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ON THE ZEROS OF A HOMOGENEOUS DIFFERENTIAL POLYNOMIAL

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1. Introduction

Concerning a question of W. K. Hayman [2] (see also [3: Problem 1.18], [4]) E. Mues [5] discussed entire functions g in which the homogeneous differential polynomial $g''g-ag'^2$ has no zeros. He proved that

$$g(z)=\exp(\alpha z+\beta), \qquad \alpha(\neq 0),\ \beta\in \boldsymbol{C}$$

are the only transcendental entire functions with this property if $a\neq 1$. Giving the following two counter-examples he also showed that the case where $a=1$ is indeed exceptional:

(a) $$g(z)=\sin z$$

and

(b) $$g(z)=\exp\{Q(z)-h(z)\},$$

provided that $h(z)$ is an arbitrary entire function and $Q(z)$ is an entire function defined by

$$Q(z)=\int_{z_0}^{z}\int_{\zeta_0}^{\zeta}\{h''(t)+\exp(2h(t))\}\,dt\,d\zeta.$$

In this paper we discuss the corresponding results to meromorphic functions g when there somewhat exist both the poles of g and the zeros of the homogeneous differential polynomial $g''g-ag'^2$. By a meromorphic function we mean a function meromorphic in the complex plane $\boldsymbol{C}$. We follow the notation and terminology of [2] and [4]. We shall explain the special symbols whenever we introduce them. In particular, if f is a meromorphic function, we shall denote by $S(r, f)$ any quantity

(1.0) $$S(r, f)=o\{T(r, f)\}$$

as $r\to\infty$, possibly outside a set of r of finite linear measure. For the sake of

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simplicity we describe this matter as (1.0) holds as $r\to\infty$, *n.e.*. For any complex number a we denote by $W_a(z)$ the homogeneous differential polynomial $g''(z)g(z)-ag'(z)^2$ in a given meromorphic function $g(z)$.

For our purpose a basic role will be played by the following result of [6]:

THEOREM A. *Let $g(z)$ be a non-constant meromorphic function and define $W_a(z)$ by*

$$W_a(z)=g''(z)g(z)-ag'(z)^2 \tag{1.1}$$

for $a\in\mathbf{C}$. If $W_a(z)$ does not vanish identially, then the inequality

$$T\left(r,\frac{g'}{g}\right)\leqq A_a m\left(r,\frac{g'}{g}\right)+B_a m\left(r,\frac{W_a'}{W_a}\right) \tag{1.2}$$

$$+C_a\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\}+U_a(r)$$

holds as $r\to\infty$, except for two cases (i) *and* (ii) *below. Here the constants A_a, B_a and C_a depend only on the number a and satisfy*

$$0\leqq A_a\leqq\begin{cases}4 & \text{if } a\neq 1,\ 1/2,\\ 2 & \text{if } a=1,\\ 1 & \text{if } a=1/2,\end{cases}\qquad 0\leqq B_a\leqq\begin{cases}5 & \text{if } a\neq 1,\ 1/2,\ 0,\\ 2 & \text{if } a=1,\\ 4 & \text{if } a=1/2,\\ 1 & \text{if } a=0,\end{cases}\qquad 0\leqq C_a\leqq 5,$$

and also $U_a(r)$ is a real-valued function on $[0, \infty)$ such that if we fix the number a, then it satisfies

$$U_a(r)=\begin{cases}O\left[\log^+T\left(r,\frac{g'}{g}\right)+\log^+m\left(r,\frac{W_a'}{W_a}\right)+\log^+\{\bar{N}(r,0,W_a)+\bar{N}(r,g)\}+\log r\right], & \\ \qquad\qquad \text{if } a\neq 1/2,\ 0, & \\ O\left[\log^+T\left(r,\frac{g'}{g}\right)+\log^+\{\bar{N}(r,0,W_a)+\bar{N}(r,g)\}+\log r\right], & \text{if } a=0,\\ O(1), & \text{if } a=1/2\end{cases}$$

as $r\to\infty$, n.e.. The exceptions are:

(i) *when $a=1/2$, $g(z)=\alpha z^2+\beta z+\gamma$, where α, β, $\gamma\in\mathbf{C}$ and $\beta^2-4\alpha\gamma\neq 0$;*

(ii) *when $a=1$, $g(z)=C_1e^{\lambda_1 z}+C_2e^{\lambda_2 z}$, where λ_1, λ_2, C_1, $C_2\in\mathbf{C}$ and $\lambda_1\neq\lambda_2$ and $C_1C_2\neq 0$.*

As an auxiliary lemma we shall make use of a corollary of Theorem A only once.

COROLLARY A. *Besides the hypotheses of Theorem A, we assume that $g(z)$ is an entire function and that*

$$m(r, W_a)=o\{m(r, g)\}$$

as $r\to\infty$, *n. e.. Then* $W_a(z)$ *must be a constant* $(\neq 0)$ *and* $g(z)$ *is at least one of the following*:

(i) *when* $a=1/2$, $g(z)=\alpha z^2+\beta z+\gamma$, *where* $\alpha, \beta, \gamma\in \boldsymbol{C}$ *with* $\beta^2-4\alpha\gamma\neq 0$;

(ii) *when* $a=1$, $g(z)=C_1e^{\lambda z}+C_2e^{-\lambda z}$, *where* λ, C_1, $C_2\in \boldsymbol{C}-\{0\}$; *and*

(iii) *when* $a\neq 0$, $1/2$, $g(z)=\alpha z+\beta$, *where* $\alpha(\neq 0)$, $\beta\in \boldsymbol{C}$.

First of all we shall separate the sections according to whether $a=1$ or not. In each section we consider some applications of this theorem. In the next two sections we shall reform Theorem A for $a\neq 1$ (Theorem 1) and improve a result due to Mues (Corollary 1). Section 4 is devoted to a comparison between the cases $a\neq 1$ and $a=1$ (Theorem 2 and Corollary 2).

