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RIEMANNIAN SUBMERSION WITH ISOMETRIC REFLECTIONS WITH RESPECT TO THE FIBERS

Dedicated to Professor Yoji Hatakeyama on his sixtieth birthday

BY FUMIO NARITA

1. Introduction

The concept of Riemannian submersion was introduced by O'Neil [10] and is discussed by him and others ([4], [8], etc). A Riemannian submersion with totally geodesic fibers often appears in the differential geometry.

On the other hand, in [3] Chen and Vanhecke introduced the notion of the reflections with respect to submanifolds. And there are some studies of reflections with respect to the fibers in a Riemannian submersion or local fibering of a Sasakian manifold (e.g. [2], [9], [11]).

In this paper, we shall consider a Riemannian submersion $\pi: M \to N$ with fibers of dimension one. In Section 2, we give some properties of the integrability tensor A with respect to π. In Section 3, we shall consider the isometric reflections with respect to the fibers in Riemannian submersion which satisfies certain conditions. Our result is a generalization of the result of Kato and Motomiya [6], [11]. And particularly, in the case of 3-dimension, we get the following result: the reflections with respect to the fibers are isometries if and only if M admits a Sasakian locally ϕ-symmetric structure. Finally, we give a complete classification of 3-dimensional Riemannian manifolds with isometric reflections with respect to the fibers.

2. Riemannian submersion

In this section we collect some results on Riemannian submersions. Let $\pi: M \to N$ be a Riemannian submersion. Let X denote a tangent vector at $x \in M$. Then X decomposes as $\mathcal{V}X + \mathcal{H}X$, where $\mathcal{V}X$ is tangent to the fiber through x and $\mathcal{H}X$ is perpendicular to it. If $X = \mathcal{V}X$, X is called a vertical vector. If $X = \mathcal{H}X$, it is called horizontal. Let ∇ and $\tilde{\nabla}$ denote the Riemannian connections of M and N respectively.

We define tensors T and A associated with the submersion by

$$T_E F = \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F, \tag{1}$$

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(2) $$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F,$$

for arbitrary vector fields E and F on M. T and A satisfy the following properties ([10]).

(i) T_E and A_E are skew symmetric linear operator on the tangent space of M, and reverse the horizontal and vertical parts.

(ii) $T_E = T_{\nu E}$ while $A_E = A_{\mathcal{H}E}$.

(iii) For V, W vertical, $T_V W$ is symmetric, i.e. $T_V W = T_W V$. For X, Y horizontal, $A_X Y$ is skew-symmetric, i.e. $A_X Y = -A_Y X$.

A vector field X on M is said to be basic if X is horizontal and π-related to a vector field $\tilde{X}$ on N. Every vector field $\tilde{X}$ on N has a unique horizontal lift X to M, and X is basic. We denote it by $X = h.l.(\tilde{X})$. Let g and $\tilde{g}$ be the metrics of M and N respectively.

LEMMA 1 ([10]). *Let X and Y be horizontal vector fields and V and W are vertical vector fields on M. Then*

(i) $\nabla_V W = T_V W + \mathcal{V}\nabla_V W$,

(ii) $\nabla_V X = \mathcal{H}\nabla_V X + T_V X$,

(iii) $\nabla_X V = A_X V + \mathcal{V}\nabla_X V$,

(iv) $\nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y$.

Furthermore, if X is basic, then $\mathcal{H}\nabla_V X = A_X V$.

Denote by R the curvature tensor of M. The horizontal lift of the curvature tensor $\tilde{R}$ is defined as follows: if X_1, X_2, X_3, X_4 are horizontal tangent vectors to M, we set

$$g(\tilde{R}_{X_1X_2}(X_3), X_4) = \tilde{g}(\tilde{R}_{\tilde{X}_1\tilde{X}_2}(\tilde{X}_3), \tilde{X}_4)\circ\pi,$$

where $\tilde{X}_i = \pi(X_i)$.

Let $\pi: M \to N$ be a Riemannian submersion with totally geodesic fibers. Then $T = 0$.

LEMMA 2 ([10]). *Let $\pi: M \to N$ be a Riemannian submersion with totally geodesic fibers. Let X, Y, Z and H be horizontal vector fields and V and W be vertical vector fields on M. Then*

(i) $R(X, V, Y, W) = g((\nabla_V A)_X Y, W) + g(A_X V, A_Y W)$,

(ii) $R(X, Y, Z, V) = g((\nabla_Z A)_X Y, V)$,

(iii) $R(X, Y, Z, H) = \tilde{R}(X, Y, Z, H) - 2g(A_X Y, A_Z H) + g(A_Y Z, A_X H) + g(A_Z X, A_Y H)$.

