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SELF MAPS OF $\Sigma^k CP^3$ FOR $k \geqq 1$

Dedicated to Prof. Shôro Araki on his 60th birthday

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§1. Introduction.

Throughout this note, all spaces, maps and homotopies are assumed to be based, and we will not distinguish the map and its homotopy class.

For two topological spaces X and Y, we denote by $[X, Y]$ the set of homotopy classes of maps from X to Y.

If $X=Y$, then the set $[X, X]$ becomes a monoid with its multiplication induced from the composition of maps and we put $M(X)=[X, X]$.

Let $\mathcal{E}(X)$ be the group consisting of all invertible elements of $M(X)$ and we call it the group of self-homotopy equivalences of X.

The group $\mathcal{E}(X)$ has been studied by several authors since the paper of W. D. Barcus and M. G. Barratt [1] appeared.

However, we have not yet obtained an effective method for calculating it except classical ones, and its structure also has not been clarified sufficiently. Furthermore, very little is known about it even when X is a simply connected CW complex with three cells which is not a H-space.

Then the purpose of this note is to study the multiplicative structure of $M(\Sigma^k CP^3)$ and determine the group $\mathcal{E}(\Sigma^k CP^3)$ for $k \geqq 1$, where CP^n is the complex n dimensional projective space and Σ^k denotes the k-times iterated suspension.

We denote by Z_n (resp. Z/n) the multiplicative (resp. additive) cyclic group of order n.

Our main results are stated as follows:

THEOREM A. (*The case* $k=1$)
(1) *There is an exact sequence*

$$0 \longrightarrow Z \xrightarrow{\nabla} \mathcal{E}(\Sigma CP^3) \xrightarrow{\Sigma} Z_2 \times Z_2 \times Z_2 \longrightarrow 1.$$

(2) $\mathcal{E}(\Sigma CP^3)=Z \ltimes (Z_2 \times Z_2)$ (*semidirect product*).

Next we consider the case $k \geqq 2$.

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Then $\Sigma^k CP^3$ is a double suspension space and it will be proved that $M(\Sigma^k CP^3)$ becomes a ring whose addition and multiplication are induced from the track addition and the composition of maps.

THEOREM B. (*The case* $k=2, 3$)

(1) *The suspension homomorphism* $\Sigma : M(\Sigma^2 CP^3) \to M(\Sigma^3 CP^3)$ *is an isomorphism as a ring.*

(2) *As an abelian group,*

$$M(\Sigma^2 CP^3) = Z\{id\} \oplus Z\{\Sigma\mu_1\} \oplus Z\{\Sigma\mu_2\} \oplus Z/2\{\Sigma\mu_3\},$$
$$M(\Sigma^3 CP^3) = Z\{id\} \oplus Z\{\Sigma^2\mu_1\} \oplus Z\{\Sigma^2\mu_2\} \oplus Z/2\{\Sigma^2\mu_3\}.$$

(3) *Let* φ *and* ψ *be two elements of* $M(\Sigma^2 CP^3)$ *of the following forms:*

$$\varphi = a(id) + b(\Sigma\mu_1) + c(\Sigma\mu_2) + u(\Sigma\mu_3),$$
$$\psi = d(id) + e(\Sigma\mu_1) + f(\Sigma\mu_2) + v(\Sigma\mu_3),$$

where $a, b, c, d, e, f \in Z$, *and* $u, v \in Z/2$. *Then*

$$\varphi \circ \psi = (ad)id + (ae + bd + 2be)\Sigma\mu_1 + (af + cd + 6cf)\Sigma\mu_2 + (av + ud)\Sigma\mu_3.$$

(4) $\mathcal{E}(\Sigma^2 CP^3) = \mathcal{E}(\Sigma^3 CP^3) = Z_2 \times Z_2 \times Z_2$.

THEOREM C. (*The case* $k \geqq 4$). *We assume* $k \geqq 4$.

(1) *As an abelian group,*

$$M(\Sigma^k CP^3) = Z\{id\} \oplus Z\{\Sigma^{k-1}\mu_1\} \oplus Z\{\Sigma^{k-1}\mu_2\}.$$

(2) *Let* φ *and* ψ *be two elements of* $M(\Sigma^k CP^3)$ *of the following forms:*

$$\varphi = a(id) + b(\Sigma^{k-1}\mu_1) + c(\Sigma^{k-1}\mu_2),$$
$$\psi = d(id) + e(\Sigma^{k-1}\mu_1) + f(\Sigma^{k-1}\mu),$$

where $a, b, c, d, e, f \in Z$. *Then*

$$\varphi \circ \psi = (ad)id + (ae + bd + 2be)\Sigma^{k-1}\mu_1 + (af + cd + 6cf)\Sigma^{k-1}\mu_2.$$

(3) $\mathcal{E}(\Sigma^k CP^3) = Z_2 \times Z_2$.

COROLLARY D. *Let* $k \geqq 4$. *Then the homomorphism* $D : M(\Sigma^k CP^3) \to D(3, Z)$ *is an isomorphism of rings, where the ring* $D(3, Z)$ *and the homomorphism* D *are defined in* (5.8) *and* (5.10).

This paper is organized as follows:

In section 2, we will calculate the homotopy groups $\pi_*(\Sigma^k CP^2)$ and $\pi_*(\Sigma^k CP^3)$ for $1 \leqq k \leqq 4$. In section 3, we will determine the additive structure of $M(\Sigma^k CP^3)$, and in section 4, we will study the multiplicative structure of

$M(\Sigma^k CP^3)$. In section 5, we will consider the natural representation

$$\Phi = deg : M(\Sigma^k CP^n) \longrightarrow \mathrm{End}(H_*(\Sigma^k CP^n, Z)) \cong Z^n \quad (k \geqq 2n-2)$$

which is defined by $\Phi(\theta) = H_*(\theta, Z)$ for $\theta \in M(\Sigma^k CP^n)$.

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§ 2. Homotopy Groups.

Let $\iota_n \in \pi_n(S^n)$ be the identity map of S^n, and $\eta_2 \in \pi_3(S^2)$ and $\nu_4 \in \pi_7(S^4)$ be the Hopf maps.

We put $\eta_n = \Sigma^{n-2}\eta_2$, $\eta_n^2 = \eta_n \circ \eta_{n+1}$, $\eta_n^3 = \eta_n \circ \eta_{n+1} \circ \eta_{n+2}$ for $n \geqq 2$ and $\nu_m = \Sigma^{m-4}\nu_4$ for $m \geqq 4$.

Let $\omega \in \pi_6(S^3)$ be the Blakers-Massey element, and $\rho : S^3 \to RP^3 = SO(3)$ be the double covering projection.

Then the following is well-known:

LEMMA 2.1. (H. Toda, [16])

(1) $\pi_n(S^n) = Z\{\iota_n\}$, *and* $\pi_m(S^n) = 0$ *for* $n > m$.

(2) $\pi_3(S^2) = Z\{\eta_2\}$, *and* $\pi_{n+1}(S^n) = Z/2\{\eta_n\}$ *for* $n \geqq 3$.

(3) $\pi_{n+2}(S^n) = Z/2\{\eta_n^2\}$ *for* $n \geqq 2$.

(4) $\pi_5(S^2) = Z/2\{\eta_2^3\}$, $\pi_6(S^3) = Z/12\{\omega\}$,

$\pi_7(S^4) = Z\{\nu_4\} \oplus Z/12\{\Sigma\omega\}$, *and* $\pi_{n+3}(S^n) = Z/24\{\nu_n\}$ *for* $n \geqq 5$.