2. Preparation

We shall need to quote a lemma of Mues [5], which we reform as follows.

LEMMA. *Suppose that* $f(z)$ *is a non-constant meromorphic function such that* $f(z)+bz$ *does not vanish identically for a non-zero constant* b. *Then*

$$m\left(r, \frac{1}{f(z)+bz}\right)\leqq 2\left\{\overline{N}\left(r, \frac{1}{f'}\right)+\overline{N}_2(r, f)\right\}+S(r, f)+O(1) \tag{2.1}$$

as $r\to\infty$, *where* $\overline{N}_2(r, f)$ *denotes the counting function with respect to the multiple poles of* $f(z)$, *each pole being counted only once.*

In particular, $S(r, f)=O\{\log^+T(r, f)+\log r\}$ *as* $r\to\infty$, *n. e. if* $f(z)$ *is transcendental, and* $S(r, f)=o(1)$ *as* $r\to\infty$ *if* $f(z)$ *is rational.*

Proof. Inequality (2.1) evidently holds if $f(z)$ is a polynomial and not of $-bz$. In fact then we obtain

$$m\left(r, \frac{1}{f(z)+bz}\right)=O(1)$$

as $r\to\infty$. Suppose that $f(z)$ is not a polynomial. By the same reason as the above we need not consider the case

$$f(z)=-\frac{C_1}{z-z_0}+C_2, \tag{2.2}$$

where z_0, C_1, $C_2\in \boldsymbol{C}$ and $C_1\neq 0$. By $\overline{N}_0(r, 1/f'')$ we denote the counting function with respect to the only distinct zeros of $f''(z)$ not corresponding to zeros of $f'(z)$. Then the inequality

$$\overline{N}(r, f)\leqq \overline{N}_0\left(r, \frac{1}{f''}\right)+\overline{N}\left(r, \frac{1}{f'}\right)+2\overline{N}_2(r, f)+S(r, f)+O(1) \tag{2.3}$$

holds as $r\to\infty$. In fact, suppose that $f(z)$ has at least a simple pole since there is nothing to prove otherwise. As mentioned in [5] the function

$$\frac{f'(z)f'''(z)}{f''(z)^2}-\frac{3}{2}=\left(-\frac{f'(z)}{f''(z)}\right)'-\frac{1}{2}$$

has at least a double zero at every simple pole of $f(z)$ if it does not vanish identically; otherwise, we deduce (2.2) excluded above, which satisfies (2.1) but indeed fails to satisfy (2.3). Thus $F(z):=-f'(z)/f''(z)$ is now not a constant and $F'(z)$ is not constantly equal to 1/2. Moreover,

$$\begin{aligned}2\{\overline{N}(r, f)-\overline{N}_2(r, f)\} &\leqq N\left(r, \frac{1}{2}, F'\right)\\ &\leqq T(r, F')+O(1)\\ &\leqq \overline{N}(r, F)+T(r, F)+m\left(r, \frac{F'}{F}\right)+O(1),\end{aligned}$$

as $r\to\infty$. Clearly all the poles of $F(z)$ can occur at and only at the zeros of $f''(z)$ other than zeros of $f'(z)$. Thus

$$\overline{N}(r, F)=\overline{N}_0\left(r, \frac{1}{f''}\right).$$

Since $F(z)$ has a necessarily simple zero when and only when either $f'(z)$ has a zero or $f(z)$ has a pole,

$$N\left(r, \frac{1}{F}\right)=\overline{N}\left(r, \frac{1}{f'}\right)+\overline{N}(r, f).$$

Using $F'/F=f''/f'-f'''/f''$ we obtain $m(r, F'/F)=S(r, f)$. Similarly $m(r, 1/F)=S(r, f)$. Therefore as r tends to infinity,

$$\begin{aligned}2\{\overline{N}(r, f)-\overline{N}_2(r, f)\} &\leqq \overline{N}(r, F)+m\left(r, \frac{1}{F}\right)+N\left(r, \frac{1}{F}\right)+m\left(r, \frac{F'}{F}\right)+O(1)\\ &=\overline{N}_0\left(r, \frac{1}{f''}\right)+\overline{N}\left(r, \frac{1}{f'}\right)+\overline{N}(r, f)+S(r, f)+O(1),\end{aligned}$$

from which (2.3) immediately follows.

Next, following Mues [5] and noting $b\neq 0$ we make use of Nevanlinna's fundamental inequality

$$m\left(r, \frac{1}{f'+b}\right)+m\left(r, \frac{1}{f'}\right)\leqq m\left(r, \frac{1}{f''}\right)+S(r, f)+O(1)$$

as $r\to\infty$, where $S(r, f)$ is such a quantity as mentioned in the second part of this Lemma (cf. Hayman [2: § 2.1.1]). Thus

$$\begin{aligned}
m\left(r, \frac{1}{f(z)+bz}\right) &\leqq m\left(r, \frac{1}{f'+b}\right)+S(r, f) \\
&\leqq m\left(r, \frac{1}{f''}\right)-m\left(r, \frac{1}{f'}\right)+S(r, f)+O(1) \\
&=T(r, f'')-N\left(r, \frac{1}{f''}\right)-m\left(r, \frac{1}{f'}\right)+S(r, f)+O(1) \\
&\leqq \bar{N}(r, f)+T(r, f')-N\left(r, \frac{1}{f''}\right)-m\left(r, \frac{1}{f'}\right)+S(r, f)+O(1) \\
&=\bar{N}(r, f)+m\left(r, \frac{1}{f'}\right)+N\left(r, \frac{1}{f'}\right)-N\left(r, \frac{1}{f''}\right)-m\left(r, \frac{1}{f'}\right) \\
&\quad +S(r, f)+O(1) \\
&=N\left(r, \frac{1}{f'}\right)-N\left(r, \frac{1}{f''}\right)+\bar{N}(r, f)+S(r, f)+O(1)
\end{aligned}$$