LEMMA 3. *Let $\pi: M \to N$ be a Riemannian submersion with totally geodesic fibers. Let X and Y be horizontal vector fields and V and W be vertical vector fields on M. Then*

(i) $g((\nabla_V A)_X Y, W) = -g(\nabla_V(A_X W), Y) - g(A_X Y, \nabla_V W) - g(\nabla_V X, A_Y W)$,

(ii) $g((\nabla_V A)_X Y, V) = 0$.

Proof. By Lemma 1 and the property of A, we get

$$\begin{aligned}
g((\nabla_V A)_X Y,\, W) &= g(\nabla_V(A_X Y),\, W) - g(A_{\nabla_V X} Y,\, W) - g(A_X(\nabla_V Y),\, W) \\
&= Vg(A_X Y,\, W) - g(A_X Y,\, \nabla_V W) + g(A_Y(\nabla_V X),\, W) \\
&\quad + g(\nabla_V Y,\, A_X W) \\
&= -Vg(Y,\, A_X W) - g(A_X Y,\, \nabla_V W) - g(\nabla_V X,\, A_Y W) \\
&\quad + g(A_X W,\, \nabla_V Y) \\
&= -g(\nabla_V Y,\, A_X W) - g(Y,\, \nabla_V(A_X W)) - g(A_X Y,\, \nabla_V W) \\
&\quad - g(\nabla_V X,\, A_Y W) + g(A_X W,\, \nabla_V Y) \\
&= -g(\nabla_V(A_X W),\, Y) - g(A_X Y,\, \nabla_V W) - g(\nabla_V X,\, A_Y W)
\end{aligned}$$

Next, we put $V=W$ in Lemma 2 (i), then $g((\nabla_V A)_X Y,\, V)$ is symmetric with respect to X and Y. On the other hand, since A has the alternation property $A_X Y = -A_Y X$ and $\nabla_V X$, $\nabla_V Y$ are horizontal, $g((\nabla_V A)_X Y,\, V)$ is skew-symmetric with respect to X and Y. Therefore we see that $g((\nabla_V A)_X Y,\, V)=0$. ∎

From these Lemmas, we have the following.

PROPOSITION 1. *Let $\pi: M \to N$ be a Riemannian submersion with totally geodesic fibers of dimension one. If X is a basic vector field on M, then $A_X V$ is a basic vector field where V is a vertical vector field on M such that $\nabla_V V=0$.*

Proof. By Lemma 3, for basic vector field X and any basic vector field B, we get

$$\begin{aligned}
Vg(A_X V,\, B) &= g(\nabla_V(A_X V),\, B) + g(A_X V,\, \nabla_V B) \\
&= g(\nabla_V(A_X V),\, B) + g(\nabla_V X,\, A_B V) \\
&= 0\,.
\end{aligned}$$

This means that $A_X V$ is a basic vector field. ∎

3. Isometric reflection

Let M be a Riemannian manifold and B a connected embedded submanifold which is relatively compact. The (local) reflection φ_B with respect to B is defined as the local geodesic symmetry for normal geodesics to B in a sufficiently small tubular neighbourhood of B. The reflection φ_B is a local diffeomorphism ([3]).

Next, we give the definition of a Sasakian locally ϕ-symmetric space. A Riemannian manifold (M, g) is said to be a Sasakian manifold if there exist a tensor field ϕ of type (1, 1), a unit vector field V and a 1-form η such that

$$\phi(V)=0, \tag{3}$$

$$\eta(\phi X)=0, \tag{4}$$

$$\phi^2(X)=-X+\eta(X)V, \tag{5}$$

$$g(\phi X, \phi Y)=g(X, Y)-\eta(X)\eta(Y), \tag{6}$$

$$(\nabla_X\phi)Y=g(X, Y)V-\eta(Y)X \tag{7}$$

for any vector fields X, Y on M, where ∇ is the Riemannian connection for g. Let R be the curvature tensor of M. A Sasakian manifold M is said to be a locally ϕ-symmetric space if $\phi^2[(\nabla_X R)(Y, Z)H]=0$ for any vector fields X, Y, Z, H orthogonal to V.

