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THE CHARACTERISTICS OF BMOA ON RIEMANN SURFACES

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In this paper we give a John-Nirenberg type theorem for BMOA on general open Riemann surfaces. Using Ba spaces we give a new characteristic for BMOA on Riemann surfaces in this paper too.

1. Introduction.

In [7], T.A. Metzger asked whether the John-Nirenberg theorem for BMOA on the unit disk is true on Riemann surfaces. We have given a positive answer for compact bordered Riemann surfaces in [4]. In this paper we will give a John-Nirenberg type theorem for BMOA on general open Riemann surfaces. Some new characteristics of BMOA on Riemann surfaces will be given in this paper too.

2. John-Nirenberg type theorem for BMOA on Riemann surfaces.

Let R be an open Riemann surface which possesses a Green's function, i.e., $R\notin O_G$. Let $G_R(w, a)$ be the Green's function of R with logarithmic singularity at $a\in R$. We firstly give an important lemma as follows:

LEMMA 2.1. *Let $R_1\subset R_2\subset\cdots\subset R_k\to R$ be an exhaustion of the Riemann surface R, where R_k are compact bordered Riemann surfaces $(1\leqq k<\infty)$. F is an analytic function on R. Let the least harmonic majorant of the subharmonic function $|F(w)|^p$ on R(or R_k) be $H(w)$ (or $H_k(w)$). Then*

$$H(w)=\sup_{k\geqq 1}H_k(w)=\lim_{k\to\infty}H_k(w).$$

If $F(w)$ has no harmonic majorant on R (or R_k) we denote $H(w)=\infty$ (or $H_k(w)=\infty$).

Proof. It is easy to verify that $\{H_k(w)\}$ is an increasing sequence. By Hanack theorem we get that $\lim_{k\to\infty}H_k(w)=H_0(w)$ is a harmonic function, or $H_0(w)$

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$=\infty$.

Because $\{R_k\}$ is an exhaustion of R, there is a $k \geqq 1$ such that $w \in R_k$ for given $w \in R$, then $H_0(w) \geqq H_k(w) \geqq |F(w)|^p$. So $H_0(w)$ is a harmonic majorant of $|F(w)|^p$ on R.

In the next step we will prove that $H_0(w)$ is the least harmonic majorant of $|F(w)|^p$ on R, i.e., if $\tilde{H}(w)$ is another harmonic majorant of $|F(w)|^p$ on R, we must show that $\tilde{H}(w) \geqq H_0(w)$ for each $w \in R$.

Suppose there is a $\zeta \in R$ such that $\tilde{H}(\zeta) < H_0(\zeta)$. Because $\{R_k\}$ is the exhaustion of R we know there is a $k_1 > 0$ such that $\zeta \in R_{k_1}$. Because $H_0(\zeta) = \lim_{k \to \infty} H_k(\zeta)$, there is a $k_2 \geqq 1$ such that $\tilde{H}(\zeta) < H_{k_2}(\zeta)$. Taking $k_0 = \max(k_1, k_2)$, then $\zeta \in R_{k_1} \subset R_{k_0}$ and $\tilde{H}(\zeta) < H_{k_2}(\zeta) \leqq H_{k_0}(\zeta)$. Thus $\tilde{H}(w)$ is a harmonic majorant of $|F(w)|^p$ on R_{k_0} and $\tilde{H}(\zeta) < H_{k_0}(\zeta)$ on $\zeta \in R_{k_0}$. This conclusion contradicts to the fact that $H_{k_0}(w)$ is the least harmonic majorant of $|F(w)|^p$ on R_{k_0}.

Thus $H_0(w)$ is the least harmonic majorant of $|F(w)|^p$ on R, i.e., $H(w) \equiv H_0(w) = \sup_{k \geqq 1} H_k(w) = \lim_{k \to \infty} H_k(w)$. This completes the proof.