as $r\to\infty$. This together with (2.3) gives further

$$\begin{aligned}
(2.4) \qquad m\left(r, \frac{1}{f(z)+bz}\right) &\leqq N\left(r, \frac{1}{f'}\right)-N\left(r, \frac{1}{f''}\right)+\bar{N}_0\left(r, \frac{1}{f''}\right)+\bar{N}\left(r, \frac{1}{f'}\right) \\
&\quad +2\bar{N}_2(r, f)+S(r, f)+O(1) \\
&=2\left\{\bar{N}\left(r, \frac{1}{f'}\right)+\bar{N}_2(r, f)\right\}-\tilde{N}(r)+S(r, f)+O(1)
\end{aligned}$$

as $r\to\infty$, where

$$\tilde{N}(r):= N\left(r, \frac{1}{f''}\right)+\bar{N}\left(r, \frac{1}{f'}\right)-\bar{N}_0\left(r, \frac{1}{f''}\right)-N\left(r, \frac{1}{f'}\right).$$

Let z_0 be a zero of $f''(z)$ and $m_0(\geqq 1)$ its multiplicity. If $f'(z_0)=0$, $f'(z)$ has the point z_0 as a zero of multiplicity m_0+1. Then the contribution of z_0 to the function $\tilde{N}(r)$ is of

$$m_0+1-0-(m_0+1)=0 .$$

If otherwise, it is of

$$m_0+0-1-0=m_0-1\geqq 0 .$$

Let z_1 be a zero of $f'(z)$ and $m_1(\geqq 1)$ its multiplicity. If $m_1\geqq 2$, then $f''(z_1)=0$ and z_1 is counted in $\tilde{N}(r)$

$$(m_1-1)+1-0-m_1=0$$

times. Also if $m_1=1$, $f''(z_1)\neq 0$ and the counting number is

$$0+1-0-1=0 .$$

Therefore the zeros of $f'(z)$ contribute nothing toward $\tilde{N}(r)$ and

$$\tilde{N}(r)=N_0\left(r, \frac{1}{f''}\right)-\bar{N}_0\left(r, \frac{1}{f''}\right)\geqq 0,$$

where in $N_0(r, 1/f'')$ only the zeros of $f''(z)$ which are not of $f'(z)$ are to be considered together with their multiplicities. Hence the estimate (2.4) gives (2.1) as desired. The second part is also verified by virtue of the Lemma on logarithmic derivatives. We have thus proved lemma.

By using this result we can estimate $m(r, g'/g)$ and consequently $m(r, W_a'/W_a)$ in terms of $\bar{N}(r, 0, W_a)$ and $\bar{N}(r, g)$. Mues [5] applied his lemma to the constant $b=a-1$ and the function

$$f(z)=-\frac{g(z)}{g'(z)}-(a-1)z.$$

It is possible in our discussion as well. In fact, both $f'(z)=W_a(z)/g'(z)^2$ and $f(z)+bz=-g(z)/g'(z)$ do not vanish identically under the hypotheses of Theorem A. Thus Inequality (2.1) holds. Every zero of $f'(z)$ can occur possibly at either a zero of $W_a(z)$ or a pole of $g(z)$, while a multiple pole of $f(z)$ occurs at a zero of $g'(z)$. We shall investigate these points more carefully.

(a) *With respect to a zero z_0 of $W_a(z)$*: In order that $f'(z)=W_a(z)/g'(z)^2=\{g''(z)g(z)-ag'(z)^2\}/g'(z)^2$ assumes the value 0 at $z=z_0$, it needs that either $g'(z_0)\neq 0$, or $g'(z)$ has a zero with multiplicity $n(\geqq 1)$ and $W_a(z)$ also has a zero with multiplicity at least $2n+1$ there. The latter situation occurs only if $g(z_0)=0$. Thus for z near z_0,

$$g(z)=c_0(z-z_0)^{n+1}\{1+O(z-z_0)\}, \qquad c_0\in \boldsymbol{C}-\{0\}$$

$$g'(z)=(n+1)c_0(z-z_0)^n\{1+O(z-z_0)\},$$

and

$$g''(z)=n(n+1)c_0(z-z_0)^{n-1}\{1+O(z-z_0)\}.$$

Putting $m=n+1$ $(\geqq 2)$, which is the multiplicity of the zero of $g(z)$ at $z=z_0$, we see that

$$W_a(z)=mC_0^2\{m(1-a)-1\}(z-z_0)^{2(m-1)}+O\{(z-z_0)^{2m-1}\}$$

as $z\to z_0$ and that $W_a(z)$ has a zero with multiplicity $\geqq 2n+1=2m-1$ only if $a=(m-1)/m$ and thus $m=-1/(a-1)$.

(b) *With respect to a pole z_1 of $g(z)$*: Let $m(\geqq 1)$ be its multiplicity and expand $g(z)$ in a neighborhood of $z=z_1$ as follows;

$$g(z)=\frac{C_1}{(z-z_1)^m}\{1+O(z-z_1)\}, \qquad C_1\in \boldsymbol{C}-\{0\}.$$

Thus

$$\frac{g'(z)}{g(z)}=\frac{-m}{z-z_1}+O(1),$$

$$\left(\frac{g(z)}{g'(z)}\right)'=-\frac{1}{m}\{1+O(z-z_1)\}$$

and therefore

$$f'(z)=\left(-\frac{g(z)}{g'(z)}-(a-1)z\right)'=\left\{\frac{1}{m}-(a-1)\right\}+O(z-z_1)$$

as $z\to z_1$. Hence $f'(z_1)$ is different from zero, unless $a=(m+1)/m$ and thus $m=1/(a-1)$.