THEOREM 1. *Let M be an orientable connected $(2n+1)$-dimensional Riemannian manifold and $\pi: M\to N$ be fiber dimension one Riemannian submersion satisfying the condition $A_{A_X W}W=-\rho g(W, W)X$, where ρ is a positive function and X is any horizontal vector field and W a vertical vector field on M. Then the reflections with respect to the fibers are isometries if and only if M admits a Sasakian locally ϕ-symmetric structure.*

Proof. We assume that the reflections with respect to the fibers are isometries. Then the fibers are totally geodesic submanifolds in M. Let V be a unit vertical vector field such that $\nabla_V V=0$. We define a (1, 1)-tensor ϕ by

$$\phi E := -\frac{1}{\sqrt{\rho}} A_E V \tag{8}$$

where E is any vector field on M. Let η be the one-form dual to V. By the definition of ϕ and η, we get

$$\phi(V)=0, \tag{9}$$

$$\eta(\phi E)=0. \tag{10}$$

By the condition $A_{A_X V}V=-\rho g(V, V)X=-\rho X$, for any vector field E on M, we get

$$\begin{aligned}\phi^2(E)&=\phi^2(\mathcal{H}E+\mathcal{V}E)=\phi^2(\mathcal{H}E)=\frac{1}{\rho}A_{A_{\mathcal{H}E}V}V=-\mathcal{H}E\\ &=-(\mathcal{H}E+\mathcal{V}E)+\eta(\mathcal{H}E+\mathcal{V}E)V=-E+\eta(E)V.\end{aligned} \tag{11}$$

Moreover, for vector fields E and F on M, we get

$$
\begin{aligned}
(12) \qquad g(\phi E, \phi F) &= \frac{1}{\rho} g(A_{\mathcal{H}E}V, A_{\mathcal{H}F}V) = -\frac{1}{\rho} g(V, A_{\mathcal{H}E}(A_{\mathcal{H}F}V)) \\
&= \frac{1}{\rho} g(V, A_{A_{\mathcal{H}F}V}\mathcal{H}E) = -\frac{1}{\rho} g(A_{A_{\mathcal{H}F}V}V, \mathcal{H}E) = g(\mathcal{H}F, \mathcal{H}E) \\
&= g(\mathcal{H}E + \mathcal{V}E, \mathcal{H}F + \mathcal{V}F) - \eta(\mathcal{H}E + \mathcal{V}E)\eta(\mathcal{H}F + \mathcal{V}F) \\
&= g(E, F) - \eta(E)\eta(F).
\end{aligned}
$$

Thus (M, ϕ, V, η, g) admits an almost contact metric structure.

Since the reflections with respect to the fibers are isometries, for horizontal vector fields X, Y, Z, we have $R(X, Y, Z, V)=0$ and $(\nabla_X R)(X, V, X, V)=0$ ([3]). Since the fibers are totally geodesic submanifolds in M, by Lemma 2 and Lemma 3, we have $R(X, V, Y, V) = g(A_X V, A_Y V) = \rho g(V, V) g(X, Y) = \rho g(X, Y)$, where X, Y are horizontal vector fields. For any horizontal vector field X, we get

$$
\begin{aligned}
0 &= (\nabla_X R)(X, V, X, V) \\
&= XR(X, V, X, V) - 2R(\nabla_X X, V, X, V) - 2R(X, \nabla_X, V, X, V) \\
&= X(\rho g(X, X)) - 2\rho g(\nabla_X X, X) \\
&= (X\rho) g(X, X).
\end{aligned}
$$

Therefore, we get $X\rho=0$. Moreover, when $g(X, X)=1$, using $\nabla_V V=0$ and Lemma 3, we get

$$
\begin{aligned}
V\rho &= V(g(A_X V, A_X V)) = 2g(\nabla_V(A_X V), A_X V) \\
&= -2g((\nabla_V A)_X(A_X V), V) - 2g(\nabla_V X, A_{A_X V} V) \\
&= 2\rho g(\nabla_V X, X) = 0.
\end{aligned}
$$

Therefore ρ is constant.

We set $\bar{g}=\rho g$, $\bar{V}=(1/\sqrt{\rho})V$ and $\bar{\eta}=\sqrt{\rho}\,\eta$. Then we have the following equations

$$\phi(\bar{V})=0, \tag{13}$$

$$\bar{\eta}(\phi E)=0, \tag{14}$$

$$\phi^2(E) = -E + \bar{\eta}(E)\bar{V}, \tag{15}$$

$$\bar{g}(\phi E, \phi F) = \bar{g}(E, F) - \bar{\eta}(E)\bar{\eta}(F), \tag{16}$$

where E and F are vector fields on M.