For the BMOA on the above Riemann surfaces, T. A. Metzger has given the definition as follows [6]: Let F be an analytic function on R. We say $F \in BMOA(R)$ if

$$B_R^2(F) = \frac{2}{\pi} \sup_{a \in R} \iint_R |F'(w)|^2 G_R(w, a) dw d\bar{w} < \infty .$$

We have the next lemma:

LEMMA 2.2. *Let $R_1 \subset R_2 \subset \cdots \subset R_k \to R$ be an exhaustion of R, where R_k are compact bordered Riemann surfaces, then*

$$B_{R_k}(F) \leqq B_R(F).$$

Proof. Let $G_k(w, a)$ be the Green's function on R with logarithmic singularity at $a \in R_k$. Let

$$G(w) = G_R(w, a) - G_k(w, a).$$

It is easy to verify that $G(w)$ is a harmonic function on R_k, and $G(w)|_{\partial R_k} = G_R(w, a)|_{\partial R_k} > 0$. Using the maximum principle of harmonic function we have $G(w) > 0$ for each $w \in R_k$, i.e.,

$$G_k(w, a) < G_R(w, a).$$

Thus

$$\iint_{R_k} |F'(w)|^2 G_k(w, a) dw d\bar{w} \leqq \iint_R |F'(w)|^2 G_R(w, a) dw d\bar{w} .$$

Taking supremum we have

$$B_{R_k}^2(F) \leqq B_R^2(F),$$

which is the conclusion of the lemma.

We will give an equivalent definition of BMOA on Riemann surfaces by the exhaustion process in the following theorem.

THEOREM 2.3. *Let R be a Riemann surface which possesses a Green's function. For each $a \in R$ we possess an exhaustion $R_{a,1} \subset R_{a,2} \subset \cdots \subset R_{a,k} \to R$ such that $a \in R_{a,1}$. Where $R_{a,k}$ are compact bordered Riemann surfaces, $1 \leqq k < \infty$. F is an analytic function on R. Thus we have*

$$B_R^2(F)=\sup_{a\in R}\sup_{k\geqq 1} H_{a,k}(a),$$

where $H_{a,k}(w)$ is the least harmonic majorant of $|F(w)-F(a)|^2$ on $R_{a,k}$.

Proof. Suppoae $H_a(w)$ be the least harmonic majorant of $|F(w)-F(a)|^2$ on R. From lemma 2.1 we have $H_a(w)=\lim_{k\to\infty} H_{a,k}(w)$. So $H_a(a)=\lim_{k\to\infty} H_{a,k}(a)=\sup_{k\geqq 1} H_{a,k}(a)$. Following lemma 1 of [5] we have

$$B_R^2(F)=\sup_{a\in R} H_a(a).$$

Thus

$$B_R^2(F)=\sup_{a\in R}\sup_{k\geqq 1} H_{a,k}(a).$$

The proof of theorem 2.3 is completed.

As an application of lemma 2.1 we can give an equivalent definition of $VMOA$ on Riemann surfaces. Firstly let us recall the definition of $VMOA(R)$. Let R be a Riemann surface which possesses a Green's function $G_R(w, a)$. F is analytic on R. We say $F \in VMOA(R)$ if

$$\lim_{a\to\partial R}\iint_R |F'(w)|^2 G_R(w, a)\, dw\, d\bar{w}=0,$$

where ∂R is the ideal border of R.

Using lemma 2.1 and the lemma 1 of [5] we have the following conclusion immediately:

PROPOSITION 2.4. *Let F be an analytic function on Riemann surface R, then $F \in VMOA(R)$ if and only if*

$$\lim_{a\to\partial R}\lim_{k\to\infty} H_{a,k}(a)=0.$$

The meaning of $H_{a,k}(w)$ is the same as that of theorem 2.3.

Next we will give a John-Nirenberg type theorem for BMOA on Riemann surfaces.