(c) *With respect to a zero z_2 of $g'(z)$*: At this point $f(z)$ can have a multiple pole only if $g(z_2)\neq 0$ and further $g'(z)$ has a multiple zero at $z=z_2$. It is easy to see that $W_a(z)$ assumes the value 0 at such a point z_2.

Summarizing (a), (b), and (c) we get the following inequality:

$$\bar{N}\left(r, \frac{1}{f'}\right)+\bar{N}_2(r, f)\leqq \bar{N}(r, 0, W_a)+\bar{N}_{1/(a-1)}(r, g),$$

where $\bar{N}_{1/(a-1)}(r, g)$ denotes the counting function with respect to the distinct poles having multiplicity $1/(a-1)$ if it is a positive integer, while define $\bar{N}_{1/(a-1)}(r, g)\equiv 0$ otherwise. Thus (2.1) together with this estimate gives

$$m\left(r, \frac{g'}{g}\right)\leqq 2\{\bar{N}(r, 0, W_a)+\bar{N}_{1/(a-1)}(r, g)\}+S(r, f)+O(1) \tag{2.5}$$

as $r\to\infty$. If g'/g is a transcendental function,

$$\begin{aligned}S(r, f)&=O\{\log^+ T(r, f)+\log r\}\\&=O\left\{\log^+ T\left(r, \frac{g'}{g}\right)+\log r\right\}\end{aligned}$$

as $r\to\infty$, n. e., and if otherwise, $S(r, f)=o(1)$ as $r\to\infty$ as stated in Lemma.

Next we are concerned with $m(r, W_a'/W_a)$. By $W_a=f'\cdot(g')^2$,

$$\frac{W_a'}{W_a}=\frac{f''}{f'}+2\frac{g''}{g'}$$

and thus

$$m\left(r, \frac{W_a'}{W_a}\right)\leqq m\left(r, \frac{f''}{f'}\right)+m\left(r, \frac{g''}{g'}\right)+O(1).$$

If g'/g is transcendental,

$$m\left(r, \frac{f''}{f'}\right)=O\{\log^+T(r, f')+\log r\}$$
$$=O\{\log^+T(r, f)+\log r\}$$
$$=O\left\{\log^+T\left(r, \frac{g'}{g}\right)+\log r\right\}$$

and

$$m\left(r, \frac{g''}{g'}\right)=m\left(r, \frac{(g'/g)'}{g'/g}+\frac{g'}{g}\right)$$
$$\leqq m\left(r, \frac{(g'/g)'}{g'/g}\right)+m\left(r, \frac{g'}{g}\right)+O(1)$$
$$=m\left(r, \frac{g'}{g}\right)+O\left\{\log^+T\left(r, \frac{g'}{g}\right)+\log r\right\}$$

as $r\to\infty$, n. e.. Then we obtain

$$m\left(r, \frac{W_a'}{W_a}\right)\leqq m\left(r, \frac{g'}{g}\right)+O\left\{\log^+T\left(r, \frac{g'}{g}\right)+\log r\right\} \tag{2.6}$$

as $r\to\infty$, n. e., which is clearly valid even if g'/g is a rational function.

We are now ready to discuss the case $a\neq 1$.

3. Results for the case $a\neq 1$

Combining Theorem A with the observation in the previous section we shall prove the following:

THEOREM 1. *Let $g(z)$ be a non-constant meromorphic function. Suppose that the homogeneous differential polynomial $W_a(z)$ defined by* (1.1) *does not vanish identically for $a\in \boldsymbol{C}-\{1\}$. Then*

$$T\left(r, \frac{g'}{g}\right)\leqq \alpha_a\bar{N}(r, 0, W_a)+\beta_a\bar{N}(r, g)+V_a(r) \tag{3.1}$$

as $r\to\infty$, where α_a and β_a are constants depending only on a and satisfying

$$0\leqq\alpha_a\leqq\begin{cases}22 & \left(a\neq\dfrac{1}{2}\right),\\ 15 & \left(a=\dfrac{1}{2}\right),\end{cases}\qquad 0\leqq\beta_a\leqq\begin{cases}23 & \left(a=\dfrac{m+1}{m}\ \text{for a positive integer } m\right),\\ 5 & (\text{otherwise}),\end{cases} \tag{3.2}$$

and $V_a(r)$ is a real-valued function defined on $[0, \infty)$ such that for any fixed a,

$$V_a(r)=O\left\{\log^+T\left(r, \frac{g'}{g}\right)+\log^+\bar{N}(r, 0, W_a)+\log r\right\}$$

as $r\to\infty$, *n. e.*.

Proof. Firstly $g(z)$ given in (i) of Theorem A clearly satisfies (3.1) by virtue of the definition of $V_a(r)$. We may thus suppose that the inequality (1.2) is given in the present case. Using (2.5) and (2.6) we obtain

$$T\left(r, \frac{g'}{g}\right)\leqq 2(A_a+B_a)\{\bar{N}(r, 0, W_a)+\bar{N}_{1/(a-1)}(r, g)\}$$
$$+C_a\{\bar{N}(r, 0, W_a)+\bar{N}(r, g)\}+U_a(r)+O\left\{\log^+T\left(r, \frac{g'}{g}\right)+\log r\right\}$$

as $r\to\infty$, n. e.. By (2.6) and the trivial inequality $T(r, g'/g)\geqq\bar{N}(r, g)$

$$U_a(r)=O\left\{\log^+T\left(r, \frac{g'}{g}\right)+\log^+\bar{N}(r, 0, W_a)+\log r\right\}$$

as $r\to\infty$, n. e.. Thus

$$T\left(r, \frac{g'}{g}\right)\leqq(2A_a+2B_a+C_a)\bar{N}(r, 0, W_a)+C_a\bar{N}(r, g)$$
$$+2(A_a+B_a)\bar{N}_{1/(a-1)}(r, g)+V_a(r)$$

as $r\to\infty$, n. e., provided that $V_a(r)$ is such a function as claimed in this theorem, If $1/(a-1)$ is a positive integer, we set

$$0\leqq\beta_a := 2A_a+2B_a+C_a\ ,$$

and if otherwise, we set $0\leqq\beta_a := C_a$. Further we set $0\leqq\alpha_a := 2A_a+2B_a+C_a$ in each care of the value a. Then (3.2) immediately follows under the definition of the numbers A_a, B_a and C_a. This completes the proof of Theorem 1.