Let $\bar{\nabla}$ be the Riemannian connection and $\bar{R}$ the curvature tensor with respect to $\bar{g}$. Let $\bar{A}$ be the integrability tensor with respect to $\bar{\nabla}$. Since ρ is constant, we have $\bar{\nabla}_E F = \nabla_E F$ and $\bar{A}_E F = A_E F$. We shall show the following equation

(17) $$(\overline{\nabla}_E\phi)F=\bar{g}(E,\ F)\bar{V}-\bar{\eta}(F)E\,.$$

Since the fibers are totally geodesic submanifolds in M, we get $\mathcal{V}\overline{\nabla}_{\nu E}\mathcal{H}F=0$ and $\mathcal{H}\overline{\nabla}_{\nu E}\mathcal{V}F=0$. Let X, Y and Z be horizontal vector fields. Since A_YZ inventical, we obtain

$$\begin{aligned}
0&=\bar{R}(Y,\ Z,\ X,\ \bar{V})\\
&=\bar{g}((\overline{\nabla}_X\bar{A})_YZ,\ \bar{V})\\
&=\bar{g}(\overline{\nabla}_X(\bar{A}_YZ),\ \bar{V})-\bar{g}(\bar{A}_{\overline{\nabla}_XY}Z,\ \bar{V})-\bar{g}(\bar{A}_Y(\overline{\nabla}_XZ),\ \bar{V})\\
&=X\bar{g}(\bar{A}_YZ,\ \bar{V})-\bar{g}(\bar{A}_YZ,\ \overline{\nabla}_X\bar{V})-\bar{g}(\bar{A}_{\overline{\nabla}_XY}Z,\ \bar{V})+\bar{g}(\overline{\nabla}_XZ,\ \bar{A}_Y\bar{V})\\
&=-X\bar{g}(Z,\ \bar{A}_Y\bar{V})-\bar{g}(\bar{A}_{\overline{\nabla}_XY}Z,\ \bar{V})+\bar{g}(\overline{\nabla}_XZ,\ \bar{A}_Y\bar{V})\\
&=-\bar{g}(\overline{\nabla}_XZ,\ \bar{A}_Y\bar{V})-\bar{g}(Z,\ \overline{\nabla}_X(\bar{A}_Y\bar{V}))-\bar{g}(\bar{A}_{\overline{\nabla}_XY}Z,\ \bar{V})+\bar{g}(\overline{\nabla}_XZ,\ \bar{A}_Y\bar{V})\\
&=-\bar{g}(\overline{\nabla}_X(\bar{A}_Y\bar{V}),\ Z)+\bar{g}(\bar{A}_{\overline{\nabla}_XY}\bar{V},\ Z),
\end{aligned}$$

and we get $\mathcal{H}(\overline{\nabla}_X(\bar{A}_Y\bar{V}))=\bar{A}_{\overline{\nabla}_XY}\bar{V}$. Using this equation and Lemma 3, for any vector fields E, F, D, we have the following equation

$$\begin{aligned}
\bar{g}((\overline{\nabla}_E\phi)F,\ D)&=\bar{g}(\overline{\nabla}_E(\phi F),\ D)-\bar{g}(\phi(\overline{\nabla}_EF),\ D)\\
&=-\bar{g}(\overline{\nabla}_E(A_F\bar{V}),\ D)+\bar{g}(A_{\overline{\nabla}_EF}\bar{V},\ D)\\
&=-\bar{g}(\overline{\nabla}_{\mathcal{H}E}(A_{\mathcal{H}F}\bar{V}),\ \mathcal{H}D)-\bar{g}(\overline{\nabla}_{\mathcal{H}E}(A_{\mathcal{H}F}\bar{V}),\ \mathcal{V}D)-\bar{g}(\overline{\nabla}_{\nu E}(A_{\mathcal{H}F}\bar{V}),\ \mathcal{H}D)\\
&\quad+\bar{g}(A_{\overline{\nabla}_{\nu E}\mathcal{H}F}\bar{V},\ \mathcal{H}D)+\bar{g}(A_{\overline{\nabla}_{\mathcal{H}E}\mathcal{H}F}\bar{V},\ \mathcal{H}D)+\bar{g}(A_{\overline{\nabla}_{\mathcal{H}E}\nu F}\bar{V},\ \mathcal{H}D)\\
&=-\bar{g}(\overline{\nabla}_{\nu E}(A_{\mathcal{H}F}\bar{V}),\ \mathcal{H}D)+\bar{g}(A_{\overline{\nabla}_{\nu E}\mathcal{H}F}\bar{V},\ \mathcal{H}D)\\
&\quad-\bar{g}(\overline{\nabla}_{\mathcal{H}E}(A_{\mathcal{H}F}\bar{V}),\ \mathcal{V}D)+\bar{g}(A_{\overline{\nabla}_{\mathcal{H}E}\nu F}\bar{V},\ \mathcal{H}D)\\
&=-\bar{g}((\overline{\nabla}_{\nu E}A)_{\mathcal{H}F}\bar{V}+A_{\mathcal{H}F}(\overline{\nabla}_{\nu E}\bar{V}),\ \mathcal{H}D))\\
&\quad+\bar{g}(A_{\mathcal{H}F}\bar{V},\ A_{\mathcal{H}E}\mathcal{V}D)-\bar{g}(A_{\mathcal{H}D}\bar{V},\ A_{\mathcal{H}E}\mathcal{V}F)\\
&=\bar{g}((\overline{\nabla}_{\nu E}A)_{\mathcal{H}F}\mathcal{H}D,\ \bar{V})\\
&\quad+\bar{g}(\bar{V},\ \mathcal{V}D)\bar{g}(A_{\mathcal{H}F}\bar{V},\ A_{\mathcal{H}E}\bar{V})-\bar{g}(\bar{V},\ \mathcal{V}F)\bar{g}(A_{\mathcal{H}D}\bar{V},\ A_{\mathcal{H}E}\bar{V})\\
&=\bar{g}(\bar{V},\ \mathcal{V}D)\bar{g}(\mathcal{H}E,\ \mathcal{H}F)-\bar{g}(\bar{V},\ \mathcal{V}F)\bar{g}(\mathcal{H}D,\ \mathcal{H}E),
\end{aligned}$$

