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REAL ZEROS OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction

We consider the second order linear differential equation

$$f''+Af=0, \tag{1.1}$$

where A is an entire function. For an entire function f, let $\rho(f)$ be its order, $\mu(f)$ its lower order, $\lambda(f)$ the exponent of convergence of its zeros and $\lambda_{NR}(f)$ the exponent of convergence of its non-real zeros. In addition, we assume that the reader is familiar with the standard notation of Nevanlinna theory (see [4]).

When A is a polynomial, the distribution of zeros of solutions of (1.1) has been studied extensively. The following theorem is well-known ([1]).

THEOREM A. *If A is a polynomial of degree $n\geqq 1$, then every solution $f\not\equiv 0$ of* (1.1) *satisfies*

$$\rho(f)=(n+2)/2, \tag{1.2}$$

and if f_1, f_2 are two linearly independent solutions of (1.1), *then*

$$\lambda(f_1f_2)=(n+2)/2. \tag{1.3}$$

Furthermore, G. Gundersen proved the following ([2]).

THEOREM B. *Under the hypothesis of Theorem A,*

$$\lambda_{NR}(f_1f_2)=(n+2)/2. \tag{1.4}$$

When A is transcendental, we apply the lemma on the logarithmic derivative in Nevanlinna theory to (1.1) and can easily deduce that any solution $f\not\equiv 0$ of (1.1) satisfies

$$\rho(f)=+\infty. \tag{1.5}$$

By analogy with Theorem A and Theorem B, we may hope that

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$$\lambda(f_1 f_2)=+\infty \tag{1.6}$$

or

$$\lambda_{NR}(f_1 f_2)=+\infty\,, \tag{1.7}$$

where f_1 and f_2 are linearly independent solutions of (1.1). However, examples in [1] show that (1.6) and (1.7) may not hold if $\rho(A)$ is infinite or equal to a positive integer. When the growth of A is suitably restricted, (1.6) and (1.7) hold.

Before stating the following results of J. Rossi, we make some definitions.

Let $n_+(r, 1/f)$ $(n_-(r, 1/f))$ be the number of zeros of f in $\{z: |z-(1/2)ir| <(1/2)r\}$ $(\{z: |z+(1/2)ir|<(1/2)r\})$, where $|z|>1$ and $r>0$. Define

$$\lambda_1(f)=\overline{\lim_{r\to\infty}}\frac{\log(n_+(r, 1/f)+n_-(r, 1/f))}{\log r}.$$

Obviously $\lambda_1(f)\leqq\lambda_{NR}(f)$. The lower exponent of convergence $\lambda_*(f)$ of the zeros of an entire function f is defined by

$$\lambda_*(f)=\underline{\lim_{r\to\infty}}\frac{\log n(r, 1/f)}{\log r},$$

where $n(r, 1/f)$ is the number of zeros of f in $|z|<r$.

J. Rossi proved the following ([8]).

THEOREM C. *If* $\rho(A)\leqq 1/2$ *and* f_1, f_2 *are linearly independent solutions of* (1.1), *then*

$$\lambda_1(f_1 f_2)=+\infty\,,$$

and

$$\lambda_*(f_1 f_2)=+\infty\,.$$

In this paper, we prove

THEOREM 1. *Let* A *be a transcendental entire function of order* $\rho<+\infty$ *with* k *distinct finite asymptotic values. Suppose that* $k=2\rho$. *If* f_1 *and* f_2 *are linearly independent solutions of* (1.1), *then*

$$\lambda_1(f_1 f_2)=+\infty\,.$$

THEOREM 2. *Under the hypothesis of Theorem* 1,

$$\lambda_*(f_1 f_2)=+\infty\,.$$

2. The Tsuji Characteristic

In [9] (c.f. [6] and [7]) M. Tsuji introduced a characteristic for a function f meromorphic in the upper half-plane based on the following Jensen-type formula:

$$\int_1^r \frac{n_+(t, 0)}{t^2}dt - \int_1^r \frac{n_+(t, \infty)}{t^2}dt$$

$$=(2\pi)^{-1}\int_{\sin^{-1}(r^{-1})}^{\pi-\sin^{-1}(r^{-1})} \log|f(r(\sin\theta)e^{i\theta})|\frac{d\theta}{r\sin^2\theta}+O(1). \tag{2.1}$$

Here $n_+(t, 0)$ ($n_+(t, \infty)$) denotes the number of zeros (poles) of f in $\{z : |z-(1/2)it| \leqq (1/2)t,\ |z| \geqq 1\}$.