Remark. There is an analogy between Inequality (3.1) and the following:

$$T\left(r, \frac{g'}{g}\right)\leqq 3\{\bar{N}(r, g)+\bar{N}(r, 0, g)\} \tag{3.3}$$
$$+4\{\bar{N}(r, 0, W_a)+\bar{N}_{1/(a-1)}(r, g)\}+S\left(r, \frac{g'}{g}\right)+O(1).$$

This is a consequence of Hayman's inequality (see [2: pp. 60-62])

$$T(r, f)\leqq\left(2+\frac{1}{l}\right)\bar{N}(r, 0, f)+\left(2+\frac{2}{l}\right)\bar{N}\left(r, \frac{1}{f^{(l)}-c}\right)+S(r, f)+O(1)$$

with $c\neq 0$, which we apply with $f(z) := g(z)/g'(z)$, $c=1-a$ and $l=1$, and use the following estimate established above:

$$\bar{N}\left(r, \frac{1}{f'-c}\right)=\bar{N}\left(r, 0, \frac{W_a}{g'^2}\right)\leqq\bar{N}(r, 0, W_a)+\bar{N}_{1/(a-1)}(r, g).$$

We may regard (3.1) as an improvement of (3.3) in view of suppressing the use of $\bar{N}(r, 0, g)$.

As an immediate consequence of this theorem we have

COROLLARY 1. *Under the hypotheses of Theorem* 1 *suppose that there is a constant* γ_a, $0 \leqq \gamma_a < 1/23$ *such that*

$$\bar{N}(r, 0, W_a)+\bar{N}(r, g) \leqq \gamma_a T\left(r, \frac{g'}{g}\right)+S\left(r, \frac{g'}{g}\right)+O(\log r) \tag{3.4}$$

holds as $r \to \infty$. *Then*

$$g(z)=R(z)e^{P(z)}, \tag{3.5}$$

where $R(z)$ *is a rational function,* $\not\equiv 0$, *and* $P(z)$ *is a polynomial with* $P(0)=0$.

In particular, if

$$\bar{N}(r, 0, W_a)+\bar{N}(r, g)=S\left(r, \frac{g'}{g}\right),$$

then $g(z)$ *is at least one of the following*:

(i) *when* $a=1/2$, $g(z)=\alpha z^2+\beta z+\gamma$ *with* $4\alpha\gamma-\beta^2 \neq 0$ *and* $\alpha \neq 0$;

(ii) *when* $a=1$, $g(z)=\alpha z+\beta$ *with* $\alpha \neq 0$;

(iii) $g(z)=\beta e^{\alpha z}$ *with* $\alpha \cdot \beta \neq 0$, *for any* a *different from* 1,

where $\alpha, \beta, \gamma \in \boldsymbol{C}$.

Proof. Inequality (3.1) together with (3.4) leads us to

$$(1-23\gamma_a)T\left(r, \frac{g'}{g}\right)=S\left(r, \frac{g'}{g}\right)+O(\log r)$$

as $r \to \infty$, n. e.. Since $1-23\gamma_a>0$, it follows that

$$T\left(r, \frac{g'}{g}\right)=O(\log r)$$

as $r \to \infty$, n. e., so that $g'(z)/g(z)$ must be a rational function. This proves the first part of this corollary.

Also in the second part $g'(z)/g(z)$ is rational and thus

$$S\left(r, \frac{g'}{g}\right)=o(\log r)$$

as $r \to \infty$, n. e.. Then we deduce that $W_a(z)$ has no zero and $R(z)$ is a polynomial in (3.5). Following Mues [5: §5, p. 340] we can arrive at the conclusion. Another way of verification is the following: Let us write $W_a(z)=\exp\{Q(z)\}$ with a polynomial Q. Then by

$$\frac{W_a(z)}{g(z)^2}=\left(\frac{g'(z)}{g(z)}\right)'-(a-1)\left(\frac{g'(z)}{g(z)}\right)^2,$$

we obtain

$$\frac{e^{Q(z)-2P(z)}}{R(z)^2}=\left(\frac{R'(z)}{R(z)}+P'(z)\right)'-(a-1)\left(\frac{R'(z)}{R(z)}+P'(z)\right)^2.$$

It immediately follows that $Q(z)-2P(z)$ is a constant. If $R(z)$ is also a constant, $P(z)$ must be a linear polynomial in order that both sides of this equation may be reduced to a constant. This gives $g(z)$ as in (iii) above. Also if $R(z)$ is a polynomial and not constant, $P'(z)\equiv 0$ since the right-hand side of this equation tends to 0 as $z\to\infty$. Then $g(z)=R(z)$ and $W_a(z)$ is a constant, so that

$$m(r, W_a)=o\{m(r, g)\}$$

as $r\to\infty$. By virtue of Corollary A we can show that then only (i) and (ii) occur. We have thus proved Corollary 1.

Remarks 1°. When we suppose

$$\bar{N}(r, 0, W_a)+\bar{N}(r, g)\equiv 0$$

in this corollary, we have the result of Mues [5].

2°. When $a=0$, $W_0(z)=g''(z)g(z)$. In this case Inequality (3.3) gives a better estimate than (3.1), since it now needs to consider the zeros of g. In any case but $a=1$ the similar result can be proved by using (3.3) if we include $\bar{N}(r, 0, g)$ in the hypotheses. In particular, if

$$\bar{N}(r, 0, g)+\bar{N}(r, 0, W_a)+\bar{N}(r, g)=S\left(r, \frac{g'}{g}\right),$$

then only (iii) occurs. Frank and Hennekemper [1] proved that this remains valid if $W_a(z)$ is some homogeneous differential polynomial of higher order, which is defined by means of a Wronskian as well.