(5) $\pi_6(S^2) = Z/12\{\eta_2 \circ \omega\}$, $\pi_7(S^3) = Z/2\{\omega \circ \eta_6\}$,

$\pi_8(S^4) = Z/2\{\nu_4 \circ \eta_7\} \oplus Z/2\{\Sigma\omega \circ \eta_7\}$,

$\pi_9(S^5) = Z/2\{\nu_5 \circ \eta_8\}$, *and* $\pi_{n+4}(S^n) = 0$ *for* $n \geqq 6$.

(6) $J(\rho) = \pm\omega$, *where* $J : \pi_3(SO(3)) = Z\{\rho\} \to \pi_6(S^3)$ *denotes the J-homomorphism.*

(7) $[\iota_4, \iota_4] = 2\nu_4 - \Sigma\omega$ *and* $\eta_3 \circ \nu_4 = \omega \circ \eta_6$, *where* $[\ ,\ ]$ *denotes the Whitehead product.*

Consider the following three cofibre sequences:

(2.2) $$S^3 \xrightarrow{\eta_2} S^2 \xrightarrow{i} CP^2 \xrightarrow{p} S^4 \xrightarrow{\eta_3} S^3 \xrightarrow{\Sigma i} \Sigma CP^2 \xrightarrow{\Sigma p} S^5$$

(2.3) $$CP^2 \xrightarrow{j} CP^3 \xrightarrow{q} S^6 \longrightarrow \Sigma CP^2 \xrightarrow{\Sigma j} \Sigma CP^3 \longrightarrow S^7$$

(2.4) $$S^2 \xrightarrow{j \circ i} CP^3 \xrightarrow{\pi} CP^3/S^2 = S^4 \vee S^6 \longrightarrow S^3 \xrightarrow{\Sigma j \circ \Sigma i} \Sigma CP^3$$

Since the order of η_3 is two, there exists a coextension of $2\iota_4$, $\widetilde{2\iota_4} \in \pi_5(\Sigma CP^2)$ such that,

$$\Sigma p \circ \widetilde{2\iota_4} = 2\iota_5 . \tag{2.5}$$

We recall the following two results:

LEMMA 2.6. ([19]; J. Mukai, [8]). *There exists some element* $\beta \in \pi_7(\Sigma CP^2)$ *satisfying the following conditions:*

(1) $SU(3) = \Sigma CP^2 \cup_\beta e^8$.

(2) $\Sigma j \circ \beta = [\alpha_5, \iota_3]_r$, *where* $\alpha_5 \in \pi_5(\Sigma CP^2, S^3)$ *denotes the characteristic map of the* 5*-cell in* ΣCP^2 *and the* $[\ ,\ ]_r$ *the relative Whitehead product.*

(3) $\Sigma\beta = \Sigma^2 i \circ \nu_4 \circ \eta_7$.

(4) $\Sigma p \circ \beta = 0$.

Proof. The assertions (1), (2) and (4) follow from (16) in [19] and (3) follows from (8.5) in [8]. Q.E.D.

LEMMA 2.7. ([19], (1.7))

(1) $\pi_k(\Sigma CP^2) = 0$ *for* $k = 1, 2, 4$.

(2) $\pi_3(\Sigma CP^2) = Z\{\Sigma i\}$.

(3) $\pi_5(\Sigma CP^2) = Z\{\widetilde{2\iota_4}\}$.

(4) $\pi_6(\Sigma CP^2) = Z/6\{\Sigma i \circ \omega\}$.

(5) $\pi_7(\Sigma CP^2) = Z\{\beta\}$.

Next, we compute $\pi_*(\Sigma^k CP^2)$ for $2 \leqq k \leqq 4$.

LEMMA 2.8.

(1) $\pi_k(\Sigma^2 CP^2) = 0$ *for* $k = 1, 2, 3, 5$.

(2) $\pi_4(\Sigma^2 CP^2) = Z\{\Sigma^2 i\}$.

(3) $\pi_6(\Sigma^2 CP^2) = Z\{\Sigma \widetilde{2\iota_4}\}$.

(4) $\pi_7(\Sigma^2 CP^2) = Z\{\Sigma^2 i \circ \nu_4\} \oplus Z/6\{\Sigma^2 i \circ \Sigma\omega\}$.

(5) $\pi_8(\Sigma^2 CP^2) = Z/2\{\Sigma^2 i \circ \nu_4 \circ \eta_7\}$.

Proof. Consider the homotopy exact sequence of the pair $(\Sigma^2 CP^2, S^4)$. Since the pair $(\Sigma^2 CP^2, S^4)$ is 5-connected, $\pi_k(\Sigma^2 CP^2) = 0$ for $1 \leqq k \leqq 3$, $\pi_4(\Sigma^2 CP^2) = Z\{\Sigma^2 i\}$ and we have the exact sequence:

$$\pi_9(\Sigma^2 CP^2, S^4) \xrightarrow{\partial_9} \pi_8(S^4) \longrightarrow \pi_8(\Sigma^2 CP^2) \longrightarrow \pi_8(\Sigma^2 CP^2, S^4) \xrightarrow{\partial_8} \tag{2.9}$$

$$\pi_7(S^4) \longrightarrow \pi_7(\Sigma^2 CP^2) \longrightarrow \pi_7(\Sigma^2 CP^2, S^4) \xrightarrow{\partial_7} \pi_6(S^4) \longrightarrow \pi_6(\Sigma^2 CP^2)$$
$$\longrightarrow \pi_6(\Sigma^2 CP^2, S^4) \xrightarrow{\partial_6} \pi_5(S^4) \longrightarrow \pi_5(\Sigma^2 CP^2) \longrightarrow 0.$$

Let $\alpha_6 \in \pi_6(\Sigma^2 CP^2, S^4)$ be the characteristic map of the 6-cell in $\Sigma^2 CP^2$.

Then using the excision theorem and [2], it is easy to see the following:

(2.10) (1) $\pi_6(\Sigma^2 CP^2, S^4) = Z\{\alpha_6\}$ *and* $\partial_6(\alpha_6) = \eta_4$.

(2) $\pi_7(\Sigma^2 CP^2, S^4) = \alpha_{6*}\pi_7(D^6, S^5) \cong Z/2$.

(3) $\pi_8(\Sigma^2 CP^2, S^4) = \alpha_{6*}\pi_8(D^6, S^5) \cong Z/2$.

(4) $\pi_9(\Sigma^2 CP^2, S^4) = Z\{[\alpha_6, \iota_4]_r\} \oplus \alpha_{6*}\pi_9(D^6, S^5) \cong Z \oplus Z/24$.

Since $\partial_6(\alpha_6) = \eta_4$, it follows from (2.1) that $\pi_5(\Sigma^2 CP^2) = 0$. It is easy to see that the diagram

(2.11)
$$\begin{array}{ccc} \pi_k(\Sigma^2 CP^2, S^4) & \xrightarrow{\partial_k} & \pi_{k-1}(S^4) \\ \downarrow \alpha_{6*} & & \downarrow \eta_{4*} \\ \pi_k(D^6, S^4) & \xrightarrow{\tilde{\partial}} & \pi_{k-1}(S^5) \end{array}$$
is commutative.

Hence, using (2.1), (2.10) and (2.11), we have

(2.12) (1) ∂_7 is an isomorphism.

(2) ∂_8 is a monomorphism and its image is equal to

$$\mathrm{Im}[\partial_8 : \pi_8(\Sigma^2 CP^2, S^4) \longrightarrow \pi_7(S^4)] = Z/2\{\eta_4^3\} = Z/2\{6\Sigma\omega\}.$$

(3) $\partial_9(\alpha_{6*}\pi_9(D^6, S^5)) = \{\eta_4 \circ \nu_5\} = Z/2\{\Sigma\omega \circ \eta_7\}$.

