z}^d$ we write $A_1\leftrightarrow A_2$ if there are sites $x\in A_1$, $y\in A_2$ such that $x\leftrightarrow y$. We define a region in $\boldsymbol{Z}^d$ by

$$C(x, \xi)=\{y\in\boldsymbol{Z}^d : x\leftrightarrow y\},$$

i.e., the occupied cluster containing the site x. For a given $l\in\boldsymbol{N}$ define a measurable function $\pi_l:\mathfrak{M}\to\boldsymbol{X}$ by

$$\pi_l(\eta)(x)=\begin{cases}1 & \text{if } D_l(x)\cap\eta\neq\emptyset,\\ 0 & \text{if } D_l(x)\cap\eta=\emptyset,\end{cases}$$

$\eta\in\mathfrak{M}$, $x\in\boldsymbol{Z}^d$, where $D_l(x)$ denotes the union of the cubes

$$I(y)=\prod_{i=1}^d\left[y_i-\frac{1}{2},\ y_i+\frac{1}{2}\right)$$

with $y\in\boldsymbol{Z}^d$ and $I(y)\subset U_l(x)$. Define a probability measure P_p on $\boldsymbol{X}$ by $P_p=\mu_{\theta^{-1}(p)}\circ\pi_l^{-1}$, where $\theta(\lambda)=1-\exp(-\lambda)$ and θ^{-1} is the inverse function of θ. The

critical value is defined as follows:

$$p_c = p_c(l) = \inf\{p>0 : P_p(\# C(0)=\infty)>0\}.$$

For $\xi \in X$ we denote by $\mathcal{N}_k(\xi)$ the maximal number of disjoint occupied paths in a cube $\{-k, -k+1, \dots, k\}^d$ from $\{-k\}\times\{-k, -k+1, \dots, k\}^{d-1}$ to $\{k\}\times\{-k, -k+1, \dots, k\}^{d-1}$. In Tanemura [20] we obtain the following proposition which is an analogue of the result shown by Grimmett and Marstrand [11] in the case of the Bernoulli percolation model.

PROPOSITION 5.1. *If $p>p_c$, then there exist $c_1(p)$, $c_2(p)>0$ such that*

$$P_p(\mathcal{N}_k \geqq c_1(p)k^{d-1}) \geqq 1-\exp(-c_2(p)k^{d-1}), \qquad \forall k \in N.$$

Put $\widehat{\mathcal{G}}_k(q, m) = (2k)^{2-d}\mathcal{G}_k(q, m)$ and $\widehat{\mathcal{N}}_k(\xi) = k^{1-d}\mathcal{N}_k(\xi)$ and introduce the non-negative constant $c_3(\beta, c)$ defined by

$$c_3(\beta, c) = \inf\{\widehat{\mathcal{G}}_k(q^\beta[I[\xi]], dx) : k \in N, \xi \in X \text{ with } \widehat{\mathcal{N}}_k(\xi) \geqq c\}, \tag{5.1}$$

for $\beta>0$ and $c>0$, where $I[\xi] = \bigcup_{\xi(x)=1} I(x)$.

LEMMA 5.1. *Suppose that $\beta \in (0, 1/\sqrt{d+3})$ and $c>0$. Then, $c_3(\beta, c)>0$.*

Proof. Let $(\tilde{q}_\xi, \tilde{m})$, $\xi \in X$ be the electrical network defined by

$$\tilde{q}_\xi(x, dy) = \mathbf{1}_{Z^d}(x)\xi(x) \sum_{\substack{u \in Z^d \\ \langle u, x\rangle}} \xi(u)\delta_u(dy), \qquad \tilde{m}(dx) = \sum_{u \in Z^d} \delta_u(dx).$$

It was already shown that for any $c>0$

$$c_4(c) = \inf\{\widehat{\mathcal{G}}_k(\tilde{q}_\xi, \tilde{m}) : k \in N, \xi \in X \text{ with } \widehat{\mathcal{N}}_k(\xi) \geqq c\} > 0. \tag{5.2}$$

(See Section 6 in Grimmett and Kesten [10] or Section 3 in Chayes and Chayes [4].) Put

$$\hat{q}_\xi(x, dy) = \sum_{u, v \in Z^d} \tilde{q}_\xi(u, \{v\})\mathbf{1}_{I(u)}(x)\mathbf{1}_{I(v)}(y)dy.$$

Then, we have

$$\hat{q}_\xi(x, dy) \leqq c_5(\beta)q^\beta[I[\xi]](x, dy), \qquad \text{if } \beta \in \left(0, \frac{1}{\sqrt{d+3}}\right), \tag{5.3}$$

where $c_5(\beta) = \beta^{-d-2}d\int_{U_1(0)} |z|^2 dz$.

