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ON FACTORIZATION OF ENTIRE FUNCTIONS SATISFYING DIFFERENTIAL EQUATIONS

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Abstract

We are concerned with the factorization (in the composite sense) of $f(z)$, where $f(z)$ is an entire function that satisfies a differential equation with polynomials or functions of exponential type as the coefficients. Some sufficient conditions for the forms and pseudo-primeness of $f(z)$ have been established and some known results have been generalized.

1. Introduction.

In 1978 a very interesting and powerful result about a certain type functional equation of composite meromorphic functions was announced due to N. Steinmetz [6], see Lemma A below. Since then, many of its applications in the value-distribution theory have been derived, especially in the studies of the factorization of entire or meromorphic solutions of differential equations. For instance, Steinmetz himself proved (see Lemma D below) that any transcendental meromorphic solution $F(z)$ of a linear differential equation with rational functions as the coefficients must be pseudo-prime, i.e., in any factorization of the form $F(z)=f(g(z))$, f, g meromorphic implies that either f or g must be rational. This is clearly not true for nonlinear differential equations in general, even with constant coefficients. For example, $w(z)=\exp(e^z)$ $(=\exp z\circ e^z)$ is a solution of $ww''-ww'-w'^2=0$. However, there have been shown meromorphic solutions of certain classes of nonlinear differential equations of first order (such as Riccati type equations, see e.g. [7]) and of second order (see e.g. [2]) are pseudo-prime. Moreover, it is proved in [2] that, under certain constraints, entire solutions of a second order linear differential equation with periodic functions as the coefficients are pseudo-prime as follows.

THEOREM A ([2]). *Consider the equation*

$$w''(z)+P(e^z)w'(z)+Q(e^z)w(z)=0\,, \tag{1}$$

where $P(z)=\sum_{j=0}^{p} p_j z^j$, $Q(z)=\sum_{k=0}^{q} q_k z^k$. *If there exist rational numbers* r_1 *and* r_2

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such that

$$p_0 = -(r_1 + r_2) \quad \text{and} \quad q_0 = r_1 r_2 ,$$

then each subnormal solution of (1) *is pseudo-prime.*

A function $w(z)$ is said to be subnormal, if it satisfies

$$\overline{\lim_{r\to\infty}} \frac{\log T(r, w)}{r} = 0 . \tag{2}$$

Earlier Wittich [9] showed that of any two linearly independent solutions of (1) at most one of them can be subnormal. In this paper, we shall consider some related problems, mainly the extension of Theorem A to n-th order differential equations as well as the factorization of solutions of arbitrary second order algebraic differential equations. Among the results, we have been able to prove that an entire periodic solution of restricted growth of a second order algebraic differential equation with rational functions as the coefficients must be pseudo-prime.

2. Preliminaries.

LEMMA A (Steinmetz [7]). *Let $F_j(z)$ and $h_j(z)$ $(j=0, 1, 2, \cdots, m)$ be not identically vanishing meromorphic functions. Let $g(z)$ be a nonconstant entire function satisfying*

$$\sum_{j=0}^{m} T(r, h_j) \leqq KT(r, g) + S(r, g) ,$$

where K is a positive constant and $S(r, g) = o\{T(r, g)\}$ as $r \to \infty$ outside a set of r of finite measure.

If F_j and h_j $(j=0, 1, 2, \cdots, m)$ satisfy

$$F_0(g)h_0 + \cdots + F_m(g)h_m \equiv 0 ,$$

then there exist polynomials $P_0, P_1, \cdots, P_m$ not all identically zero such that

$$P_0(g)h_0 + P_1(g)h_1 + \cdots + P_m(g)h_m \equiv 0 .$$

Furthermore, if not all h_j's are identically zero, then there exist not all identically zero polynomaials $Q_0, Q_1, \cdots, Q_m$ such that

$$F_0 Q_0 + F_1 Q_1 + \cdots + F_m Q_m \equiv 0 .$$

LEMMA B (Valiron-Wittich [10]). *Suppose that $w(z)$ is a meromorphic solution of a linear differential equation with rational functions as the coefficients. Then the order of $w(z)$ is a positive rational number.*

LEMMA C (Strelitz-Zimogljad [8]). *Every entire transcendental solution of a second order algebraic differential equation with rational functions as the coefficients is of a positive order.*

In the sequel, we shall denote by D the class of meromorphic functions $F(z)$ that satisfy linear differential equatious with rational functions as the coefficients ([5]). Then, by an application of Lemma A, together with Lemma B, Steinmetz [7] showed the following interesting result.