Here we remark

(2.13) $$\partial_9([\alpha_6, \iota_4]_r) = \Sigma\omega \circ \eta_7.$$

In fact,

$$\partial_9([\alpha_6, \iota_4]_r) = -[\partial_6(\alpha_6), \iota_4] = [\eta_4, \iota_4] = [\iota_4 \circ \Sigma\eta_3, \iota_4 \circ \Sigma\iota_3]$$
$$= [\iota_4, \iota_4] \circ \Sigma(\eta_3 \wedge \iota_3) = (2\nu_4 - \Sigma\omega) \circ \eta_7 = \Sigma\omega \circ \eta_7.$$

Hence it follows from (2.1), (2.9), (2.10), (2.12), (2.13) and the excision theorem that we have

(2.14) (1) $\Sigma^2 p_* : \pi_6(\Sigma^2 CP^2) \longrightarrow \pi_6(S^6)$ is a monomorphism and $\mathrm{Im}(\Sigma^2 p_*) = Z\{2\iota_6\}$.

(2) $\pi_7(\Sigma^2 CP^2) = Z\{\Sigma^2 i \circ \nu_4\} \oplus Z/6\{\Sigma^2 i \circ \Sigma\omega\}$.

(3) $\pi_8(\Sigma^2 CP^2) = Z/2\{\Sigma^2 i \circ \nu_4 \circ \eta_7\}$.

Furthermore, using (2.5) we have $\Sigma^2 p_*(\Sigma \widetilde{2\iota_4}) = 2\iota_6$. Hence $\pi_6(\Sigma^2 CP^2) = Z\{\Sigma \widetilde{2\iota_4}\}$.

This completes the proof. Q.E.D.

Similar calculations show the following three results and we will omit the proofs.

LEMMA 2.15.

(1) $\pi_k(\Sigma^3 CP^2)=0$ *for* $1 \leqq k \leqq 4$ *or* $k=6$.

(2) $\pi_5(\Sigma^3 CP^2)=Z\{\Sigma^3 i\}$.

(3) $\pi_7(\Sigma^3 CP^2)=Z\{\Sigma^2 \widetilde{2\iota_4}\}$.

(4) $\pi_8(\Sigma^3 CP^2)=Z/12\{\Sigma^3 i\circ\nu_5\}$.

(5) $\pi_9(\Sigma^3 CP^2)=Z/2\{\Sigma^3 i\circ\nu_5\circ\eta_8\}$.

LEMMA 2.16.

(1) $\pi_k(\Sigma^4 CP^2)=0$ *for* $1 \leqq k \leqq 5$ *or* $k=7$.

(2) $\pi_6(\Sigma^4 CP^2)=Z\{\Sigma^4 i\}$.

(3) $\pi_8(\Sigma^4 CP^2)=Z\{\Sigma^3 \widetilde{2\iota_4}\}$.

(4) $\pi_9(\Sigma^4 CP^2)=Z/12\{\Sigma^4 i\circ\nu_6\}$.

(5) $\pi_{10}(\Sigma^4 CP^2)=0$.

COROLLARY 2.17.

(1) $\pi_0^S(S^0\cup_\eta e^2)=Z\{i\}$.

(2) $\pi_1^S(S^0\cup_\eta e^2)=0$.

(3) $\pi_2^S(S^0\cup_\eta e^2)=Z\{\widetilde{2\iota}\}$

(4) $\pi_3^S(S^0\cup_\eta e^2)=Z/12\{i\circ\nu\}$.

(5) $\pi_4^S(S^0\cup_\eta e^2)=0$.

Now we will compute the groups $\pi_*(\Sigma^k CP^3)$ for $1 \leqq k \leqq 4$. First we need the following:

LEMMA 2.18. *Let* $\beta_6 \in \pi_5(CP^2)$ *be the attaching map of the* 6*-cell in* CP^3. *Then* $\Sigma\beta_6=\pm\Sigma i\circ\omega$.

Proof. Since the space CP^3 is the total space of the S^2-bundle over S^4 with its characteristic element $\rho\in\pi_3(SO(3))=Z\{\rho\}$, it follows from (3.1) in [4] and (6) of (2.1) that $\Sigma\beta_6=\Sigma i\circ J(\rho)=\pm\Sigma i\circ\omega$. Q.E.D.

PROPOSITION 2.19.

(1) $\pi_k(\Sigma CP^3)=0$ *for* $k=1, 2, 4, 6$.

(2) $\pi_3(\Sigma CP^3)=Z\{\Sigma j\circ\Sigma i\}$.

(3) $\pi_5(\Sigma CP^3)=Z\{\Sigma j\circ\widetilde{2\iota_4}\}$.

(4) $\pi_7(\Sigma CP^3)=Z\{\Sigma j\circ\beta\}\oplus Z\{\widetilde{6\iota}\}$, *where the* $\widetilde{6\iota}$ *satisfies the condition*

$$\Sigma q\circ\widetilde{6\iota}=6\iota_7. \tag{2.20}$$

Proof. Since the pair $(\Sigma CP^3, \Sigma CP^2)$ is 6-connected, the induced homomorphism $\Sigma j_*:\pi_k(\Sigma CP^2)\to\pi_k(\Sigma CP^3)$ is an isomorphism for $1\leqq k\leqq 5$ and epimorphism for $k=6$. Thus using (2.7), it suffices only to show the case $k=6$ or 7.

Consider the homotopy exact sequence

$$\begin{aligned}\pi_8(\Sigma CP^3, \Sigma CP^2)&\xrightarrow{\tilde\partial}\pi_7(\Sigma CP^2)\longrightarrow\pi_7(\Sigma CP^3)\longrightarrow\\ \pi_7(\Sigma CP^3, \Sigma CP^2)&\xrightarrow{\partial}\pi_6(\Sigma CP^2)\longrightarrow\pi_6(\Sigma CP^3)\longrightarrow 0.\end{aligned}\tag{2.21}$$

Let $\alpha_7\in\pi_7(\Sigma CP^3, \Sigma CP^2)$ be the characteristic map of the 7-cell in ΣCP^3. Then it is easy to see

(2.22) (1) $\pi_7(\Sigma CP^3, \Sigma CP^2)=Z\{\alpha_7\}$.

(2) $\pi_8(\Sigma CP^3, \Sigma CP^2)=\alpha_{7*}\pi_8(D^7, S^6)\cong Z/2$.

Hence the boundary homomorphism $\tilde\partial$ is trivial because $\pi_7(\Sigma CP^2)=Z\{\beta\}$.

Similarly, since $\partial(\alpha_7)=\Sigma\beta_6=\pm\Sigma i\circ\omega$ and $\pi_6(\Sigma CP^2)=Z/6\{\Sigma i\circ\omega\}$, the boundary homomorphism ∂ is surjective.

Hence, $\pi_6(\Sigma CP^3)=0$ and we have the exact sequence

$$0\longrightarrow\pi_7(\Sigma CP^2)\longrightarrow\pi_7(\Sigma CP^3)\longrightarrow Z\{6\alpha_7\}\longrightarrow 0. \tag{2.23}$$

Here the induced homomorphism $(\Sigma q)_*:\pi_7(\Sigma CP^3, \Sigma CP^2)\to\pi_7(S^7)$ is an isomorphism.

Therefore, there exists some element $\widetilde{6\iota}\in\pi_7(\Sigma CP^3)$ such that

(2.24) (1) $\Sigma q\circ\widetilde{6\iota}=6\iota_7$, and

(2) $\pi_7(\Sigma CP^3)=Z\{\Sigma j\circ\beta\}\oplus Z\{\widetilde{6\iota}\}$. Q.E.D.

Similar calculation shows the following three results and the proofs are left to the reader.

PROPOSITION 2.25.

(1) $\pi_k(\Sigma^2CP^3)=0$ for $1\leqq k\leqq 3$ or $k=5$.