THEOREM 2.5. *Let R be a Riemann surface which possesses a Green's function $G_R(w, a)$. For each $a \in R$ we possess an exhaustion $R_{a,1} \subset R_{a,2} \subset \cdots \subset R_{a,k} \to R$, where $R_{a,k}$ $(1 \leqq k < \infty)$ are compact bordered Riemann surfaces and $a \in R_{a,1}$. F is an analytic function on R. Then $F \in BMOA(R)$ if and only if for every $k \geqq 1$*

and every $a\in R$,

$$\int_{E_{a,k,\lambda}}\frac{\partial G_k(w,a)}{\partial n}ds\leqq Ke^{-\beta\lambda}. \tag{2.1}$$

Where $E_{a,k,\lambda}=\{z\in\partial R_{a,k},\ |F(w)-F(a)|>\lambda\}$. $\partial/\partial n$ is the inner normal derivative with respect to $R_{a,k}$. K is an absolute constant. When $F\in BMOA(R)$, $\beta=c/B_R(F)$, c is another absolute constant.

Proof. Suppose $F\in BMOA(R)$. For $a\in R$ and the exhaustion $a\in R_{a,1}\subset R_{a,2}\subset\cdots\subset R_{a,k}\to R$, because $R_{a,k}$ are compact bordered Riemann surfaces, by lemma 2.2 we know $F\in BMOA(R_{a,k})$. From theorem 1 of [4] we know for every $k\geqq 1$ and every $a\in R$,

$$\int_{E_{a,k,\lambda}}\frac{\partial G_k(w,a)}{\partial n}ds\leqq Ke^{-\beta_k\lambda}, \tag{2.2}$$

where $\beta_k=c/B_{R_{a,k}}(F)$. From lemma 2.2 we know $B_{R_{a,k}}(F)\leqq B_R(F)$. So $\beta_k\geqq c/B_R(F)=\beta$, therefore $e^{-\beta_k\lambda}\leqq e^{-\beta\lambda}$, so we have

$$\int_{E_{a,k,\lambda}}\frac{\partial G_k(w,a)}{\partial n}ds\leqq Ke^{-\beta\lambda}.$$

To see the converse we suppose that for every $a\in R$ and every $k\geqq 1$ (2.1) is true. Let

$$\Lambda_{a,k}(\lambda)=\int_{E_{a,k,\lambda}}\frac{\partial G_k(w,a)}{\partial n}ds,$$

then if $H_{a,k}(w)$ is the least harmonic majorant of $|F(w)-F(a)|^2$ on $R_{a,k}$, we have

$$\begin{aligned}H_{a,k}(w)&=\int_{\partial R_{a,k}}|F(w)-F(a)|^2\frac{\partial G_k(w,a)}{\partial n}ds\\&=2\int_0^\infty\lambda\Lambda(\lambda)d\lambda\leqq 2K\int_0^\infty\lambda e^{-\beta\lambda}d\lambda\\&=\frac{2K}{\beta^2}<\infty.\end{aligned}$$

so

$$\sup_{a\in R}\sup_{k\geq 1}H_{a,k}(w)\leqq\frac{2K}{\beta^2}<\infty.$$

From theorem 2.3 we have $F\in BMOA(R)$ and the proof is completed.

It is analogous to the corollary of [4], using theorem 2.3 and theorem 2.5, noting the next equation [8]:

$$\int_{\partial R_{a,k}}\frac{\partial G_k(w,a)}{\partial n}ds=2\pi,$$

we have the corollary as follows:

COROLLARY 2.6. *Let R be a Riemann surface which possesses a Green's function. For each $a\in R$ we possess an exhaustion $R_{a,1}\subset R_{a,2}\subset\cdots\subset R_{a,k}\to R$, where $R_{a,k}$ $(1\leqq k<\infty)$ are compact bordered Riemann surfaces and $a\in R_{a,1}$. Thus $F\in BMOA(R)$ if and only if*

$$\sup_{a\in R}\sup_{k\geqq 1}\int_{\partial R_{a,k}}|F(w)-F(a)|^p\frac{\partial G_k(w,a)}{\partial n}ds=M_p<\infty,\quad 1\leqq p<\infty.$$

and $M_p\sim B_R^p(F)$. When $F\in BMOA(R)$, $M_p\leqq(K/C^p)\Gamma(p+1)(B_R(F))^p$.