He defined

$$m_+(r, \infty)=m_+(r, f)=(2\pi)^{-1}\int_{\sin^{-1}(r^{-1})}^{\pi-\sin^{-1}(r^{-1})} \log^+|f(r(\sin\theta)e^{i\theta})|\frac{d\theta}{r\sin^2\theta}, \tag{2.2}$$

$$m_+(r, a)=m_+(r, 1/(f-a)), \qquad a\in \boldsymbol{C}, \tag{2.3}$$

$$N_+(r, \infty)=N_+(r, f)=\int_1^r \frac{n_+(t, \infty)}{t^2}dt=\sum_{1\leq r_k\leq r\sin\phi_k}\left[\frac{\sin\phi_k}{r_k}-\frac{1}{r}\right], \tag{2.4}$$

where $r_k e^{i\phi_k}$ are the poles of f in $Im\, z>0$,

$$N_+(r, a)=N_+(r, 1/(f-a)), \qquad a\in \boldsymbol{C}, \tag{2.5}$$

and

$$T_+(r, f)=m_+(r, f)+N_+(r, f). \tag{2.6}$$

For f meromorphic in $Im\, z>0$, Tsuji proved the following properties.

(A) $m_+(r, a)+N_+(r, a)=T_+(r, f)+O(1), \quad a\in \boldsymbol{C}$.

(B) If f is also meromorphic in a neighborhood of the origin,

$$m_+(r, f'/f)=O(\log T_+(r, f)+\log r), \qquad \text{n.e.}$$

(C) $(q-2)T_+(r, f)\leqq \sum_{k=1}^{q} N_+(r, a_k)+O(\log T_+(r, f)+\log r)$ n.e. $(a_k\in \boldsymbol{C}\cup\{\infty\})$.

(n.e. means except on a set of finite linear measure.)

(D) $T_+(r, f)$ is a monotone increasing function of r. In [7, p. 332] it is also proved that

(E) $\displaystyle\int_R^\infty \frac{m_{0,\pi}(r, f)}{r^3}dr \leqq \int_R^\infty \frac{m_+(r, f)}{r^2}dr \qquad (R\geqq 1)$,

where

$$m_{0,\pi}(r, f)=(2\pi)^{-1}\int_0^\pi \log^+|f(re^{i\theta})|d\theta.$$

Remark. Properties (A), (B) and (C) are analogues of Nevanlinna's first fundamental theorem, the lemma on the logarithmic derivative and Nevanlinna's second fundamental theorem, respectively.

Similarly, we can introduce the notations T_-, m_- and N_- for the lower half-plane analogues of the Tsuji functionals.

3. Preliminary Lemmas

We need some lemmas.

LEMMA 1. *Let A be an entire function of order $\rho<+\infty$ with k distinct finite asymptotic values a_i $(1\leqq i\leqq k)$ and L_i $(1\leqq i\leqq k)$ the asymptotic paths corresponding to a_i, which are simple curves from the origin to ∞ and non-intersecting except at the origin and divide the plane $\mathbf{C}$ into k disjoint simply connected domains D_i $(1\leqq i\leqq k)$. We may assume that D_i is bounded by L_i and L_{i+1} $(1\leqq i\leqq k$; $L_{k+1}=L_1)$. Suppose that $k=2\rho$, then*

(1) *there exists in D_i a path Γ_i going to ∞ such that*

$$\lim_{\substack{|z|\to\infty \\ z\in\Gamma_i}}\frac{\log\log|A(z)|}{\log|z|}=\rho, \tag{3.1}$$

(2) *$A(z)$ has no finite deficient values.*

This lemma can be found in [10, p. 324 and p. 353].

LEMMA 2 [8]. *Let f be entire with infinite lower order such that*

$$\left.\begin{aligned} m_+(r, f)=O(r^\alpha) \\ m_-(r, f)=O(r^\alpha) \end{aligned}\right\} \quad (r\longrightarrow\infty), \tag{3.2}$$

where $0<\alpha<\infty$. Then, given λ, $0<\lambda<\infty$,

$$m(r, f)=(1+o(1))(2\pi)^{-1}\int_{E(r)}\log^+|f(re^{i\theta})|\,d\theta, \quad \text{n.e.}, \tag{3.3}$$

where the angular measure is

$$\operatorname{meas}(E(r))=O(r^{-\lambda}). \tag{3.4}$$

Applying Wiman-Valiron theory (c.f. [5]), we can deduce that

LEMMA 3. *If $\rho(A)<\rho_1<\infty$ and f is a solution of (1.1), then*

$$\log\log M(r, f)\leqq r^{\rho_1}, \qquad (r\geqq r_0). \tag{3.5}$$

LEMMA 4. *Let $\varepsilon>0$ be arbitrary and E be entire. If there exist μ_1 $(0<\mu_1<\infty)$ and a sequence $R_n\to\infty$ such that*

$$\lim_{n\to\infty}\frac{\log M(R_n, E)}{R_n^{\mu_1}}=0, \tag{3.6}$$

then

$$\text{(1)} \qquad \varlimsup_{n\to\infty}(\log R_n)^{-1}\int_{G(\varepsilon)\cap[1, R_n]}\frac{dr}{r}\leqq\varepsilon, \tag{3.7}$$

where $G(\varepsilon)=\{r: \log M(2r, E)\geqq r^{\mu_1/\varepsilon}\}$.