(2) $\pi_4(\Sigma^2CP^3)=Z\{\Sigma^2j\circ\Sigma^2i\}$.

(3) $\pi_6(\Sigma^2CP^3)=Z\{\Sigma^2j\circ\Sigma\widetilde{2\iota_4}\}$.

(4) $\pi_7(\Sigma^2CP^3)=Z\{\Sigma^2j\circ\Sigma^2i\circ\nu_4\}$.

(5) $\pi_8(\Sigma^2CP^3)=Z/2\{\Sigma^2j\circ\Sigma^2i\circ\nu_4\circ\eta_7\}\oplus Z\{\Sigma\widetilde{6\iota}\}$.

PROPOSITION 2.26.

(1) $\pi_k(\Sigma^3 CP^3)=0$ for $1\leqq k\leqq 4$ or $k=6$.

(2) $\pi_5(\Sigma^3 CP^3)=Z\{\Sigma^3 j\circ\Sigma^3 i\}$.

(3) $\pi_7(\Sigma^3 CP^3)=Z\{\Sigma^3 j\circ\Sigma^2\widetilde{2\iota_4}\}$.

(4) $\pi_8(\Sigma^3 CP^3)=Z/2\{\Sigma^3 j\circ\Sigma^3 i\circ\nu_5\}$.

(5) $\pi_9(\Sigma^3 CP^3)=Z/2\{\Sigma^3 j\circ\Sigma^3 i\circ\nu_5\circ\eta_8\}\oplus Z\{\Sigma^2\widetilde{6\iota}\}$.

PROPOSITION 2.27.

(1) $\pi_k(\Sigma^4 CP^3)=0$ for $1\leqq k\leqq 5$ or $k=7$.

(2) $\pi_6(\Sigma^4 CP^3)=Z\{\Sigma^4 j\circ\Sigma^4 i\}$.

(3) $\pi_8(\Sigma^4 CP^3)=Z\{\Sigma^4 j\circ\Sigma^3\widetilde{2\iota_4}\}$.

(4) $\pi_9(\Sigma^4 CP^3)=Z/2\{\Sigma^4 j\circ\Sigma^4 i\circ\nu_6\}$.

(5) $\pi_{10}(\Sigma^4 CP^3)=Z\{\Sigma^3 6\iota\}$.

§ 3. $M(\Sigma^k CP^3)$.

For a based topological space X, let $M(X)$ be the monoid defined by

(3.1) $$M(X)=[X, X].$$

where its multiplication is induced from the composition of maps and its identity element 1 is the identity map id. If $X=\Sigma Y$ (resp. $\Sigma^2 Y$), $M(X)$ becomes a group (resp. abelian group) with the track addition and there is a right distributive law

(3.2) $$\alpha\circ(\varphi+\psi)=\alpha\circ\varphi+\alpha\circ\psi \qquad \text{for } \alpha, \varphi, \psi\in M(X).$$

However, left distributive law

(3.3) $$(\varphi+\psi)\circ\alpha=\varphi\circ\alpha+\psi\circ\alpha \qquad \text{for } \alpha, \varphi, \psi\in M(X),$$

is valid only when α is a co H-map, e.g. a suspension map.

In this section, we will study $M(\Sigma^k CP^3)$ for $k\geqq 1$.

First, consider the cofibre sequence

(3.4) $$S^3 \xrightarrow{\Sigma(j\circ i)} \Sigma CP^3 \xrightarrow{\Sigma\pi} \Sigma CP^3/S^3=S^5\vee S^7 \longrightarrow S^4.$$

Applying $[\ ,\Sigma CP^3]$ to (3.4), we have the exact sequence

(3.5) $$\pi_4(\Sigma CP^3)\longrightarrow \pi_5(\Sigma CP^3)\oplus\pi_7(\Sigma CP^3)\xrightarrow{\Sigma\pi^*} M(\Sigma CP^3)\xrightarrow{\Sigma(j\circ i)^*}\pi_3(\Sigma CP^3).$$

Since $\pi_3(\Sigma CP^3)=Z\{\Sigma(j\circ i)\}$ and $\Sigma(j\circ i)^*(id)=\Sigma(j\circ i)$, the induced homomorphism $\Sigma(j\circ i)^*$ is surjective. Hence using $\pi_4(\Sigma CP^3)=0$, we have the following:

LEMMA 3.6. *There is a split exact sequence of groups,*

$$0 \longrightarrow \pi_5(\Sigma CP^3)\oplus\pi_7(\Sigma CP^3) \xrightarrow{\Sigma\pi^*} M(\Sigma CP^3) \xrightarrow{\Sigma(j\circ i)^*} \pi_3(\Sigma CP^3) \longrightarrow 0.$$

Remark 3.7. The group $M(\Sigma CP^3)$ is not necessarily commutative and it seems difficult to solve the extension problem of (3.6).

However, we remark the following:

LEMMA 3.8. *The group $M(\Sigma CP^3)$ is generated by the four elements id,, μ_1, μ_2 and μ_3, where we put*

(3.9) (a) $\mu_1=\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1,$

(b) $\mu_2=\widetilde{6\iota}\circ\Sigma q,$

(c) $\mu_3=\Sigma j\circ\beta\circ\Sigma q.$

Here, let $pr: CP^3/S^2=S^4\vee S^6\to S^4$ be the natural projection map to the first factor and we define the map

$$\pi_1: CP^3 \longrightarrow S^4$$

by the following composition of maps,

$$\pi_1=pr\circ\pi: CP^3 \xrightarrow{\pi} CP^3/S^2=S^4\vee S^6 \xrightarrow{pr} S^4. \tag{3.10}$$

In particular, the orders of the above four generators are all infinite.

Proof. From (2.19), we have $\pi_5(\Sigma CP^3)=Z\{\Sigma j\circ\widetilde{2\iota_4}\}$ and $\pi_7(\Sigma CP^3)=Z\{\Sigma j\circ\beta\}\oplus Z\{\widetilde{6\iota}\}$.

Hence the assertion easily follows from (3.6). Q.E.D.

Next, consider the cofibre sequence

$$S^{k+2} \xrightarrow{\Sigma^k(j\circ i)} \Sigma^k CP^3 \xrightarrow{\Sigma^k\pi} \Sigma^k(CP^3/S^2)=S^{k+4}\vee S^{k+6} \longrightarrow S^{k+3}. \quad (k\geqq 2) \tag{3.11}$$

Then we have the following:

PROPOSITION 3.12.

(1) $M(\Sigma^2CP^3)=Z\{id\}\oplus Z\{\Sigma\mu_1\}\oplus Z\{\Sigma\mu_2\}\oplus Z/2\{\Sigma\mu_3\}.$

(2) $M(\Sigma^3CP^3)=Z\{id\}\oplus Z\{\Sigma^2\mu_1\}\oplus Z\{\Sigma^2\mu_2\}\oplus Z/2\{\Sigma^2\mu_3\}.$

(3) $M(\Sigma^kCP^3)=Z\{id\}\oplus Z\{\Sigma^{k-1}\mu_1\}\oplus Z\{\Sigma^{k-1}\mu_2\}$ *for* $k\geqq 4$.