3. BMOA on regular Riemann surfaces.

Let R be a Riemann surface. $R\notin O_G$. We call R be regular if for each $w\in R$, $\lim_{a\to\partial R}G_R(w,a)=0$. A simple example of regular Riemann surface is the unit disk. For the regular Riemann surface R, John-Nirenberg type theorem for $BMOA(R)$ has a much simple form. We firstly give two lemmas:

LEMMA 3.1. [1] *If R is a regular Riemann surface. Then for a given compact set $R_0\subset R$ and $\varepsilon>0$, there is a compact set $S_0\subset R$ such that $w\in R\setminus S_0$ implies $G_R(w,a)<\varepsilon$ for every $a\in R_0$. Especially for each $a\in R$, we can take $R_0=\{a\}$.*

From lemma 3.1 it is easy to know that if R is a regular Riemann surface, then for every $a\in R$ and every $t>0$ $\Gamma_{a,t}=\{w\in R,\ G_R(w,a)=t\}$ is constructed by a finite number of analytic Jordan curves.

LEMMA 3.2. *Let R be a regular Riemann surface. For every $a\in R$ and every $t>0$, $R_{a,t}=\{w\in R,\ G_R(w,a)\geqq t\}$ is a compact bordered Riemann surface whose border is $\Gamma_{a,t}$. $G_{R_{a,t}}(w,a)=G_R(w,a)-t$ is the Green's function on $R_{a,t}$ with logarithmic singularity at a, and $\{R_{a,t}\}$ is an exhaustion of R when t decreases to zero.*

Proof. Fix $a\in R$. Suppose $\{w_n\}\subset R_{a,t}$, $w_n\to w_0$. From lemma 3.1, there is a compact set $S_0\subset R$ such that every $w\in R\setminus S_0$ implies $G_R(w,a)<t$. So $R_{a,t}\subset S_0$, i.e., $\{w_n\}\subset S_0$. Because S_0 is compact we know $w_0\in S_0\subset R$, thus $G_R(w_0,a)=\lim_{n\to\infty}G_R(w_n,a)\geqq t$. Equivalently, $w_0\in R_{a,t}$. So $R_{a,t}$ is a closed subdomain of the compact set $S_0\subset R$, then we have $R_{a,t}$ is a compact domain.

Let $w\in R$ such that $G_R(w,a)>t>0$. From the continuity of $G_R(w,a)$ we know there is a sufficient small parametric disk U_ε which contains w such that every $\zeta\in U_\varepsilon$ implies $G_R(\zeta,a)>t$. Thus $U_\varepsilon\subset R_{a,t}$. So w can not be the border point of $R_{a,t}$, and the border of $R_{a,t}$ is constructed by $\Gamma_{a,t}$, i.e., $R_{a,t}$ is a compact bordered Riemann surface.

By direct verification we can know $G_{R_{a,t}}(w,a)=G_R(w,a)-t$ is the Green's function of $R_{a,t}$ with logarithmic singularity at a.

For each $w\in R$, there is a $t_1>0$ such that $G_R(w, a)>t_1>0$. So $w\in R_{a,t_1}$, and we conclude that $\{R_{a,t}\}$ is an exhaustion of R when t decreases to zero. The proof is completed.

From lemma 3.2, for the regular Riemann surface R, we have

$$\frac{\partial G_{R_{a,t}}(w, a)}{\partial n}=\frac{\partial G_R(w, a)}{\partial n}. \tag{3.1}$$

Where $\partial/\partial n$ is the inner normal derivative with respect to the compact bordered Riemann surface $R_{a,t}$.