because $A_{\mathcal{H}E}(f\bar{V})=\mathcal{H}(\nabla_{\mathcal{H}E}(f\bar{V}))=f\mathcal{H}\nabla_{\mathcal{H}E}\bar{V}=fA_{\mathcal{H}E}\bar{V}$. On the other hand,

$$\begin{aligned}
\bar{g}(\bar{g}(E,\ F)\bar{V}-\bar{\eta}(F)E,\ D)&=\bar{g}(\mathcal{H}E,\ \mathcal{H}F)\bar{g}(\bar{V},\ \mathcal{V}D)+\bar{g}(\mathcal{V}E,\ \mathcal{V}F)\bar{g}(\bar{V},\ \mathcal{V}D)\\
&\quad-\bar{g}(\mathcal{H}E,\ \mathcal{H}D)\bar{g}(\bar{V},\ \mathcal{V}F)-\bar{g}(\mathcal{V}E,\ \mathcal{V}D)\bar{g}(\bar{V},\ \mathcal{V}F)\\
&=\bar{g}(\bar{V},\ \mathcal{V}D)\bar{g}(\mathcal{H}E,\ \mathcal{H}F)-\bar{g}(\bar{V},\ \mathcal{V}F)\bar{g}(\mathcal{H}D,\ \mathcal{H}E).
\end{aligned}$$

Therefore we get

$$(\bar{\nabla}_E\phi)F=\bar{g}(E,\ F)\bar{V}-\bar{\eta}(F)E\,.$$

Hence $(M,\ \phi,\ \bar{V},\ \bar{\eta},\ \bar{g})$ is a Sasakian manifold. We complete the proof with the following fact of [2], [11]: a necessary and sufficient condition for a Sasakian manifold to be a locally ϕ-symmetric space is that the local ϕ-geodesic symmetries (i.e. the reflections with respect to the fibers) are isometries. ∎

Remark 1. In the above theorem, if we suppose that the reflections with respect to the fibers are isometries and set $J\tilde{X}=-(1/\sqrt{\rho})\pi_*(A_XV)$ where $\tilde{X}$ is any vector field on N and $X=h.l.(\tilde{X})$, then N admits a locally symmetric Kaehlerian structure.

Let G be a semi-simple, compact and connected Lie group and g a bi-invariant Riemannian metric on G. Let G/K be a homogeneous space of a Lie group G over a connected, closed subgroup K of G, and assume that the Lie algebra $\mathfrak{g}$ of G has a family $(\mathfrak{g}_i)_{i\geqq 0}$ of subspaces of $\mathfrak{g}$ satisfying the following conditions (i)$\sim$(iv):

(i) $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2$ (direct sum),

(ii) $[\mathfrak{g}_i,\ \mathfrak{g}_j]\subset\mathfrak{g}_{i+j}+\mathfrak{g}_{|i-j|}$, where $\mathfrak{g}_l=\{0\}$ for $l>2$,

(iii) $\mathfrak{g}_0$ is the Lie algebra of K,

(iv) $\dim\mathfrak{g}_2=1$.