4. Results for the case $a=1$

It is the reason why our consideration in this section completely differs from that in the previous two that the lemma proved in § 2 does not hold here any longer.

Example 1. Consider the function

$$f(z)=-\exp\{-H(z)\},$$

where $H(z)$ is a non-constant entire function. Then $\bar{N}_2(r, f)\equiv 0$ and

$$\begin{aligned}\bar{N}\left(r, \frac{1}{f'}\right)&=\bar{N}(r, 0, H')\leqq m(r, H')+O(1)\\&\leqq O\{\log^+ m(r, e^H)+\log r\}\end{aligned}$$

as $r\to\infty$, n. e., while

$$\log r=o\{m(r, e^{H})\}=o\left\{m\left(r, \frac{1}{f}\right)\right\}$$

as $r\to\infty$. Thus $f(z)$ is a counter-example to the lemma for $b=0$.

We further set

$$f(z)=-\frac{g(z)}{g'(z)}$$

and obtain

$$g(z)=C\exp\left\{\int_0^z e^{H(\zeta)}d\zeta\right\}, \qquad C\in\boldsymbol{C}-\{0\}.$$

This is an entire function having no zero and

$$W_1(z)=\left(\frac{g'(z)}{g(z)}\right)' g(z)^2=H'(z)e^{H(z)}g(z)^2\not\equiv 0.$$

It is easily shown that $g(z)$ is a counter-example to Theorem 1 for $a=1$. If we define $H(z)$ by

$$H'(z)=e^{h(z)}$$

for any entire function h, then

$$g(z)=C\exp\left\{\int_0^z C_0\exp\left(\int_0^\zeta e^{h(t)}dt\right)d\zeta\right\}, \qquad C_0\in\boldsymbol{C}-\{0\},$$

which is a function with $g(z)\neq 0, \infty$ and $W_1(z)\neq 0$ and thus a counter-example to Corollary 1.

What can we show about $g(z)$ by use of $W_1(z)$? At first, Theorem A with $a=1$ states that excepting (ii) we have

$$\begin{aligned}(4.1)\qquad T\left(r, \frac{g'}{g}\right)\leqq 2\left\{m\left(r, \frac{g'}{g}\right)+m\left(r, \frac{W_1'}{W_1}\right)\right\}+5\{\bar{N}(r, 0, W_1)+\bar{N}(r, g)\}\\ +O\left[\log^+ T\left(r, \frac{g'}{g}\right)+\log^+ m\left(r, \frac{W_1'}{W_1}\right)\right.\\ \left.+\log^+\{\bar{N}(r, 0, W_1)+\bar{N}(r, g)\}+\log r\right]\end{aligned}$$

as $r\to\infty$, n. e.. Because of

$$W_1=\left(\frac{g'}{g}\right)'\cdot g^2(\not\equiv 0),$$

we obtain

$$m\left(r, \frac{W_1'}{W_1}\right) \leqq m\left(r, \frac{(g'/g)''}{(g'/g)'}\right) + m\left(r, \frac{g'}{g}\right) + O(1)$$

$$= m\left(r, \frac{g'}{g}\right) + O\left\{\log^+ T\left(r, \left(\frac{g'}{g}\right)'\right) + \log r\right\}$$

$$= m\left(r, \frac{g'}{g}\right) + O\left\{\log^+ T\left(r, \frac{g'}{g}\right) + \log r\right\}$$

as $r \to \infty$, n. e.. Inequality (4.1) together with this yields

$$T\left(r, \frac{g'}{g}\right) \leqq 5\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\} + 4m\left(r, \frac{g'}{g}\right) + O\left[\log^+ T\left(r, \frac{g'}{g}\right) + \log^+ \{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\} + \log r\right] \tag{4.2}$$

as $r \to \infty$, n. e.. It is well-known that

$$T\left(r, \frac{g'}{g}\right) = m\left(r, \frac{g'}{g}\right) + N\left(r, \frac{g'}{g}\right)$$

$$= \bar{N}(r, 0, g) + \bar{N}(r, g) + O\{\log^+ T(r, g) + \log r\}$$

$$\leqq 2T(r, g) + O\{\log^+ T(r, g) + \log r\}$$

as $r \to \infty$, n. e.. Thus this together with (4.2) proves the following

THEOREM 2. *Let $g(z)$ be a non-constant meromorphic function. Suppose that the homogeneous differential polynomial $W_1(z) = g''(z)g(z) - g'(z)^2$ does not vanish identically. Then*

$$T\left(r, \frac{g'}{g}\right) \leqq 5\{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\} + O[\log^+ T(r, g) + \log^+ \{\bar{N}(r, 0, W_1) + \bar{N}(r, g)\} + \log r] \tag{4.3}$$

as $r \to \infty$, n. e., unless $g(z)$ is of the following:

(i) $g(z) = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z}$,

where λ_i, $C_i \in \boldsymbol{C}(i=1, 2)$ and $\lambda_1 \neq \lambda_2$, $C_1 \cdot C_2 \neq 0$.

If $g(z)$ is of finite order in this theorem, we have a result analogous to Corollary 1.

COROLLARY 2. *Under the hypotheses of Theorem 2 suppose that $g(z)$ has finite order and satisfies*

$$\bar{N}(r, 0, W_1) + \bar{N}(r, g) \leqq \gamma_1 T\left(r, \frac{g'}{g}\right) + S\left(r, \frac{g'}{g}\right) + O(\log r) \tag{4.4}$$

as $r \to \infty$, for a constant γ_1, $0 \leqq \gamma_1 < 1/5$. Then

$$g(z)=R(z)e^{P(z)}, \tag{4.5}$$

where $R(z)$ is a rational function, $\not\equiv 0$, and $P(z)$ is a polynomial with $P(0)=0$, except for (i) *in Theorem* 2.