Here the following relation holds:

$$\Sigma\mu_3=\Sigma^2 j\circ\Sigma^2 i\circ\nu_4\circ\eta_7\circ\Sigma^2 q\,. \tag{3.13}$$

Proof. Let $2\leqq k\leqq 4$. If we apply $[\ ,\ \Sigma^k CP^3]$ to (3.11), since $\pi_{k+3}(\Sigma^k CP^3)=0$, $\pi_{k+2}(\Sigma^k CP^3)=Z\{\Sigma^k j\circ\Sigma^k i\}$ and $\Sigma^k(j\circ i)^*(id)=\Sigma^k j\circ\Sigma^k i$, we have the split exact sequence

$$\begin{aligned} 0\longrightarrow \pi_{k+4}(\Sigma^k CP^3)\oplus\pi_{k+6}(\Sigma^k CP^3) &\xrightarrow{\Sigma^k\pi^*} M(\Sigma^k CP^3)\\ &\xrightarrow{\Sigma^k(j\circ i)^*} \pi_{k+2}(\Sigma^k CP^3)\longrightarrow 0\,. \end{aligned} \tag{3.14}$$

Hence we have

$$M(\Sigma^k CP^3)=Z\{id\}\oplus\Sigma^k\pi^*(\pi_{k+4}(\Sigma^k CP^3)\oplus\pi_{k+6}(\Sigma^k CP^3))\,. \tag{3.15}$$

Using (2.25), (2.26), (2.27) and (2.6), we have the desired results for $2\leqq k\leqq 4$.

It follows from the Freudenthal suspension theorem that the suspension homomorphism $\Sigma: M(\Sigma^k CP^3)\to M(\Sigma^{k+1}CP^3)$ is an isomorphism for $k\geqq 4$. This completes the proof. Q.E.D.

COROLLARY 3.16. *The sequence*

$$0\longrightarrow Z\{2\mu_3\}\longrightarrow M(\Sigma CP^3)\xrightarrow{\Sigma} M(\Sigma^2 CP^3)\longrightarrow 0$$

is exact as a group.

Proof. This easily follows from (3.8) and (3.12). Q.E.D.

COROLLARY 3.17. *The sequence*

$$1\longrightarrow 1+Z\{2\mu_3\}\longrightarrow \mathcal{E}(\Sigma CP^3)\xrightarrow{\Sigma} \mathcal{E}(\Sigma^2 CP^3)\longrightarrow 1$$

is exact as a multiplicative group, where we put

$$1+Z\{2\mu_3\}=\{id+2m\mu_3: m\in Z\}\,. \tag{3.18}$$

Proof. It is easy to see that $\mathrm{Ker}[\Sigma:\mathcal{E}(\Sigma CP^3)\to\mathcal{E}(\Sigma^2 CP^3)]=1+Z\{2\mu_3\}$. Thus it suffices to show $\Sigma(\mathcal{E}(\Sigma CP^3))=\mathcal{E}(\Sigma^2 CP^3)$. Clearly $\Sigma(\mathcal{E}(\Sigma CP^3)\subset\mathcal{E}(\Sigma^2 CP^3)$.

Conversely, let $\theta\in\mathcal{E}(\Sigma^2 CP^3)$. Then using (3.16), there exists some element $\xi\in M(\Sigma CP^3)$ satisfying the condition $\theta=\Sigma\xi$. Since $H_*(\theta,Z)=\theta_*\in\mathrm{Aut}(H_*(\Sigma^2 CP^3,Z))$ and θ_* commutes the suspension isomorphism of homology groups, $H_*(\xi,Z)=\xi_*\in\mathrm{Aut}(H_*(\Sigma CP^3,Z))$. Because the space ΣCP^3 is simply connected, it follows from the Whitehead Theorem that $\xi\in\mathcal{E}(\Sigma CP^3)$.

Hence $\mathcal{E}(\Sigma^2 CP^3)=\Sigma(\mathcal{E}(\Sigma CP^3))$. Q.E.D.

COROLLARY 3.19. *The suspension homomorphism $\Sigma: M(\Sigma^2 CP^3)\to M(\Sigma^3 CP^3)$ is an isomorphism as a ring.*

Proof. This easily follows from (3.12). Q.E.D.

§ 4. The multiplicative structure.

In this section, we will investigate the multiplicative structure of $M(\Sigma^k CP^3)$ for $k \geqq 1$.

First, we need the following:

LEMMA 4.1. *$(a(id)+m\mu_3)\circ\mu_3 = a\mu_3$ for $a, m \in Z$.*

Proof. Since $q\circ j=0$, using (3.9) we have

$$\mu_3\circ\Sigma j=(\Sigma j\circ\beta\circ\Sigma q)\circ\Sigma j=\Sigma j\circ\beta\circ\Sigma(q\circ j)=0.$$

Hence,

$$\begin{aligned}(a(id)+m\mu_3)\circ\mu_3 &= (a(id)+m\mu_3)\circ\Sigma j\circ\beta\circ\Sigma q\\ &= (a(\Sigma j)+m(\mu_3\circ\Sigma j))\circ\beta\circ\Sigma q\\ &= a(\Sigma j\circ\beta\circ\Sigma q)=a\mu_3.\end{aligned}$$

Q.E.D.

PROPOSITION 4.2. *Let $\nabla: Z\to 1+Z\{2\mu_3\}$ be the natural bijection defined by*

(4.3) $$\nabla(m)=id+2m\mu_3 \quad for \quad m\in Z.$$

Then, $\nabla: Z\to 1+Z\{2\mu_3\}$ is an isomorphism of groups, where the multiplications of $1+Z\{2\mu_3\}$ and Z are induced from the composition of maps and the natural addition, respectively.

Proof. It suffices only to show the following:

(4.4) $$(id+m\mu_3)\circ(id+n\mu_3)=id+(m+n)\mu_3 \quad \text{for} \quad m, n\in Z.$$

Then

$$\begin{aligned}(id+m\mu_3)\circ(id+n\mu_3) &= (id+m\mu_3)\circ id+(id+m\mu_3)\circ(n\mu_3)\\ &= id+m\mu_3+n\{(id+m\mu_3)\circ\mu_3\}\\ &= id+m\mu_3+n\mu_3 \quad \text{(by (4.1))}\\ &= id+(m+n)\mu_3.\end{aligned}$$

Q.E.D.

LEMMA 4.5. *$\widetilde{2\iota_4}\circ\eta_5^2=0$.*

Proof. Since the order of η_5^2 is two, the order of $\widetilde{2\iota_4}\circ\eta_5^2$ is at most two. However, because it is contained in $\pi_7(\Sigma CP^2)\cong Z$, we have

$$\widetilde{2\iota_4}\circ\eta_5^2=0$$

Q.E.D.

PROPOSITION 4.6.

(1) $\mu_1\circ\mu_1=2\mu_1$.

(2) $\mu_2\circ\mu_2=6\mu_2$.

(3) $\mu_3\circ\mu_2=6\mu_3$.

(4) *If* $1\leqq n,\ k\leqq 3$ *and* $(n, k)\neq(1, 1), (2, 2)$ *and* (3.2), *then*

$$\mu_n\circ\mu_k=0.$$

Proof. It is easy to see the following:

$$\pi_1\circ j=p. \tag{4.7}$$

Then we have the following:

(1)
$$\begin{aligned}\mu_1\circ\mu_1&=(\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1)\circ(\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1)\\ &=\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma(\pi_1\circ j)\circ\widetilde{2\iota_4}\circ\Sigma\pi_1\\ &=\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma p\circ\widetilde{2\iota_4}\circ\Sigma\pi_1 && \text{(by (4.7))}\\ &=\Sigma j\circ\widetilde{2\iota_4}\circ(\Sigma p\circ\widetilde{2\iota_4})\circ\Sigma\pi_1\\ &=\Sigma j\circ\widetilde{2\iota_4}\circ(2\iota_5)\circ\Sigma\pi_1 && \text{(by (2.5))}\\ &=2(\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1)=2\mu_1.\end{aligned}$$