It is easy to see that for any $\xi \in X$

$$\mathcal{G}_k(\tilde{q}_\xi, \tilde{m}) = \mathcal{G}_k(\hat{q}_\xi, dx), \qquad \forall k \in N, \tag{5.4}$$

and

$$(5.5)\qquad \mathcal{J}_k(\hat{q}_\xi, dx) \leqq c_5(\beta)\mathcal{J}_k(q^\beta[I[\xi]], dx), \quad \forall k\in \boldsymbol{N},\ \forall \beta\in\left(0, \frac{1}{\sqrt{d+3}}\right),$$

from (5.3). Combining (5.4) and (5.5), we obtain Lemma 5.1 from (5.2). ■

Proof of Lemma 4.2. Note that $(\mu_\lambda, \eta)=(\mu_{\alpha^d\lambda}, \alpha\eta)$ for any $\alpha>0$. Then, we see that $\lambda_c(r)=\alpha^d\lambda_c(\alpha r)$, $\alpha>0$. Let $\lambda>\lambda_c(r)$. Choose $\delta_0>0$ such that $\lambda>(r/(r-\delta_0))^d\lambda_c(r)=\lambda_c(r-\delta_0)$. We first show that there exist α, $l\in \boldsymbol{N}$ with $\alpha>\sqrt{d+3}$ satisfying

$$(5.6)\qquad U_{r-\delta_0}(\eta)\subset \frac{1}{n\alpha} I[\pi_{nl}(n\alpha\eta)]\subset U_r(\eta), \qquad \forall \eta\in\mathfrak{M},\ \forall n\in \boldsymbol{N}.$$

From the definition we have $U_{l-\sqrt{d}}(x)\subset D_l(x)\subset U_l(x)$ for any $x\in \boldsymbol{R}^d$. Then

$$\begin{aligned} U_{l-2\sqrt{d}}(\eta)&\subset \bigcup_{\substack{x\in \boldsymbol{Z}^d\\ U_{l-\sqrt{d}}(x)\cap\eta\neq\emptyset}} I(x)\subset \bigcup_{\substack{x\in \boldsymbol{Z}^d\\ D_l(x)\cap\eta\neq 0}} I(x)\\ &=I[\pi_l\eta]\subset \bigcup_{\substack{x\in \boldsymbol{Z}^d\\ U_l(x)\cap\eta\neq\emptyset}} I(x)\subset U_{l+\sqrt{d}}(\eta), \end{aligned}$$

for any $l\in \boldsymbol{N}$ and $\eta\in\mathfrak{M}$. Hence

$$\begin{aligned} U_{(l/\alpha)-(2\sqrt{d}/n\alpha)}(\eta)&=\frac{1}{n\alpha}U_{nl-2\sqrt{d}}(n\alpha\eta)\subset \frac{1}{n\alpha}I[\pi_{nl}(n\alpha\eta)]\\ &\subset \frac{1}{n\alpha}U_{nl+\sqrt{d}}(n\alpha\eta)=U_{(l/\alpha)+(\sqrt{d}/n\alpha)}(\eta). \end{aligned}$$

Choosing l and α such that $r-\delta_0<(1/\alpha)(l-2\sqrt{d})<(1/\alpha)(l+\sqrt{d})<r$, we obtain (5.6).