LEMMA D. *If $F\in D$, then F is pseudo-prime.*

LEMMA 1 ([5]). *If $f\in D$ and $g\in D$, then $f+g\in D$.*

LEMMA 2. *If $f, g\in D$, then the product $fg\in D$.*

Proof. By assumption we have $\sum_{i=0}^{t} P_i(z)f^{(i)}(z)=0$ and $\sum_{j=0}^{s} Q_j(z)g^{(j)}(z)=0$, where t, s are positive integers, the P_i's and Q_j's are polynomials with $P_tQ_s\not\equiv 0$. Here let us note that when f satisfies the equation $\sum_{k=0}^{m} R_k(z)f^{(k)}(z)=R(z)$, then after differentiating this and eliminating $R(z)$ and $R'(z)$, we obtain the equation of the form $\sum_{i=0}^{t} P_i(z)f^{(i)}(z)=0$ $(t=m+1)$. Now, let $h=fg$ and recall that the general formula for the n-th derivative of the product fg is

$$h^{(n)}=(fg)^{(n)}=f^{(0)}g^{(n)}+{}_nc_1f^{(1)}g^{(n-1)}+{}_nc_2f^{(2)}g^{(n-2)}+\cdots+{}_nc_nf^{(n)}g^{(0)}.$$

Then clearly the set $\{f^{(i)}g^{(j)}\}$, $0\leqq i\leqq t$, $0\leqq j\leqq s$, forms a base for the space of functions generated by $\{h^{(0)}, h^{(1)}, h^{(2)}, \cdots\}$ over the field of rational functions. Note the cardinality of the base is $st+s+t+1$. It follows in particular the set $\{h^{(0)}, h^{(1)}, \cdots, h^{(st+s+t+1)}\}$ is linearly dependent over the rational functions. This also shows that $h\in D$.

Remarks. 1) Let $f(z)=e^z-1$, then $f\in D$, but $1/f\notin D$, because $1/f$ has an infinite number of poles. However, it seems plausible that if $f\in D$ and $g\in D$ $(g\not\equiv 0)$, then f/g is always pseudo-prime.

(2) If $f\in D$ and P is a polynomial, then we can easily verify that $f(P)\in D$ and, hence, $f(P)$ is pseudo-prime. It remains to be verified: If f is pseudo-prime, then so is $f(P)$, P is a nonconstant polynomial.

LEMMA 3. *We consider the following homogeneous differential equation of the second order*

$$w''(z)+P_1(e^z, z^{-z})w'(z)+P_2(e^z, e^{-z})w(z)=0\,, \tag{3}$$

where the $P_j(x, y)$ $(\not\equiv 0)$ are polynomials in its arguments x and y. Then, if either $P_1(e^z, e^{-z})$ or $P_2(e^z, e^{-z})$ is non-constant, of any two linearly independent *entire solutions of* (3) *only one of them can be subnormal.*

Proof. Let us note that if $P_1(e^z, e^{-z})\equiv$constant and $P_2(e^z, e^{-z})\not\equiv$constant, then it is easily verified that (3) has no subnormal solution. Thus we proceed to prove Lemma 3 under the assumption that $P_1(e^z, e^{-z})\not\equiv$constant. Assume now that $w_1=h(z)$ ($\not\equiv 0$) is an entire solution of (3) which is subnormal. Let $w(z)$ be any other entire solution of (3) which is linearly independent of w_1. We proceed to show that w is not subnormal. Let us consider the following auxiliary function $u(z)$

$$u=w/w_1 \qquad (\text{i.e., } w=w_1u). \tag{4}$$

Then, since w_1 and w satisfy (3),

$$\begin{aligned}\{2w_1'+P_1(e^z, e^{-z})w_1\}u'+w_1u''\\ =w''+P_1(e^z, e^{-z})w'+P_2(e^z, e^{-z})w\\ -\{w_1''+P_1(e^z, e^{-z})w_1'+P_2(e^z, e^{-z})w_1\}u=0.\end{aligned}$$

Hence, by integration, we have

$$u'=w_1^{-2}\exp\left\{-\int^z P_1(e^z, e^{-z})dz+c\right\}, \tag{5}$$

where c is a constant. Therefore,

$$\underline{\lim}_{r\to\infty}\frac{\log T(r, w_1^2u')}{r}\geqq c_0 \qquad (>0). \tag{6}$$