(2)
$$\begin{aligned}\mu_2\circ\mu_2&=(\widetilde{6\iota}\circ\Sigma q)\circ(\widetilde{6\iota}\circ\Sigma q)\\ &=\widetilde{6\iota}\circ(\Sigma q\circ\widetilde{6\iota})\circ\Sigma q\\ &=\widetilde{6\iota}\circ(6\iota_7)\circ\Sigma q && \text{(by (2.20))}\\ &=6(\widetilde{6\iota}\circ\Sigma q)=6\mu_2.\end{aligned}$$

(3)
$$\begin{aligned}\mu_3\circ\mu_2&=(\Sigma j\circ\beta\circ\Sigma q)\circ(\widetilde{6\iota}\circ\Sigma q)\\ &=\Sigma j\circ\beta\circ(\Sigma q\circ\widetilde{6\iota})\circ\Sigma q\\ &=\Sigma j\circ\beta\circ(6\iota_7)\circ\Sigma q && \text{(by (2.20))}\\ &=6(\Sigma j\circ\beta\circ\Sigma q)=6\mu_3.\end{aligned}$$

(4)
$$\begin{aligned}\mu_1\circ\mu_2&=(\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1)\circ(\widetilde{6\iota}\circ\Sigma q)\\ &=\Sigma j\circ\widetilde{2\iota_4}\circ(\Sigma\pi_1\circ\widetilde{6\iota})\circ\Sigma q.\end{aligned}$$

Since $\Sigma\pi_1\circ\widetilde{6\iota}\in\pi_7(S^5)=Z/2\{\eta_5^2\}$, we obtain

$$\Sigma\pi_1\circ\widetilde{6\iota}=\eta_5^2 \quad \text{or} \quad 0. \tag{4.8}$$

If $\Sigma\pi_1\circ\widetilde{6\iota}=0$, then $\mu_1\circ\mu_2=0$ and we may suppose that $\Sigma\pi_1\circ\widetilde{6\iota}=\eta_5^2$.
Then, $\mu_1\circ\mu_2=\Sigma j\circ(\widetilde{2\iota_4}\circ\eta_5^2)\circ\Sigma q=0$. (by (4.5))
Hence we have $\mu_1\circ\mu_2=0$.

Next, we have

$$\begin{aligned}\mu_2\circ\mu_1&=(\widetilde{6\iota}\circ\Sigma q)\circ(\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1)\\&=\widetilde{6\iota}\circ\Sigma(q\circ j)\circ\widetilde{2\iota_4}\circ\Sigma\pi_1\\&=0. && \text{(by using } q\circ j=0)\end{aligned}$$

Similarly we have the following:

$$\begin{aligned}\mu_1\circ\mu_3&=(\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1)\circ(\Sigma j\circ\beta\circ\Sigma q)\\&=\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma(\pi_1\circ j)\circ\beta\circ\Sigma q\\&=\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma p\circ\beta\circ\Sigma q && \text{(by (4.7))}\\&=0. && \text{(by (2.6))}\end{aligned}$$

$$\begin{aligned}\mu_3\circ\mu_1&=(\Sigma j\circ\beta\circ\Sigma q)\circ(\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1)\\&=\Sigma j\circ\beta\circ\Sigma(q\circ j)\circ\widetilde{2\iota_4}\circ\Sigma\pi_1\\&=0. && \text{(by using } q\circ j=0)\end{aligned}$$

$$\begin{aligned}\mu_2\circ\mu_3&=(\widetilde{6\iota}\circ\Sigma q)\circ(\Sigma j\circ\beta\circ\Sigma q)\\&=\widetilde{6\iota}\circ\Sigma(q\circ j)\circ\beta\circ\Sigma q\\&=0. && \text{(by using } q\circ j=0)\end{aligned}$$

$$\begin{aligned}\mu_3\circ\mu_3&=(\Sigma j\circ\beta\circ\Sigma q)\circ(\Sigma j\circ\beta\circ\Sigma q)\\&=\Sigma j\circ\beta\circ\Sigma(q\circ j)\circ\beta\circ\Sigma q\\&=0. && \text{(by using } q\circ j=0)\end{aligned}$$

This completes the proof. Q.E.D.

Next, it is easy to see the following:

LEMMA 4.9. *Let a, b, c, d, e, f be six integers and u, v be two elements in* $Z/2=\{0, 1\}$.
Suppose the following four conditions hold:

(i) $ad=1$.

(ii) $ae+bd+2be=0$.

(iii) $af+cd+6cf=0$.

(iv) $av+du=0$.

Then the following hold:

(1) $a=d=1$, $b=e=0, -1$, $c=f=0$ *and* $u=v=0, 1$; *or*

(2) $a=d=-1$, $b=e=0, 1$, $c=f=0$ *and* $u=v=0, 1$.

Proof. It is easy to see that $a=d=\pm 1$. Then, from the conditions (ii), (iii) and (iv), we have

$$(1\pm 2b)(1\pm 2e)=(1\pm 6c)(1\pm 6f)=1 \quad \text{and} \quad u+v=0.$$

Hence we have the desired results. Q.E.D.

Proof of Theorem B. The assertions (1) and (2) follow from (3.19) and (3.12). First, we show the statement (3).

Let φ and ψ be two elements of $M(\Sigma^2 CP^3)$ of the following forms:

$$\begin{aligned} \text{(4.10)} \qquad \varphi &= a(id)+b(\Sigma\mu_1)+c(\Sigma\mu_2)+u(\Sigma\mu_3), \\ \psi &= d(id)+e(\Sigma\mu_1)+f(\Sigma\mu_2)+v(\Sigma\mu_3), \end{aligned}$$

where $a, b, c, d, e, f \in Z$ and $u, v \in Z/2$.

Since $\Sigma: M(\Sigma CP^3) \to M(\Sigma^2 CP^3)$ is surjective, the group $M(\Sigma^2 CP^3)$ becomes a ring.

Hence, using (3.3) and (4.6) we have the following:

$$\begin{aligned} \varphi\circ\psi &= d\{a(id)+b(\Sigma\mu_1)+c(\Sigma\mu_2)+u(\Sigma\mu_3)\} \\ &\quad + e\{a(id)+b(\Sigma\mu_1)+c(\Sigma\mu_2)+u(\Sigma\mu_3)\}\circ\Sigma\mu_1 \\ &\quad + f\{a(id)+b(\Sigma\mu_1)+c(\Sigma\mu_2)+u(\Sigma\mu_3)\}\circ\Sigma\mu_2 \\ &\quad + v\{a(id)+b(\Sigma\mu_1)+c(\Sigma\mu_2)+u(\Sigma\mu_3)\}\circ\Sigma\mu_3 \\ &= \{ad(id)+bd(\Sigma\mu_1)+cd(\Sigma\mu_2)+du(\Sigma\mu_3)\} \\ &\quad + e\{a(\Sigma\mu_1)+b(\Sigma(\mu_1\circ\mu_1))+c(\Sigma(\mu_2\circ\mu_1))+u(\Sigma(\mu_3\circ\mu_1))\} \\ &\quad + f\{a(\Sigma\mu_2)+b(\Sigma(\mu_1\circ\mu_2))+c(\Sigma(\mu_2\circ\mu_2))+u(\Sigma(\mu_3\circ\mu_2))\} \\ &\quad + v\{a(\Sigma\mu_3)+b(\Sigma(\mu_1\circ\mu_3))+c(\Sigma(\mu_2\circ\mu_3))+u(\Sigma(\mu_3\circ\mu_3))\} \\ &= \{ad(id)+bd(\Sigma\mu_1)+cd(\Sigma\mu_2)+du(\Sigma\mu_3)\} \\ &\quad + e\{a(\Sigma\mu_1)+2b(\Sigma\mu_1)\}+f\{a(\Sigma\mu_2)+6c(\Sigma\mu_2)\}+av(\Sigma\mu_3) \\ &= ad(id)+(ae+bd+2be)\Sigma\mu_1+(af+cd+6cf)\Sigma\mu_2 \\ &\quad +(av+du)\Sigma\mu_3. \end{aligned}$$

Next, we prove the statement (4).