From (3.1), we can conclude immediately that for every $a\in R$ and every $t>0$,

$$\int_{\Gamma_{a,t}}\frac{\partial G_R(w, a)}{\partial n}ds=2\pi. \tag{3.2}$$

Let $F(w)$ be an analytic function on R. $H(w)$ is the least harmonic majorant of $|F(w)|^p$, from lemma 2.1 and (3.1) we have

$$H(a)=\frac{1}{2\pi}\sup_{t>0}\int_{\Gamma_{a,t}}|F(w)|^p\frac{\partial G_R(w, a)}{\partial n}ds. \tag{3.3}$$

If $|F(w)|^p$ has no harmonic majorant on R we denote $H(w)=\infty$.

From this, corresponding to theorem 2.3 and noting (3.3) we have

THEOREM 3.3. *Let F be an analytic function on a regular Riemann surface R, then*

$$B_R^2(F)=\frac{1}{2\pi}\sup_{a\in R}\sup_{t>0}\int_{\Gamma_{a,t}}|F(w)-F(a)|^2\frac{\partial G_R(w, a)}{\partial n}ds.$$

Corresponding to proposition 2.4 we have

PROPOSITION 3.4. *Let F be an analytic function on a regular Riemann surface R, then $F\in VMOA(R)$ if and only if*

$$\lim_{a\to\partial R}\lim_{t\to 0+}\int_{\Gamma_{a,t}}|F(w)-F(a)|^2\frac{\partial G_R(w, a)}{\partial n}ds=0.$$

From theorem 2.5 we can easily deduce John-Nirenberg type theorem on regular Riemann surfaces. It has a much simple form.

THEOREM 3.5. *Let R be a regular Riemann surface. F is an analytic function on R. Then $F\in BMOA(R)$ if and only if for every $a\in R$ and every $t>0$*

$$\int_{E_{a,t,\lambda}}\frac{\partial G_R(w, a)}{\partial n}ds\leq Ke^{-\beta\lambda}.$$

Where $E_{a,t,\lambda}=\{w\in\Gamma_{a,t}, |F(w)-F(a)|>\lambda\}$. $\partial/\partial n$ is the inner normal derivative with respect to $R_{a,t}$. K is an absolute constant. When $F\in BMOA(R)$, $\beta=$

$c/B_R(F)$, c is another constant.

Corresponding to corollary 2.6 we have

COROLLARY 3.6. *Let R be a regular Riemann surface. F is an analytic function on R. $F \in BMOA(R)$ if and only if*

$$\sup_{a\in R}\sup_{t>0}\int_{\Gamma_{a,t}}|F(w)-F(a)|^p\frac{\partial G_R(w,a)}{\partial n}ds=M_p<\infty,\quad 1\leqq p<\infty,$$

and $M_p \sim B_R^p(F)$, when $F\in BMOA(R)$, $M_p \leqq (K/C^p)\Gamma(p+1)(B_R(F))^p$.

As an application of theorem 3.5, we point out that the distribuion function on some area measure of $BMOA$ on the regular Riemann surface R has exponential decay.

Let $G_R^*(w, a)$ be the conjugate function of $G_R(w, a)$, then $P(w)=G_R(w, a)+iG_R^*(w, a)$ is an analytic function on $R\setminus\{a\}$. We have

COROLLARY 3.7. *Let R be a regular Riemann surface. If $F\in BMOA(R)$, then*

$$\iint_{D_{a,\lambda}}|P'(w)|^2e^{-kG_R(w,a)}dwd\bar{w}\leqq\frac{K}{k}e^{-\beta\lambda}.$$

where k is an arbitrary positive integer. $\beta=c/B_R(F)$, K, c are absolute constants. $D_{a,\lambda}=\{w\in R,\ |F(w)-F(a)|>\lambda\}$.