In view of (4), $u'=(w'w_1-w_1'w)/w_1^2$. Thus

$$w_1^2u'=-w_1'w\left\{1-\frac{w_1}{w_1'}\cdot\frac{w'}{w}\right\},$$

which is an entire function. Hence, by applying the lemma on logarithmic derivative, we deduce from the above

$$\begin{aligned}T(r, w_1^2u')=m(r, w_1^2u')\leqq T(r, w)+m\left(r, \frac{w'}{w}\right)+T(r, w_1)+2m(r, w_1')+O(1)\\ \leqq(1+o(1))T(r, w)+3(1+o(1))T(r, w_1),\end{aligned} \tag{7}$$

provided $r\notin E_0$, where E_0 is a set of r of finite linear measure. Since w_1 is a subnormal solution, noting (6) and (7), we obtain

$$\overline{\lim_{r\to\infty}}\frac{\log T(r, w)}{r}\geqq c_0>0.$$

This also shows that w is not subnormal.

Further, we need the following result.

LEMMA 4. *Consider the n-th order differential equation of the form*

$$P_0(e^z, e^{-z})w^{(n)}+P_1(e^z, e^{-z})w^{(n-1)}+\cdots+P_n(e^z, e^{-z})w=P_{n+1}(e^z, e^{-z}), \tag{8}$$

where the $P_j(x, y)$ are polynomials in the arguments x and y with $P_0(e^z, e^{-z})\not\equiv 0$. Suppose that $w=f(z)$ is an entire and subnormal solution of (8) *and that f can be expressed as $f(z)=e^{\alpha z}\psi(e^z)$, where $\psi(\zeta)$ is holomorphic in $0<|\zeta|<\infty$. Then $\psi(\zeta)=\psi_1(\zeta)+\psi_2(1/\zeta)$, where ψ_j $(j=1, 2)$ are polynomials.*

Proof. By letting $g(z)=\psi(e^z)$ and substituting $f(z)=e^{\alpha z}\psi(e^z)$ into (8), we see easily that $w=g(z)$ will satisfy a differential equation of the form

$$e^{\alpha z}\{\tilde{P}_0(e^z, e^{-z})w^{(n)}+\tilde{P}_1(e^z, e^{-z})w^{(n-1)}+\cdots+\tilde{P}_n(e^z, e^{-z})w\}=\tilde{P}_{n+1}(e^z, e^{-z}),$$

where $\tilde{P}_j$ $(j=0, 1, 2, \cdots, n+1)$ are polynomials in the arguments x and y with $\tilde{P}_0\equiv P_0$ and $\tilde{P}_{n+1}\equiv P_{n+1}$.

It is now clear that if $P_{n+1}(e^z, e^{-z})\not\equiv 0$ (hence so is $\tilde{P}_{n+1}$), then $e^{\alpha z}$ is periodic with period $2\pi i$ and α has to be an integer. However, if $P_{n+1}\equiv 0$, then after dividing both sides on the above equation by $e^{\alpha z}$, we see immediately that $w=g(z)$ satisfies an equation of the form (8). Therefore, in the sequel, we may assume that $f(z)=\psi(e^z)$ with $\psi(\zeta)=\psi_1(\zeta)+\psi_2(1/\zeta)$, where ψ_j $(j=1, 2)$ are entire functions. Differentiating $w=f(z)$, we have

$$w'=\psi_1'(e^z)e^z-\psi_2'(e^{-z})e^{-z}$$

$$w''=\psi_1''(e^z)e^{2z}+\psi_1'(e^z)e^z+\psi_2''(e^{-z})e^{-2z}+\psi_2'(e^{-z})e^{-z},$$

$$\cdots$$

$$\cdots$$

$$w^{(n)}=\psi_1^{(n)}(e^z)e^{nz}+\beta_1\psi_1^{(n-1)}(e^z)e^{(n-1)z}+\cdots+\beta_{n-1}\psi_1'(e^z)e^z$$
$$+(-1)^n\{\psi_2^{(n)}(e^{-z})e^{-nz}+\beta_1\psi_2^{(n-1)}(e^{-z})e^{-(n-1)z}+\cdots+\beta_{n-1}\psi_2'(e^{-z})e^{-z}\},$$

where β_j $(j=1, 2, \cdots, n-1)$ are constants.