Let $\lambda'\in(\lambda_c(r-\delta_0), \lambda)$. Then, for $\mu_{\lambda'}$-a.e. η, $U_{r-\delta_0}(\eta)$ contains an unbounded connected component and so $I[\pi_l(\alpha\eta)]$ contains an unbounded connected component. Hence, $I[\pi_l(\eta)]$ contains an unbounded connected component for $\mu_{\alpha^{-d}\lambda'}$-a.e. η, and so $\theta(\alpha^{-d}\lambda)>\theta(\alpha^{-d}\lambda')\geqq p_c(l)$. In view of Proposition 5.1 putting $c_1'=c_1(\theta(\alpha^{-d}\lambda))$ and $c_2'=c_2(\theta(\alpha^{-d}\lambda))$, we obtain

$$P_{\theta(\alpha^{-d}\lambda)}(\widehat{\mathfrak{N}}_k\geqq c_1')\geqq 1-\exp(-c_2'k^{d-1}), \qquad \forall k\in \boldsymbol{N},$$

and hence

$$\mu_\lambda(\eta\in\mathfrak{M}: \widehat{\mathfrak{N}}_k(\pi_l(\alpha\eta))\geqq c_1')\geqq 1-\exp(-c_2'k^{d-1}), \qquad \forall k\in \boldsymbol{N}.$$

Noting that $nI[\pi_l(\eta)]\subset I[\pi_{nl}(n\eta)]$, we obtain $\widehat{\mathfrak{N}_{nk}}(\pi_{nl}(n\eta))\geqq\widehat{\mathfrak{N}}_k(\pi_l(\eta))$. Accordingly,

$$(5.7)\qquad \mu_\lambda(\eta\in\mathfrak{M}: \widehat{\mathfrak{N}_{nk}}(\pi_{nl}(n\alpha\eta))\geqq c_1')\geqq 1-\exp(c_2'k^{d-1}), \qquad \forall k,\ \forall n\in \boldsymbol{N}.$$

Since $q_\eta^n(x, dy)\geqq q^n[(1/n\alpha)I[\pi_{nl}(n\alpha\eta)]](x, dy)$ from (5.6), we get

$$\mathcal{J}_k(q_\eta^n,\ dx) \geqq \mathcal{J}_k\Big(q^n\Big[\frac{1}{n\alpha} I[\pi_{nl}(n\alpha\eta)]\Big],\ dx\Big), \qquad \forall \eta \in \mathfrak{M}. \tag{5.8}$$

We see that $\widehat{\mathcal{J}_k}(q^\beta[A],\ dx) \geqq \widehat{\mathcal{J}_{nk}}(q^{\beta/n}[nA],\ dx)$ by simple calculation. Then

$$\widehat{\mathcal{J}_k}\Big(q^n\Big[\frac{1}{n\alpha} I[\pi_{nl}(n\alpha\eta)]\Big],\ dx\Big) \geqq \widehat{\mathcal{J}_{n\alpha k}}(q^{1/\alpha}[I[\pi_{nl}(n\alpha\eta)]],\ dx), \quad \forall \eta \in \mathfrak{M}. \tag{5.9}$$

From (5.1), (5.7), (5.8) and (5.9) we have

$$\begin{aligned}
&\mu_\lambda\Big(\eta \in \mathfrak{M} : \widehat{\mathcal{J}_k}(q_\eta^n,\ dx) \geqq c_3\Big(\frac{1}{\alpha},\ c_1'\Big)\Big) \\
&\geqq \mu_\lambda\Big(\eta \in \mathfrak{M} : \widehat{\mathcal{J}_k}\Big(q^n\Big[\frac{1}{n\alpha} I[\pi_{nl}(n\alpha\eta)]\Big],\ dx\Big) \geqq c_3\Big(\frac{1}{\alpha},\ c_1'\Big)\Big) \\
&\geqq \mu_\lambda\Big(\eta \in \mathfrak{M} : \widehat{\mathcal{J}_{n\alpha k}}(q^{1/\alpha}[I[\pi_{nl}(n\alpha\eta)]],\ dx) \geqq c_3\Big(\frac{1}{\alpha},\ c_1'\Big)\Big) \\
&\geqq \mu_\lambda(\eta \in \mathfrak{M} : \widehat{\mathcal{N}_{n\alpha k}}(\pi_{nl}(n\alpha\eta)) \geqq c_1') \\
&\geqq 1 - \exp(-c_2' k^{d-1}).
\end{aligned}$$

Since $c_3(1/\alpha,\ c_1') > 0$ from Lemma 5.1, Lemma 4.2 is obtained by Borel-Cantelli's Lemma. ■

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