Let H be the connected and closed subgroup of G with the Lie algebra $\mathfrak{g}_0+\mathfrak{g}_2$ and $K\subset H$. We shall consider the following diagram

$$G\xrightarrow[\pi]{}G/K\xrightarrow[\mu]{}G/H.$$

G/K and G/H inherit natural metrics through the projections $\pi: G\rightarrow G/K$ and $\eta=\mu\circ\pi: G\rightarrow G/H$ respectively. Then η, π and μ are real analytic Riemannian submersions with compact connected totally geodesic fibers. Moreover, we assume the following condition

(v) For any $X\in\mathfrak{g}_1$ and $V\in\mathfrak{g}_2$,

$$[[X,\ V],\ V]=-\rho g(V,\ V)X,$$

where ρ is a positive function on G/K.

Then, we get

COROLLARY 1. *In the above Riemannian submersion $\mu: G/K\rightarrow G/H$, we assume that the Lie algebra $\mathfrak{g}$ of G has a family $(\mathfrak{g}_i)_{i\geqq 0}$ of subspaces of $\mathfrak{g}$ satisfying the conditions* (i)$\sim$(v). *Then G/K admits a Sasakian locally ϕ-symmetric structure.*

Proof. By Example 2 in [9], when a family $(\mathfrak{g}_i)_{i\geqq 0}$ of subspaces of $\mathfrak{g}$ satisfies the conditions (i)$\sim$(iii), the reflections with respect to the fibers are isometries. Let A be the integrability tensor with respect to μ. Then, for $V\in\mathfrak{g}_2$, $X\in\mathfrak{g}_1$, we get $A_{A_XV}V=(1/4)[[X,\ V],\ V]_{\mathfrak{g}_1}=-(1/4)\rho g(V,\ V)X$. Therefore, by Theorem 1, G/K admits a Sasakian locally ϕ-symmetric structure. ■

Remark 2. The above Theorem 1 is a generalization of the following result of Kato and Motomiya [6], [11]:

Let G/K be a homogeneous space of a semi-simple, compact and simply connected Lie group G over a connected, closed subgroup K of G, and assume that the Lie algebra $\mathfrak{g}$ of G has a family $(\mathfrak{g}_i)_{i\geq 0}$ of subspaces of $\mathfrak{g}$ satisfying the following conditions (i)~(vi):

(i) $\mathfrak{g}=\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2$ (direct sum),

(ii) $[\mathfrak{g}_i, \mathfrak{g}_j]\subset\mathfrak{g}_{i+j}+\mathfrak{g}_{|i-j|}$, where $\mathfrak{g}_l=\{0\}$ for $l>2$,

(iii) $\mathfrak{g}_0$ is the Lie algebra of K, and $[\mathfrak{g}_0, \mathfrak{g}_0]=\mathfrak{g}_0$,

(iv) $\dim\mathfrak{g}_2=1$,

(v) There is an element V of $\mathfrak{g}_2$ such that

$$[[X, V], V]=-X \quad \text{for all} \quad X\in\mathfrak{g}_1,$$

(vi) $\mathrm{Ad}(g)\mathfrak{g}_i=\mathfrak{g}_i$, and $\mathrm{Ad}(g)V=V$ for all $g\in K$, where $\mathrm{Ad}(g)$ denotes the adjoint representation of K in $\mathfrak{g}$.

Let H be the connected Lie subgroup of G with the Lie algebra $\mathfrak{g}_0+\mathfrak{g}_2$.

Then G/K is a Sasakian locally ϕ-symmetric space and G/K is a principal circle bundle over a Hermitian symmetric space G/H with Kaehlerian structure.

Next, we consider the case where the dimension of M is three.

THEOREM 2. *Let M be an orientable connected 3-dimensional Riemannian manifold and $\pi: M\to N$ be fiber dimension one Riemannian submersion satisfying $A_XV\neq 0$, where X is a horizontal vector field and V a vertical vector field on M. Then the reflections with respect to the fibers are isometries if and only if M admits a Sasakian locally ϕ-symmetric structure.*

Proof. By $g(A_XV, X)=-g(V, A_XX)=0$, A_XV is orthogonal to X and horizontal. Therefore, $\{V, X, A_XV\}$ is a local basis of tangent space of M. By $g(A_{A_XV}V, A_XV)=-g(V, A_{A_XV}(A_XV))=0$, $A_{A_XV}V$ is orthogonal to V and A_XV. Therefore we have $A_{A_XV}V=-\rho g(V, V)X$, where ρ is a positive function on M, because $g(A_{A_XV}V, X)=-g(V, A_{A_XV}X)=g(V, A_X(A_XV))=-g(A_XV, A_XV)$ $=-\rho g(V, V)g(X, X)$. We show that ρ is independent of the choice of X. So, we consider another horizontal vector field $Y=\alpha X+\beta A_XV$. Then $A_{A_YV}V=$ $A_{A_{(\alpha X+\beta A_XV)}V}V=\alpha A_{A_XV}V+\beta A_{A_{A_XV}V}V=-\alpha\rho g(V, V)X-\beta\rho g(V, V)A_XV=$ $-\rho g(V, V)Y$. Hence, for any horizontal vector field X, we have the following equation $A_{A_XV}V=-\rho g(V, V)X(\rho>0)$. Therefore, by Theorem 1, M admits a Sasakian locally ϕ-symmetric structure. ■