In particular, if

$$\bar{N}(r, 0, W_1)+\bar{N}(r, g)=S\left(r, \frac{g'}{g}\right),$$

then $g(z)$ is at least one among (i) *above and the following*:

(ii) $g(z)=(C_2z+C_1)e^{\lambda z}$, *where* C_1, $C_2(\neq 0)$, $\lambda\in \boldsymbol{C}$;

(iii) $g(z)=C\exp(\alpha z^2+\beta z)$, *where* $C(\neq 0)$, $\alpha(\neq 0)$, $\beta\in\boldsymbol{C}$.

Remark. The conclusion above also holds without the assumption that $g(z)$ should be of finite order, if we suppose

$$\bar{N}(r, 0, W_1)+\bar{N}(r, g)+\frac{4}{5}m\left(r, \frac{g'}{g}\right)\leqq\gamma_1 T\left(r, \frac{g'}{g}\right)+S\left(r, \frac{g'}{g}\right)+O(\log r) \tag{4.6}$$

instead of (4.4). As mentioned below, there is no difference in the procedue of their proofs but in this case we shall use (4.2) instead of (4.3).

Proof of Corollary 2. Suppose that a function $g(z)$ is different from the one given by (i) in Theorem 2. By (4.3) and (4.4) we obtain

$$(1-5\gamma_1)T\left(r, \frac{g'}{g}\right)\leqq S\left(r, \frac{g'}{g}\right)+O\{\log^+T(r, g)+\log r\}$$

as $r\to\infty$, n.e.. Since $g(z)$ is of finite order, the right-hand side of this inequality is reduced to $S(r, g'/g)+O(\log r)$ (, which is also deduced from (4.2) and (4.6)). Because of $1-5\gamma_1>0$, it thus follows

$$T\left(r, \frac{g'}{g}\right)=O(\log r)$$

as $r\to\infty$, n.e., which shows that g'/g is a rational function and g has the form (4.5). This proves the first part of the corollary.

In the second part we now see that $g(z)$ is an entire function and $W_1(z)$ has no zero. Therefore $R(z)$ is a polynomial in (4.5). In order that

$$W_1(z)=\left(\frac{g'(z)}{g(z)}\right)'g(z)^2=\left\{\left(\frac{R'(z)}{R(z)}\right)'+P''(z)\right\}R(z)^2e^{2P(z)}$$

is different from zero, the polynomial $\{(R'/R)'+P''\}R^2=R''R-R'^2+P''R^2$ must be reduced to a non-zero constant C_1, say. If $R(z)$ is not a constant, then $P''(z)\equiv 0$ follows by considering the behavior of

$$\left(\frac{R'(z)}{R(z)}\right)'+P''(z)=\frac{C_1}{R(z)^2}$$

at infinity. Moreover $R(z)$ must be a linear polynomial but a constant. Then

(ii) occurs. Also if $R(z)\equiv C(\neq 0)$, then

$$P''(z)\equiv\frac{C_1}{C_2}=2\alpha,\ \text{say},\ \in \boldsymbol{C}-\{0\},$$

and thus $P(z)=\alpha z^2+\beta z$, $\beta\in\boldsymbol{C}$ because of $P(0)=0$. This is $g(z)$ as mentioned in (iii). Since

$$g(z)=C_1e^{\lambda_1 z}+C_2e^{\lambda_2 z},\qquad \lambda_i,\ C_i\in\boldsymbol{C}$$

is an entire function of order 1 if $\lambda_1\neq\lambda_2$ and $C_1\cdot C_2\neq 0$, and $W_1(z)=C_1C_2(\lambda_1-\lambda_2)^2e^{(\lambda_1+\lambda_2)z}$, this is the last one of the functions as claimed in the second part of this corollary. We have thus completed the proof.

Remark. We shall mention the counter-examples (a) and (b) in § 1 given by Mues [5]. The function (a), $g(z)=\sin z$, is obtained by setting $C_1=\mp C_2=1/(2i)$ and $\lambda_1=-\lambda_2=i$ in (i) above. When $g(z)$ is an entire function and $W_1(z)$ has never a zero, there is an entire function $Q(z)$ such that

$$W_1(z)=e^{2Q(z)}.$$

Further if $g(z)$ as well as $W_1(z)$ has no zero, then we may denote it by

$$g(z)=\exp\{Q(z)-h(z)\}$$

for an entire function $h(z)$. It follows

$$Q''(z)=e^{2h(z)}+h''(z).$$

The function (b) is thus obtained. This function has finite order only if $h(z)$ is a constant, so that $Q(z)$ is a polynomial of degree 2. Then it gives (iii) of Corollary 2.

The function $g(z)$ in *Example* 1 as well as (b) has the property that $g(z)\neq 0$, ∞ and $W_1(z)\neq 0$. This $g(z)$ however has infinite order and satisfies $T(r, g'/g)=m(r, g'/g)$. Hence a constant γ_1 in (4.6) cannot be replaced by a number not smaller than 4/5.