Let $\theta \in M(\Sigma^2 CP^3)$ be the element of the form

$$\theta = a(id) + b(\Sigma\mu_1) + c(\Sigma\mu_2) + u(\Sigma\mu_3), \tag{4.11}$$

where $a, b, c \in Z$ and $u \in Z/2$.

Then using (3), it is easy to see the following:

(4.12) *$\theta \in \mathcal{E}(\Sigma^2 CP^3)$ if and only if there are integers $d, e, f \in Z$ and $v \in Z/2$ satisfying the following:*

(i) $ad = 1$.

(ii) $ae + bd + 2be = 0$.

(iii) $af + cd + 6cf = 0$.

(iv) $av + du = 0$.

Hence it follows from (4.9) that

$$\mathcal{E}(\Sigma^2 CP^3) = \{\pm id + u(\Sigma\mu_3),\ \pm(id - \Sigma\mu_1) + v(\Sigma\mu_3) : u, v \in Z/2\}. \tag{4.13}$$

Since $\theta \circ \theta = id$ for any element $\theta \in \mathcal{E}(\Sigma^2 CP^3)$, we have

$$\mathcal{E}(\Sigma^2 CP^3) = Z_2 \times Z_2 \times Z_2. \qquad \text{Q.E.D.}$$

Proof of Theorem C. The assertions (1) and (2) follow from (3.19) and (3.12).

Similar method as above also shows the statement (3).

Now we suppose that $k \geqq 4$.

Then, it follows from the modified proof of (4) of Theorem B that $\mathcal{E}(\Sigma^k CP^3) = \{\pm id, \pm(id - \Sigma^{k-1}\mu_1)\}$ and $\theta \circ \theta = id$ for any $\theta \in \mathcal{E}(\Sigma^k CP^3)$. Thus we have $\mathcal{E}(\Sigma^k CP^3) = Z_2 \times Z_2$. Q.E.D.

Proof of Theorem A. It follows from (3.17) that the sequence

$$1 \longrightarrow 1 + Z\{2\mu_3\} \longrightarrow \mathcal{E}(\Sigma CP^3) \xrightarrow{\Sigma} \mathcal{E}(\Sigma^2 CP^3) \longrightarrow 1$$

is exact.

Here, from (4.2) and Theorem B, there are isomorphisms of groups,

$$1 + Z\{2\mu_3\} \cong Z \quad \text{and} \quad \mathcal{E}(\Sigma^2 CP^3) \cong Z_2 \times Z_2 \times Z_2.$$

Hence we have the exact sequence

$$0 \longrightarrow Z \xrightarrow{\nabla} \mathcal{E}(\Sigma CP^3) \xrightarrow{\Sigma} Z_2 \times Z_2 \times Z_2 \longrightarrow 1.$$

Furthermore, since the suspension homomorphism $\Sigma : M(\Sigma CP^3) \to M(\Sigma^2 CP^3)$ induces the isomorphism $M(\Sigma CP^3)/Z\{\mu_3\} \to M(\Sigma^2 CP^3)$, it is easy to construct the splitting $s : Z_2 \times Z_2 \to \mathcal{E}(\Sigma CP^3)$ and we have the semidirect product $\mathcal{E}(\Sigma CP_3) = Z \ltimes (Z_2 \times Z_2)$. Q.E.D.

Remark 4.14. The statement (3) of Theorem C was already obtained by S. Sasao [13] using the another method.

In fact, since the space CP^3 is the total space of the S^2-bundle over S^4, it follows from Theorem A in [13] that the sequence

$$(4.15)\qquad 0 \longrightarrow \pi_4^S(S^0\cup_\eta e^2) \longrightarrow \mathcal{E}(\Sigma^k CP^3) \xrightarrow{\varphi} \mathcal{E}(\Sigma^k CP^2) \longrightarrow 1$$

is exact if $k\geqq 4$.

Then it follows from (2.17) and (4.1) in [11] that $\mathcal{E}(\Sigma^k CP^3)=\varepsilon(\Sigma^k CP^2)=Z_2\times Z_2$ if $k\geqq 4$. Q.E.D.

The following remark was suggested by the referee and the author would like to thank him for his valuable advices.

Remark 4.16. Using the Barcus-Barratt method (e.g. (2.11) in [11]), there are two exact sequences,

$$(4.17)\qquad 0 \longrightarrow Z/2 \xrightarrow{\lambda} \mathcal{E}(\Sigma^k CP^3) \xrightarrow{\varphi} Z_2\times Z_2 \longrightarrow 1$$

for $k=2$ *or* 3, *and*

$$(4.18)\qquad 0 \longrightarrow \pi_7(\Sigma CP^2) \xrightarrow{\lambda} \mathcal{E}(\Sigma CP^3) \xrightarrow{\varphi} Z_2\times Z_2 \longrightarrow 1$$

where $\pi_7(\Sigma CP^3)=Z\{\beta\}$ *and* φ *denotes the restriction homomorphism*

$$\varphi : \mathcal{E}(\Sigma^k CP^3) \longrightarrow \mathcal{E}(\Sigma^k CP^2)=Z_2\times Z_2 \qquad \textit{for}\quad k=1, 2, 3.$$

Here we remark the following two viewpoints:

(a) If we only use the Barcus-Barratt method, we must consider the extension problem of (4.17).

(b) At first sight, it seems that (4.18) contradicts the assertion of Theorem A. However, they are equivalent. In fact, this follows from the following:

From the definition of the homomorphism λ (e.g. (1.2) in [11]), it is easy to see

$$(4.19)\qquad \lambda(m\beta)=id+m\mu_3 \qquad \textit{for}\quad m\in Z.$$

Hence, from (4.18) we have the exact sequence

$$(4.20)\qquad 0 \longrightarrow 1+Z\{\mu_3\} \longrightarrow \mathcal{E}(\Sigma CP^3) \xrightarrow{\varphi} Z_2\times Z_2 \longrightarrow 1,$$

where we put

$$1+Z\{\mu_3\}=\{id+m\mu_3 : m\in Z\}.$$

Furthermore, it follows from (4.4) that $1+Z\{\mu_3\}\cong Z$. Hence the exact sequence of Theorem A and (4.18) are equivalent to the following two exact sequence:

$$(4.21)\qquad 0 \longrightarrow Z \xrightarrow{\nabla} \mathcal{E}(\Sigma CP^3) \xrightarrow{\varphi\times \mathrm{proj}} Z_2\times Z_2\times Z_2 \longrightarrow 1,$$

$$0 \longrightarrow Z \xrightarrow{\lambda} \mathcal{E}(\Sigma CP^3) \xrightarrow{\varphi} Z_2\times Z_2 \dashrightarrow 1,$$

where

$\lambda(2m)=\nabla(m)=id+2m\mu_3$ *for* $m\in Z$, *and the sequence* $0 \longrightarrow Z \xrightarrow{2} Z \xrightarrow{\mathrm{proj}} Z \longrightarrow 1$ *is exact.*

Hence the sequence given in Theorem A is equivalent to (4.18).

§5. The natural representation.