Example 1. Let H be the Heisenberg group:

$$H=\left\{\begin{pmatrix}1 & s & u\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix};\ s,\ t,\ u\in \boldsymbol{R}\right\}.$$

Heisenberg group H is not semi-simple. We identify H and $\boldsymbol{R}^3$ as manifolds. We denote the elements of H as $m=(s, t, u)$, the multiplication being

$$(s, t, u)(s', t', u')=(s+s', t+t', u+u'+st').$$

The vector fields

$$X_1=\frac{\partial}{\partial s},\qquad X_2=\frac{\partial}{\partial t}+s\frac{\partial}{\partial u},\qquad V_1=\frac{\partial}{\partial u}$$

are left invariant and $[X_1, X_2]=V_1$, $[X_1, V_1]=[X_2, V_1]=0$. With respect to the standard coordinates (s, t, u) in $\boldsymbol{R}^3$, we set $g=ds^2+dt^2+(du-sdt)^2$. Then g is a Riemannian metric such that vector fields X_1, X_2 and V_1 are orthonormal. g is left invariant, but not right invariant. Let ∇ be the Riemannian connection associated to g. Then we get

$$\nabla_{X_1}X_2=-\nabla_{X_2}X_1=\frac{1}{2}V_1\quad \nabla_{X_2}V_1=\nabla_{V_1}X_2=\frac{1}{2}X_1,\quad \nabla_{V_1}X_1=\nabla_{X_1}V_1=-\frac{1}{2}X_2,$$

$$\nabla_{X_1}X_1=\nabla_{X_2}X_2=\nabla_{V_1}V_1=0.$$

We consider the subgroup K of H:

$$K=\left\{\begin{pmatrix}1 & 0 & u\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix};\ u\in \boldsymbol{R}\right\}.$$

Then, $\pi: H\to H/K$ is a Riemannian submersion with totally geodesic fibers [5]. V_1 is a vertical vector field and X_1 and X_2 are horizontal vector fields. Let R be the Riemannian curvature tensor associated to g. Since g is left invariant, for any left invariant vector fields $C, E, F, I, L,$ we get $Cg(E, F)=0$ and $CR(E, F, I, L)=0$. By

$$(\nabla_{X_1}R)(X_1, V_1, X_1, X_2)=R(X_1, \nabla_{X_1}V_1, X_1, X_2)+R(X_1, V_1, X_1, \nabla_{X_1}X_2)=\frac{1}{2},$$

H is not a locally symmetric space. By the following equations

$$R(X_1, X_2, X_1, V_1)=0,\quad R(X_2, X_1, X_2, V_1)=0,\quad (\nabla_{X_1}R)(X_1, X_2, X_1, X_2)=0,$$

$$(\nabla_{X_2}R)(X_1, X_2, X_1, X_2)=0,\qquad (\nabla_{X_1}R)(X_1, V_1, X_1, V_1)=0,$$

$$(\nabla_{X_1}R)(X_1, V_1, X_2, V_1)=0,\qquad (\nabla_{X_1}R)(X_2, V_1, X_2, V_1)=0,$$

we get

$$R(X, Y, X, V)=0, \quad (\nabla_X R)(X, Y, X, Z)=0, \quad (\nabla_X R)(X, V, X, V)=0,$$

where X, Y, Z are horizontal vector fields and V is a vertical vector field. Let $q\in H/K$ and $p\in\pi^{-1}(q)$. Let x, y, z be horizontal vectors at p and v be a vertical vector at p. Let V be a vertical vector field such that $V_p=v$. We extend x, y, z to horizontal vector fields X, Y, Z such that X, Y and Z are linear combination of X_1 and X_2 with constant coefficients. Then, we get

$$\nabla_X X=0, \qquad \nabla_X Y\in\mathcal{V}, \qquad \nabla_X\nabla_Y Z\in\mathcal{H},$$

$$\nabla_X V=\nabla_V X\in\mathcal{H}, \qquad \nabla_X\nabla_V Y\in\mathcal{V}.$$