By substituting these results into (8) and observing the fact $w=\psi_1(e^z)+\psi_2(e^{-z})$, we have

$$P_0(e^z, e^{-z})\{[\psi_1^{(n)}(e^z)e^{nz}+\beta_1\psi_1^{(n-1)}(e^z)e^{(n-1)z}+\cdots+\beta_{n-1}\psi_1'(e^z)e^z]$$
$$+(-1)^n[\psi_2^{(n)}(e^{-z})e^{-nz}+\beta_1\psi_2^{(n-1)}(e^{-z})e^{-(n-1)z}+\cdots$$
$$+\beta_{n-1}\psi_2'(e^{-z})e^{-z}]\}+\cdots+P_n(e^z, e^{-z})\{\psi_1(e^z)+\psi_2(e^{-z})\}$$
$$=P_{n+1}(e^z, e^{-z}).$$

We obtain, by rewriting this,

$$P_0(e^z, e^{-z})e^{nz}\psi_1^{(n)}(e^z)+(P_1(e^z, e^{-z})+\beta_1P_0(e^z, e^{-z}))e^{(n-1)z}\psi_1^{(n-1)}(e^z)+$$
$$\cdots +P_n(e^z, e^{-z})\psi_1(e^z)$$
$$=(-1)^{n+1}P_0(e^z, e^{-z})e^{-nz}\psi_2^{(n)}(e^{-z})+(-1)^n(P_1(e^z, e^{-z})-\beta_1P_0(e^z, e^{-z}))\cdot$$
$$\cdot e^{-(n-1)z}\psi_2^{(n-1)}(e^{-z})+ \cdots +(-1)P_n(e^z, e^{-z})\psi_2(e^{-z})+P_{n+1}(e^z, e^{-z}).$$

By changing e^z to ζ, we have the following identity:

$$(9)\quad P_0\Big(\zeta, \frac{1}{\zeta}\Big)\zeta^n\psi_1^{(n)}(\zeta)+\Big(P_1\Big(\zeta, \frac{1}{\zeta}\Big)+\beta_1P_0\Big(\zeta, \frac{1}{\zeta}\Big)\Big)\zeta^{n-1}\psi_1^{(n-1)}(\zeta)+ \cdots +P_n\Big(\zeta, \frac{1}{\zeta}\Big)\psi_1(\zeta)$$
$$=(-1)^{n+1}P_0\Big(\zeta, \frac{1}{\zeta}\Big)\zeta^{-n}\psi_2^{(n)}\Big(\frac{1}{\zeta}\Big)+ \cdots +(-1)P_n\Big(\zeta, \frac{1}{\zeta}\Big)\psi_2\Big(\frac{1}{\zeta}\Big)+P_{n+1}\Big(\zeta, \frac{1}{\zeta}\Big).$$

Now we shall treat two cases separately.

Case 1: One and only one of ψ_1 and ψ_2 is transcendental.

Case 2: Both ψ_1 and ψ_2 are transcendental.

In case 1, it can be assumed without loss of generality that ψ_1 is transcendental and ψ_2 is a polynomial. Then it follows from (9) immediately that $w=\psi_1(\zeta)$ satisfies an n-th order linear differential equation with rational functions as coefficients. Hence by Lemma B, the order of ψ_1, $\rho(\psi_1)$, is positive. Then it is easy to show that $\psi_1(e^z)$ is not subnormal and so is $f(z)=\psi_1(e^z)+\psi_2(e^{-z})$, a contradiction. Here let us note that the subnormality (2) of an entire function $w(z)$ can also be equivalently defined by using the maximum modulus of w, $M(r, w)$ as follows

$$\overline{\lim_{r\to\infty}}\frac{\log\log M(r, w)}{r}=0.$$

We suppose now case 2 holds. In this case we may assume further that the left-hand side of (9) will not be reduced to a rational function (cf. the argument used in case 1), therefore it can be expressed as $\zeta^{-l}\Phi(\zeta)$, where l is a non-negative integer and Φ is a transcendental entire function. By interchanging ζ with $1/\zeta$ in (9), we have

$$(-1)^{n+1}P_0\Big(\frac{1}{\zeta}, \zeta\Big)\zeta^n\psi_2^{(n)}(\zeta)+ \cdots +(-1)P_n\Big(\frac{1}{\zeta}, \zeta\Big)\psi_2(\zeta)+P_{n+1}\Big(\frac{1}{\zeta}, \zeta\Big)=\zeta^l\Phi\Big(\frac{1}{\zeta}\Big).$$

Now by comparing the nature of the singularity at the origin $\zeta=0$, an inconsistency will be derived from the two sides of the equation. Thus neither of the cases 1 or 2 can occur. It follows that both ψ_1 and ψ_2 are polynomials as to be proved.