For a topological space X, let $\Phi: M(X)\to \mathrm{End}(H_*(X, Z))$ and $\Phi: \mathcal{E}(X)\to \mathrm{Aut}(H_*(X, Z))$ be the natural representation defined by

$$(5.1)\qquad \Phi(\theta)=H_*(\theta, Z) \quad \text{for} \quad \theta\in M(X) \text{ or } \mathcal{E}(X).$$

In this section, we consider the represntation Φ for $X=\Sigma^k CP^n$. Let $x_m\in H_{2m+k}(\Sigma^k CP^n, Z)\cong Z$ be the generator for $1\leqq m\leqq n$. For $\theta\in M(\Sigma^k CP^n)$ and $1\leqq m\leqq n$, let $d_m(\theta)\in Z$ be the m-th degree defined by

$$(5.2)\qquad \theta_*(x_m)=d_m(\theta)x_m.$$

Then we define the homomorphism $deg: M(\Sigma^k CP^n)\to Z^n$ (n-times product of Z) by the following:

$$(5.3)\qquad deg(\theta)=(d_1(\theta), d_2(\theta), \cdots, d_n(\theta)) \quad \text{for} \quad \theta\in M(\Sigma^k CP^n).$$

It is easy to see that we can identify Φ with deg:

$$(5.4)\qquad \Phi=deg: M(\Sigma^k CP^n) \longrightarrow \mathrm{End}(H_*(\Sigma^k CP^n, Z))\cong Z^n.$$

LEMMA 5.5. *The map deg is an additive and multiplicative homomorphism for* $k\geqq 1$.

Proof. It is clear. Q.E.D.

PROPOSITION 5.6. (The case $n=3$)

(1) $deg(id)=(1, 1, 1)$ *for any* k.

(2) $deg(\mu_1)=(0, 2, 0)$.

(3) $deg(\mu_2)=(0, 0, 6)$.

(4) $deg(\mu_3)=(0, 0, 0)$.

Proof. The statement (1) is obvious. Since $\mu_1=\Sigma j\circ\widetilde{2\iota_4}\circ\Sigma\pi_1$, the assertion (2) follows from (2.5). Similarly, the assertion (3) follows from (2.24). Since $\mu_3\circ\mu_3=0$, the statement (4) is clear. Q.E.D.

COROLLARY 5.7. (The case $n=3$). *Let $k\geqq 4$ and $n=3$. Then the following holds:*

$$d_1(\theta)=a,\ d_2(\theta)=a+2b \text{ and } d_3(\theta)=a+6c \text{ for}$$
$$\theta=a(id)+b(\Sigma^{k-1}\mu_1)+c(\Sigma^{k-1}\mu_2)\in M(\Sigma^k CP^3).$$

Proof. This follows from (5.4) and (5.5). Q.E.D.

For each natural number n, let $M_n(Z)$ be the ring consisting of all (n, n)-matrices with integer coefficients. Similarly, for each n, let $D(n, Z)$ be the subring of $M_n(Z)$ consisting of all diagonal matrices of the following form:

(5.8) $$D(n, Z)=\{\mathrm{diag}(a_1, a_1+2!a_2, \cdots, a_1+n!a_n): a_i\in Z\},$$

where we put

(5.9) $$\mathrm{diag}(x_1, x_2, \cdots, x_n)=\begin{pmatrix} x_1 & & & & \\ & x_2 & & & \\ & & x_3 & & \\ & & & \ddots & \\ & & & & x_n \end{pmatrix}$$

Then we define the ring homomorphism $D: M(\Sigma^k CP^n)\to M_n(Z)$ by the following:

(5.10) $$D(\theta)=\mathrm{diag}(d_1(\theta), d_2(\theta), \cdots, d_n(\theta)) \quad \text{for} \quad \theta\in M(\Sigma^k CP^n).$$

Proof of Corollary D. The assertion easily follows from (5.7) and Theorem C. Q.E.D.

Problem 5.11. Let $k\geqq 2n-2$. Then, does the homomorphism D induce the monomorphism of rings, $D: M(\Sigma^k CP^n)\to M_n(Z)$?

Remark 5.12. The above problem (5.11) and $\mathrm{Im}[D: M(\Sigma^k CP^n)\to M_n(Z)]=D(n, Z)$ are true for $1\leqq n\leqq 3$. In fact, the case $n=1$ is trivial and the case $n=2$ was proved by S. Oka in [10]. The case $n=3$ is obtained by Corollary D.

Now we define the subring of $M_n(Z)$, $I(n)$, defined by

(5.13) $$I(n)=\bigoplus_{1\leq m\leq n} Z\{\mathrm{diag}(m, m^2, m^3, \cdots, m^n)\}.$$

Then, the following is well-known:

PROPOSITION 5.14. (C.A. McGibbon, [7]). *If $k\geqq 2n-2$, $\mathrm{Im}[D: M(\Sigma^k CP^n)\to M_n(Z)]=I(n)$.*

Proof. This follows from Theorem 3.4 in [7]. Q.E.D.

Remark 5.15. It is easy to see that $I(n)=D(n, Z)$ for $1\leqq n\leqq 3$ and $I(4)\neq D(4, Z)$. Hence, in general, the ring $I(n)$ is not always equal to $D(n, Z)$.

REFERENCES

[1] W.D. BARCUS AND M.G. BARRATT, On the homotopy classification of a fixed map, Trans. Amer. Math. Soc., **88** (1958), 57-74.

[2] I.M. JAMES, On the homotopy groups of certain pairs and triads, Quart. J. Math. Oxford, **5** (1954), 260-270.

[3] I.M. JAMES, Note on cup products, Proc. Amer. Math. Soc., **8** (1957), 374-383.

[4] I.M. JAMES, On sphere bundles over spheres, Comment. Math. Helv., **35** (1961), 126-135.

[5] I.M. JAMES AND J.H.C. WHITEHEAD, The homotopy theory of sphere bundles over spheres (I), Proc. London Math. Soc., **4** (1954), 196-218.

[6] I.M. JAMES AND J.H.C. WHITEHEAD, The homotopy theory of sphere bundles over spheres (II), Proc. London Math. Soc., **5** (1955), 148-166.

[7] C.A. MCGIBBON, Self-maps of projective spaces, Trans. Amer. Math. Soc., **271** (1982), 325-346.

[8] J. MUKAI, The S^1-transfer map and homotopy groups of suspended complex projective spaces, Math. J. Okayama Univ., **24** (1982), 179-200.

[9] J. MUKAI, On the stable homotopy of the real projective space of even low dimension, Publ. RIMS. Kyoto Univ., **22** (1986), 81-95.

[10] S. OKA, Groups of self-equivalences of certain complexes, Hiroshima Math. J., **2** (1972), 285-293,

[11] S. OKA, N. SAWASHITA AND M. SUGAWARA, On the group of self-equivalences of a mapping cone, Hiroshima Math. J., **4** (1974), 9-28.

[12] J.W. RUTTER, The group of homotopy self-equivalence classes of *CW* complexes, Math. Proc. Camb. Phil. Soc., **93** (1983), 275-293.

[13] S. SASAO, The stable group of self-homotopy equivalences of sphere bundles over the sphere, Kodai Math. J., **4** (1981), 231-238.

[14] S. SASAO, On self-homotopy equivalences of the total spaces of a sphere bundle over a sphere, Kodai Math. J., **7** (1984), 365-381.

[15] A.J. SIERADSKI, Twisted self-homotopy equivalences, Pacific J. Math., **34** (1970), 789-802.

[16] H. TODA, Composition methods in homotopy groups of spheres, Annals of Math. Studies, **49** (1962), Princeton Univ. Press.

[17] K. YAMAGUCHI, The group of self-homotopy equivalences of S^2-bundles over S^4, I, Kodai Math. J., **9** (1986), 308-326.

[18] K. YAMAGUCHI, The group of self-homotopy equivalences of S^2-bundles over S^4, II; Applications, Kodai Math. J., **10** (1987), 1-8.

[19] K. YAMAGUCHI, Self-homotopy equivalences and highly connected Poincare complexes, Lecture Notes in Math., **1425** (1990), 157-169, Springer-Verlag.

[20] G.W. WHITEHEAD, Elements of homotopy theory, Springer Verlag G.T.M., **61** (1979),

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