Proof. Because $G_R(w, a)$ is a constant on $\Gamma_{a,t}$, we have

$$iP'(w)dw=-\frac{\partial G_R^*(w,a)}{\partial s}ds=\frac{\partial G_R(w,a)}{\partial n}ds$$

along $\Gamma_{a,t}$, where ds is the arc length and $\partial/\partial n$ is the inner normal derivative with respect to $R_{a,t}$. Thus we have

$$|P'(w)|^2=\left(\frac{\partial G_R(w,a)}{\partial n}\right)^2$$

along $\Gamma_{a,t}$. Because $G_R(w, a)=t$ along $\Gamma_{a,t}$ we have $(\partial G_R(w, a)/\partial n)dn=dt$. Thus

$$\begin{aligned}&\iint_{D_{a,\lambda}}|P'(w)|^2e^{-kG_R(w,a)}dwd\bar{w}\\&=\iint_{D_{a,\lambda}}\left(\frac{\partial G_R(w,a)}{\partial n}\right)^2e^{-kG_R(w,a)}dsdn\\&=\int_0^\infty\int_{E_{a,t,\lambda}}\frac{\partial G_R(w,a)}{\partial n}e^{-kt}dsdt\leqq\int_0^\infty Ke^{-\beta\lambda}e^{-kt}dt\\&=\frac{K}{k}e^{-\beta\lambda}.\end{aligned}$$

Where $\beta=c/B_R(F)$, K, c are absolute constants, and the proof is completed.

4. *BMOA* and Ba spaces on Riemann surfaces.

Ba spaces was introduced by Ding Xiaxi and Luo Peizhu in [2]. In [4], we have discussed a special class of Ba spaces H^{Ba} on compact bordered Riemann surfaces and have given a new characteristic of BMOA on compact bordered Riemann surfaces. Next we will show that on the general open Riemann surface R which possesses a Green's function, the above conclusion is still true.

We firstly recall Hardy spaces $H^p(R)$ on the Riemann surface $R(1\leqq p<\infty)$.

$$H^p(R)=\{F,\ F \text{ is analytic on } R \text{ and } |F(w)|^p \text{ has a harmonic majorant on } R\}.$$

We have known [3] if $H(w)$ is the least harmonic majorant of the subharmonic funtion $|F(w)|^p$, the norm of $H^p(R)$ can be defined by

$$\|F\|_p=|H(a)|^{1/p}, \qquad a\in R. \tag{4.1}$$

If we exchange the reference point we get equivalent norms. Notice by lemma 2.1, for an arbitrary exhaustion $R_1\subset R_2\subset\cdots\subset R_k\to R$, which $a\in R_1$, we have

$$\|F\|_p^p=\lim_{k\to\infty}|H_k(a)|=\sup_{k\geqq 1}|H_k(a)|, \tag{4.2}$$

where $H_k(w)$ is the least harmonic majorant of $|F(w)|^p$ on R_k.

Now let $E(\zeta)=\sum_{m=1}^{\infty}a_m\zeta^m$ be a finite order $(\rho<\infty)$ and mean type $(\sigma<\infty)$ entire function, and $a_m\geqq 0$. The sequence $\{p_m\}$ has the property $1\leqq p_1\leqq p_2\leqq\cdots\leqq p_m\to\infty$ and

$$\overline{\lim_{m\to\infty}}\frac{p_m}{m^{1/p}}=p^*<\infty.$$

For $F(w)\in\bigcap_{m=1}^{\infty}H^{p_m}(R)$ we set

$$I(F,\alpha)=\sum_{m=1}^{\infty}a_m\|F\|_{p_m}^m\alpha^m, \tag{4.3}$$

and use d_F to denote the convergence radius of (4.3). Define

$$H^{Ba}(R)=\left\{F,\ F\in\bigcap_{m=1}^{\infty}H^{p_m}(R),\text{ and } d_F>0\right\}.$$

The norm of F in $H^{Ba}(R)$ is defined by

$$\|F\|_{Ba}=\inf\left\{\frac{1}{|\alpha|},\ I(F,|\alpha|)\leqq 1\right\}.$$

Set

$$|\|F\||=\sup_{a\in R}\{\|F(w)-F(a)\|_{Ba}\}.$$

Thus we have

THEOREM 4.1. *Let F be an analytic function on the Riemann surface R, then there is a constant c such that*

$$c^{-1}B_R(F)\leqq|\|F\||\leqq cB_R(F).$$

The proof of this theorem is similar to that of theorem 2 in [4], so we omit it here.

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