3. Main Results.

Our results begin with the following one.

THEOREM 1. *If* $f_j \in D$, $g_j \in D$, *and* Q_j *be a polynomial* ($j=1, 2, \cdots, n$), *then*

$$F(z) \equiv \sum_{j=1}^{n} Q_j(z) f_j(z) g_j(z) \in D.$$

Hence F *is pseudo-prime.*

In particular, if $f \in D$ *and* P *is a nonconstant polynomial, then* $G(z) \equiv P(f(z)) \in D$ *and is pseudo-prime.*

Proof. The assertion can be verified by Lemma D and Lemmas 1 and 2, as well as the fact that if $f \in D$ and $Q(z)$ is a polynomial, then $Q(z)f(z) \in D$.

Now we prove the following result as a generalization of Theorem A.

THEOREM 2. *Consider the following second order homogeneous differential equation:*

$$w''(z) + P_1(e^z, e^{-z})w'(z) + P_2(e^z, e^{-z})w(z) = 0, \tag{10}$$

where $P_j(x, y)$ *is a polynomial of the two variables* x *and* y ($j=1, 2$) *such that either* $P_1(e^z, e^{-z})$ *or* $P_2(e^z, e^{-z})$ *is non-constant. If* $w=h(z)$ *is an entire and subnormal solution of* (10), *then* h *can be expressed as*

$$h(z) = e^{\alpha z}\phi(e^z), \tag{11}$$

where $\phi(z) = \phi_1(z) + \phi_2(1/z)$ *and that* ϕ_j ($j=1, 2$) *are polynomials. Moreover,* $h(z)$ *is pseudo-prime.*

Proof. Assume that h is entire ($\not\equiv$ constant) and $w=h(z)$ is a subnormal solution of (10). Then so is $h(z+2\pi i)$. By Lemma 3, $h(z)$ and $h(z+2\pi i)$ are linearly dependent and hence $h(z+2\pi i) = e^{2\pi i\alpha}h(z)$ for some α. Therefore, $h(z)$ can be expressed as

$$h(z) = e^{\alpha z}\phi(e^z).$$

where $\phi(\zeta)$ is holomorphic in $0<|\zeta|<\infty$. Then by Lemma 4 it follows that the conclusion of Theorem 2 holds. Therefore,

$$\phi(\zeta) = \phi_1(\zeta) + \phi_2(1/\zeta),$$

where ϕ_j ($j=1, 2$) are polynomials. The pseudo-primeness of $h(z)$ follows from Theorem 1. This also completes the proof of Theorem 2.

As an extension of Theorem A in a different consideration, we have obtained the following result.

THEOREM 3. *Consider the following differential equation*

(12) $$w''(z)+(P_1(e^z)+Q_1(e^{-z}))w'(z)+(P_2(e^z)+Q_2(e^{-z}))w(z)=0\,,$$

where P_j and Q_j are polynomials. Assume that

(13) (i) $\deg P_1>\deg P_2$

and

(14) (ii) $\deg Q_1>\deg Q_2$ (or $\deg Q_1<\deg Q_2$).

Then any nonconstant entire solution of (12) *is not subnormal.*

Proof. First of all, we assure that if $w(z)$ ($\not\equiv$ const.) is a subnormal and entire solution of (12) under the condition (13), then,
(i) $w'(z)$ is bounded in sector $D: -\pi/2+\delta\leqq\arg z\leqq\pi/2-\delta$, $(0<\delta<\pi/2)$:
(ii) $w'(z)\to 0$ and $w''(z)\to 0$ as $z\to\infty$ in D;
(iii) $w(z)\to B$, a nonzero constant as $z\to\infty$ in D.
The verification of the above properties of $w(z)$ can be obtained by consulting Ozawa's argument [4, Theorem 2] and observing that if $f(z)$ is entire and subnormal, then, for any positive number ε,

$$\left|\frac{f'(z)}{f(z)}\right|,\qquad \left|\frac{f''(z)}{f(z)}\right|\leqq e^{\varepsilon|z|}$$

hold for $|z|$ outside of intervals of finite total length (by an obviously slight modification of the proof of Theorem 4.5.1 in Hille's book [3]).