(2) *there exists a positive integer* $q=q(\varepsilon)$ *such that*

$$|(E'/E)^2(re^{i\theta})-2(E''/E)(re^{i\theta})|\leqq r^q \quad \text{for } r\geqq r_0>1\,, \tag{3.8}$$

$r\overline{\in}G(\varepsilon^2)$ *and* $\theta\overline{\in}J_r$, *where* meas$(J_r)\leqq\varepsilon\pi$.

We remark that Lemma 4 is due to J. Rossi [8, Lemma 5 and 6]. But in his paper, he miswrote

$$m(r, (E'/E)^2-2E''/E)=O(\log T(r, E)+\log r) \quad \text{for all } r\in \boldsymbol{R},$$

it should be written as

$$m(r, (E'/E)^2-2E''/E)=O(\log T(2r, E)+\log r) \quad \text{for all } r\in \boldsymbol{R}.$$

4. Proof of Theorem 2

Let f_1, f_2 be linearly independent solutions of (1.1). Set $E=f_1f_2$, and we note as in [1] that

$$-4A=(c/E)^2-(E'/E)^2+2(E''/E)\,, \tag{4.1}$$

where c is the constant Wronskian of f_1 and f_2. Applying Nevanlinna theory to (4.1), we have

$$T(r, E)=N(r, 1/E)+\frac{1}{2}T(r, A)+O(\log T(r, E)+\log r), \quad \text{n.e.} \tag{4.2}$$

Suppose that $\mu(E)<+\infty$, then there exist μ_1 such that $\mu(E)<\mu_1<+\infty$, and $R_n\to\infty$ such that

$$\lim_{n\to\infty}\frac{\log M(R_n, E)}{R_n^{\mu_1}}=0\,. \tag{4.3}$$

Fix $\varepsilon>0$. Since A has no finite deficient values, it must have infinitely many zeros. Let $b_1, b_2, \cdots, b_{q+1}$ be $q+1$ zeros of A with $q=q(\varepsilon)$ as in Lemma 4. Define

$$H(z)=A(z)\Big/\prod_{i=1}^{q+1}(z-b_i),$$

then H is entire and of order $\rho(H)=\rho(A)=k/2$.

Set

$$D(H)=\{z: |H(z)|>1\}\,,$$

$$D(E)=\{z: |E(z)|>1\},$$

$$D(\varepsilon^2)=\{z=re^{i\theta}: 0\leqq\theta\leqq 2\pi,\ r\overline{\in}G(\varepsilon^2)\},$$

$$D=\{z=re^{i\theta}: \theta\in J_r,\ r\overline{\in}G(\varepsilon^2)\},$$

with J_r, $G(\varepsilon^2)$ as in Lemma 4.

From (3.8) and (4.1), we deduce that

$$4|A(z)| \leqq |c|^2 + |z|^q, \qquad z \in D(E) \cap D(\varepsilon^2) \setminus D, \quad (|z| \geqq r_0). \tag{4.4}$$

But for $z \in D(H) \cap D(\varepsilon^2) \setminus D$,

$$|A(z)| > \left(\frac{1}{2}|z|\right)^{q+1}, \qquad (2|z| \geqq \max_{1 \leqq i \leqq q+1} |b_i|), \tag{4.5}$$

From (4.4) and (4.5), we have for r large enough $(r \geqq r_* \geqq r_0)$

$$\{\theta : re^{i\theta} \in D(H) \cap D(E) \cap D(\varepsilon^2)\} \subseteqq J_r . \tag{4.6}$$

Set

$$L = \bigcup_{i=1}^{k} L_i$$

with L_i $(1 \leqq i \leqq k)$ as in Lemma 1. It is easy to see that

$$D(H) \cap \{z : |z| > r\} \cap L = \phi ,$$

if r is large enough. Without loss of generality, we may assume that $r=0$.

By Lemma 1, there exists point $z_i \in D_i$ $(1 \leqq i \leqq k)$ such that

$$|H(z_i)| > e .$$

Let Ω_i $(1 \leqq i \leqq k)$ be the connected component of $D(H)$ containing the point z_i, then $\Omega_i \subset D_i$ $(1 \leqq i \leqq k)$. By the maximum modulus principle, we conclude that Ω_i $(1 \leqq i \leqq k)$ is unbounded.