By the equation

$$\begin{aligned}(\nabla^2_{XX}R)(X, Y, X, V)=&\nabla_X((\nabla_X R)(X, Y, X, V))-(\nabla_X R)(\nabla_X X, Y, X, V)\\ &-(\nabla_X R)(X, \nabla_X Y, X, V)-(\nabla_X R)(X, Y, \nabla_X X, V)\\ &-(\nabla_X R)(X, Y, X, \nabla_X V)-(\nabla_{\nabla_X X}R)(X, Y, X, V),\end{aligned}$$

using above property, at point p, we obtain $(\nabla^2_{xx}R)(x, y, x, v)=0$. Next, by the mathematical induction, we can prove the following equations

$$(\nabla^{2k+1}_{x\cdots x}R)(x, y, x, z)=0, \qquad (\nabla^{2k+1}_{x\cdots x}R)(x, v, x, v)=0,$$

$$(\nabla^{2k+2}_{x\cdots x}R)(x, y, x, v)=0 \qquad (k\in\boldsymbol{N}).$$

Therefore, the reflections with respect to the fibers are isometries ([3]). Hence H admits a Sasakian locally ϕ-symmetric structure.

Remark 3. The above example is an example of Theorem 1 which can not be covered by the result of Kato and Motomiya.

Next, under the same notation in Corollary 1, we get

COROLLARY 2. *Let the Lie algebra* $\mathfrak{g}$ *of* G *has a family* $(\mathfrak{g}_i)_{i\geq 0}$ *of subspaces of* $\mathfrak{g}$ *satisfying conditions* (i)~(iv). *In a Riemannian submersion* $\mu: G/K\rightarrow G/H$, *suppose* $\dim(G/K)=3$ *and* $A_X V\neq 0$ *where* $X\in\mathfrak{g}_1$, $V\in\mathfrak{g}_2$ *Then* G/K *admits a Sasakian locally* ϕ-*symmetric structure.*

Proof. Since the reflections with respect to the fibers are isometries [9], by Theorem 2, G/K admits a Sasakian locally ϕ-symmetric structure. ∎

Next, we shall consider a three-dimensional Lie group.

COROLLARY 3. *Let* G *be a three-dimensional semi-simple, compact and connected Lie group and* K *be a one-dimensional closed subgroup of* G. *Let* g *be a bi-invariant metric of* G. *In Riemannian submersion* $\pi: G\rightarrow G/K$, *suppose* $R(X, Y, Z, V)=0$ *and* $A_X V\neq 0$, *where* X, Y, Z *are horizontal vector fields and*

V is a vertical vector field on G. Then G admits a Sasakian locally ϕ-symmetric structure.

Proof. Since G is a symmetric space and $R(X, Y, Z, V)=0$, the reflections with respect to the fibers are isometries ([3]). By Theorem 2, G admits a Sasakian locally ϕ-symmetric structure. ■

Example 2. We consider a Riemannian submersion;

$$\pi: SU(2) \longrightarrow SU(2)/S(U(1)\times U(1)).$$

The decomposition of the Lie algebra $\mathfrak{g}$ of $SU(2)$ is given by

$$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2 \qquad \text{(direct sum)},$$

where

$$\mathfrak{g}_1=\left\{\begin{pmatrix} 0 & -\bar{\xi} \\ \xi & 0 \end{pmatrix};\ \xi\in \boldsymbol{C}^1\right\}$$

and

$$\mathfrak{g}_2=\left\{\begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix};\ \alpha+\bar{\alpha}=0\right\}.$$

Then, for $X, Y, Z\in\mathfrak{g}_1$ and $V\in\mathfrak{g}_2$, we have $R(X, Y, Z, V)=0$ and $A_X V\neq 0$. Therefore, $SU(2)$ admits a Sasakian locally ϕ-symmetric structure. In this example $\rho=1$.

Finally, we give a complete classification of 3-dimensional Riemannian manifolds with isometric reflections with respect to the fibers. A simply connected complete Sasakian locally ϕ-symmetric space is a naturally reductive homogeneous space [2]. Using the result of Theorem 2 and the explicit classification of naturally reductive homogeneous spaces in dimension three (cf. [2], [12]), we get the following:

THEOREM 3. *Let M be a three-dimensional orientable connected simply connected complete Riemannian manifold and $\pi: M\rightarrow N$ be fiber dimension one Riemannian submersion satisfying $A_X V\neq 0$, where X is a horizontal vector field and V is a vertical vector field on M. Then all the reflections with respect to the fibers are isometries if and only if M is isometric to one of the following spaces:*

(i) *the unit sphere S^3 in $\boldsymbol{R}^4$;*
(ii) *$SU(2)$;*
(iii) *Heisenberg group H;*
(iv) *the universal covering space of $SL(2, \boldsymbol{R})$.*

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