Now assume that $w=f(z)$ ($\not\equiv$ const.) is an entire and subnormal solution of (12), so is $f(z+2\pi i)$. Then, by Lemma 3, we have

$$f(z+2\pi i)=e^{2\pi i\alpha}f(z)$$

for some constant α. It follows, by the remark made at the beginning of the proof that both $f(z+2\pi i)$ and $f(z)$ tend to B ($\neq 0$) as $z\to\infty$ in D. Hence, from (15), we conclude that

$$e^{2\pi i\alpha}=1\,.$$

This implies that $f(z)$ is periodic with period $2\pi i$. Therefore, $f(z)$ can be expressed as

(16) $$f(z)=\phi_1(e^z)+\phi_2(e^{-z})\,,$$

where ϕ_j ($j=1, 2$) are entire functions. However, by Lemma 4, both ϕ_j ($j=1, 2$) are polynomials. Assume that ϕ_1 (or ϕ_2) is nonconstant. Then, by substituting $w=\phi_1(e^z)+\phi_2(e^{-z})$ into (12) and then comparing the growth in the right (or left(half-plane of two sides of the following equation

$$(P_1(e^z)+Q_1(e^{-z}))w'=-w''-(P_2(e^z)+Q_2(e^{-z}))w\,,$$

we easily arrive at a contradiction under the condition (13) (or (14)). Hence both ψ_1 and ψ_2 are constant. This yields f is a constant, contrary to the assumption. Theorem 3 is thus proved.

Finally we show the following result.

THEOREM 4. *Consider the following algebraic differential equation with rational functions as coefficients*

$$P(z, w, w', w'')=0. \tag{16}$$

Let $w=h(z)$ *be an entire and periodic function which satisfies* (16). *Suppose that*

$$\varlimsup_{r\to\infty}\frac{\log\log T(r, h)}{\log r}=0 \tag{17}$$

(*i.e.* h *is of hyperorder zero*), *or* $h(z)$ *is subnormal and of finite lower order, then* h *is pseudo-prime*

Proof. Assume, on the contrary, that h is not pseudo-prime. Then

$$h(z)=f(g(z)),$$

where either (i) both f and g are entire transcendental, or (ii) f is meromorphic (non-entire) transcendental and g is entire transcendental. In case (i), we only need to consider the following two situations separately: subcase 1, $\mu(f)=0$; and subcase 2, $\mu(f)>0$, $\mu(f)$ the lower order of f.

In subcase 1, g must be periodic (see e.g. Gross [1], p. 106), and hence

$$M(r, g)\geqq e^{c_0 r} \qquad (c_0>0,\ r\geqq r_0).$$

Note that the order of f is positive by virtue of Lemmas A and C. Hence h will not be subnormal as can be seen by applying Polya's lemma. Also h is of positive hyperorder.

In subcase 2, since g is transcendental, h is of infinite lower order. Also, again by Lemmas A and C, the order of $g=\rho(g)>0$, it follows that h is of positive hyperorder. Thus case (i) is impossible to hold.

As to case (ii), it is relatively easy to deal with. In fact, since h is entire, then

$$f(z)=\frac{\varphi(z)}{(z-b)^m} \quad \text{and} \quad g(z)=b+e^{\psi(z)},$$

where b is a complex number, m a natural number, and φ and ψ are entire with φ being transcendental and $\psi\not\equiv$constant. Then clearly φ satisfies a second order algebraic differential equation with rational functions as the coefficients since f is such a function. It follows that $\rho(\varphi)>0$ by Lemma C. Also $M(r, g)\geqq e^{c_0 r}$ ($r\geqq r_0$, $c_0>0$). Thus it is easily verified that $\varphi(g)$ and hence $h=f(g)$ is not subnormal. This also completes the proof of Theorem 4.

4. Concluding Remarks.

1. A natural extension about Theorem 4 of the present paper will be the question: whether any finite order entire solution of an algebraic differential equation of arbitrary order n with rational functions as the coefficients must be pseudo-prime? Now, evidently, the answer to the equestion is affirmative for $n=1, 2$, by noting Lemmas A and C.

2. We have also realized thus far that the nonlinear differential equations which yield non-pseudo-prime entire solutions are homogeneous differential equations. To avoid the trivial case, we ask: Does there exist a finite order entire but non-pseudo-prime function $f(z)$ that satisfies an algebraic differential equation of the form $\sum_{k=1}^{n} P_k(z)D_k(z, w)=0$, where $n\geqq 2$, the $P_j(z)$ are polynomials $\not\equiv 0$, and $D_k(z, w)$ denotes a homogeneous differential polynomials in w (with rational functions as the coefficients) of degree t_k such that $t_i\neq t_j$ for $i\neq j$, and no two $D_k(z, w)$ can identically vanish simultaneously for the same entire function $w(z)$ $(\not\equiv 0)$?

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