We can also give an example of entire functions $g(z)$ such that it has an infinite number of the zeros, while $W_1(z)$ has never a zero (which has of course infinite order):

Example 2. Define an entire function $h(z)$ by the equation

$$h''(z)=\frac{e^{\pi i \sin z}+1}{\cos^2 z}. \tag{4.7}$$

In fact, it is easy to show that all the zeros of $\cos^2 z$ are cancelled by those of $\exp(\pi i \sin z)+1$. Set

$$g(z)=(\cos z)e^{h(z)}.$$

Then

$$W_1(z)=\left(\frac{g'(z)}{g(z)}\right)' g(z)^2=\left\{h'(z)-\frac{\sin z}{\cos z}\right\}'(\cos^2 z)e^{2h(z)}$$
$$=\exp\{2h(z)+\pi i \sin z\}.$$

Evidently $g(z)$ has infinitely many zeros but

$$\bar{N}(r, 0, g)=N(r, 0, \cos z)=O\{\log^+ T(r, e^h)\}=O\{\log^+ T(r, g)\}$$

as $r\to\infty$, n. e., while $W_1(z)$ has the value 0 as an exceptional value. The order (as well as the hyper-order) of $g(z)$ is infinite, since so is that of $h(z)$. By

$$\frac{g'(z)}{g(z)}=h'(z)-\frac{\sin z}{\cos z} \tag{4.8}$$

and

$$\left(\frac{g'(z)}{g(z)}\right)'=h''(z)-\frac{1}{\cos^2 z}=-\frac{e^{\pi i \sin z}}{\cos^2 z}$$

we see that for $r\to\infty$, n. e.,

$$T\left(r, \left(\frac{g'}{g}\right)'\right)\leqq T\left(r, \frac{g'}{g}\right)+N\left(r, \frac{g'}{g}\right)+O\left\{\log^+ T\left(r, \frac{g'}{g}\right)+\log r\right\}$$
$$\leqq T\left(r, \frac{g'}{g}\right)+T(r, \cos z)+O\left\{\log^+ T\left(r, \frac{g'}{g}\right)+\log r\right\}$$

and thus

$$T\left(r, \frac{g'}{g}\right)\geqq T\left(r, \left(\frac{g'}{g}\right)'\right)-T(r, \cos z)-O\left\{\log^+ T\left(r, \frac{g'}{g}\right)+\log r\right\}$$
$$\geqq T(r, e^{\pi i \sin z})-3T(r, \cos z)-O\left\{\log^+ T\left(r, \frac{g'}{g}\right)+\log r\right\}.$$

It is well-known that

$$\lim_{r\to\infty}\frac{T(r, e^{\pi i \sin z})}{T(r, \sin z)}=+\infty.$$

(See for example, Hayman [2: p. 54, § 2.9, Excercise (ii)].) Since $\cos z=(\sin z)'$,

$$T(r, \cos z)\leqq(1+o(1))T(r, \sin z)$$

as $r\to\infty$. Also $\log r=o\{\log^+ T(r, g'/g)\}$ as $r\to\infty$. Thus

$$(1-o(1))T\left(r, \frac{g'}{g}\right)\geqq T(r, e^{\pi i \sin z})-3(1+o(1))T(r, \sin z)$$
$$=(1-o(1))T(r, e^{\pi i \sin z})$$

as $r\to\infty$, n. e.. This shows

$$\bar{N}(r, 0, g)=N(r, 0, \cos z)\leqq m(r, \cos z) \qquad (\text{since } \cos 0=1)$$

$$=m\left(r, \frac{1}{\pi i}\cdot\frac{(e^{\pi i \sin z})'}{e^{\pi i \sin z}}\right)$$
$$=O\{\log^+ T(r, e^{\pi i \sin z})+\log r\}$$
$$=O\left\{\log^+ T\left(r, \frac{g'}{g}\right)+\log r\right\}$$
$$=O\left\{\log^+ T\left(r, \frac{g'}{g}\right)\right\}$$

as $r\to\infty$, n. e.. Although this function $g(z)$ satisfies

$$\bar{N}(r, 0, g)+\bar{N}(r, 0, W_1)+\bar{N}(r, g)=O\left\{\log^+ T\left(r, \frac{g'}{g}\right)\right\},$$

(particularly, $\bar{N}(r, 0, W_1)+\bar{N}(r, g)\equiv 0,$) the logarithmic derivative $g'(z)/g(z)$ is a transcendental function. This is another example which shows us the failure of Corollary 2 without the assumption of $g(z)$ to be of finite order. Further it immediately follows

$$T\left(r, \frac{g'}{g}\right)=m\left(r, \frac{g'}{g}\right)+\bar{N}(r, 0, g)$$
$$=m\left(r, \frac{g'}{g}\right)+O\left\{\log^+ T\left(r, \frac{g'}{g}\right)\right\}$$

as $r\to\infty$, n. e.. Therefore this $g(z)$ shows again that we cannot replace a constant γ_1 in (4.6) by any number not smaller than 4/5.

There is an example of meromorphic functions $g(z)$ such that $g(z)$ as well as $W_1(z)$ has never a zero but it has an infinite number of poles and finite order.

Example 3. Consider the function

$$g(z)=\frac{1}{\cos z},$$

then

$$W_1(z)=\left(\frac{g'(z)}{g(z)}\right)' g(z)^2=\frac{1}{\cos^4 z}.$$

This is such a function as mentioned above. Another example is given by choosing $h(z)$ as an entire function obtained in (i) of Theorem 2 and setting

$$g(z)=\frac{1}{h(z)^m}$$

with a positive integer m. More generally, the only matter required of $h(z)$ is that it should be an entire function of finite order and having infinite many zeros but the meromorphic function

$$\frac{h''(z)h(z)-ah'(z)^2}{h(z)^{2(m+1)}} \tag{4.9}$$

has never a zero with $a=1$.

Remark. As far as the author knows it is however an open question whether there exists an entire function $h(z)$ other than those in (i) of Theorem 2 such that it has infinitely many zeros but the function (4.9) has no zero or not. This question makes us return to Hayman's original one in [2: § 3.6] of meromorphic functions $f(z)$ such that $f(z)f''(z)\neq 0$, for setting $f(z)=1/h(z)$ we have

$$f''(z)=-\frac{h''(z)h(z)-2h'(z)^2}{h(z)^2}$$

and therefore we are led to the similar question to the above. As mentioned in [2: Appendix] this solved by Mues when $f(z)$ and thus $h(z)$ are of finite order.

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