Let

$$r_1 = \max\{r_*, |z_1|, |z_2|, \cdots, |z_k|\}$$

and θ_{it} $(1 \leqq i \leqq k ; r_1 \leqq t < \infty)$ be the part of the circle $|z|=t$ in $\bar{\Omega}_i$ and $t\theta_i(t)$ the the linear measure of θ_{it}. We have

LEMMA 5.

$$\overline{\lim_{n\to\infty}} (\log R_n)^{-1} \pi \int_{2r_1}^{(1/2)R_n} \left(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\right) \frac{dt}{t} = \frac{k^2}{2} . \tag{4.7}$$

Proof. By a theorem of Tsuji [9], we have

$$\log |H(z_i)| \leqq 9\sqrt{2} \exp\left(-\pi \int_{2r_1}^{(1/2)R_n} \frac{dt}{t\theta_i(t)}\right) \log M(R_n, H), \tag{4.8}$$

for $R_n > 4r_1$ and $1 \leqq i \leqq k$.

(4.8) gives

$$\pi \int_{2r_1}^{(1/2)R_n} \left(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\right) \frac{dt}{t} \leqq k \log\log M(R_n, H) + k \log (9\sqrt{2}) . \tag{4.9}$$

Noting that

$$k^2 \leqq \Big(\sum_{i=1}^{k} \theta_i(t)\Big)\Big(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\Big) \leqq 2\pi\Big(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\Big), \tag{4.10}$$

from (4.9), we have

$$k^2/2\int_{2r_1}^{(1/2)R_n} \frac{dt}{t} \leqq \pi\int_{2r_1}^{(1/2)R_n} \Big(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\Big)\frac{dt}{t}$$

$$\leqq k \operatorname{loglog} M(R_n, H) + k \log(9\sqrt{2}). \tag{4.11}$$

The desired conclusion follows from (4.11) and $\rho(H)=k/2$.

Let

$$\Delta(\varepsilon) = \Big\{r :\ \sum_{i=1}^{k} \theta_i(r) < (2-\varepsilon)\pi\Big\}$$

and

$$\beta = \varlimsup_{n\to\infty} (\log R_n)^{-1} \int_{\Delta(\varepsilon)\cap[2r_1, (1/2)R_n]} \frac{dt}{t},$$

then we have

LEMMA 6.

$$\beta = 0. \tag{4.12}$$

Proof. First we note that

$$\pi\int_{2r_1}^{(1/2)R_n} \Big(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\Big)\frac{dt}{t}$$

$$= \pi\int_{\Delta(\varepsilon)\cap[2r_1, 1/2)R_n]} \Big(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\Big)\frac{dt}{t} + \pi\int_{[2r_1, (1/2)R_n]-\Delta(\varepsilon)} \Big(\sum_{i=1}^{k} \frac{1}{\theta_i(t)}\Big)\frac{dt}{t}$$

$$\geqq \pi\int_{\Delta(\varepsilon)\cap[2r_1, (1/2)R_n]} \frac{k^2}{(2-\varepsilon)\pi}\frac{dt}{t} + \pi\int_{[2r_1, (1/2)R_n]-\Delta(\varepsilon)} \frac{k^2}{2\pi}\frac{dt}{t}$$

$$= \Big(\frac{k^2}{2-\varepsilon} - \frac{k^2}{2}\Big)\int_{\Delta(\varepsilon)\cap[2r_1, (1/2)R_n]} \frac{dt}{t} + \frac{k^2}{2}\int_{2r_1}^{(1/2)R_n} \frac{dt}{t}. \tag{4.13}$$

From (4.7) and (4.13), we have

$$\frac{k^2}{2} \geqq \frac{k^2}{2} + \Big(\frac{k^2}{2-\varepsilon} - \frac{k^2}{2}\Big)\beta. \tag{4.14}$$

We note that the right-hand side of (4.14) is strictly greater than $k^2/2$ unless $\beta=0$. Hence $\beta=0$.

Let $\Omega(E)$ be a connected component of $D(E)$ and θ_t be the part of the circle $|z|=t$ in $\Omega(E)$ and $t\theta(t)$ the linear measure of θ_t, then again by the theorem of Tsuji [9], we have

$$\log\log M(R_n, E) \geqq \pi \int_{2r_1}^{(1/2)R_n} \frac{dt}{t\theta(t)}$$

$$\geqq \pi \int_{[2r_1, (1/2)R_n]-G(\varepsilon^2)-\Delta(\varepsilon)} \frac{dt}{t\theta(t)}$$

$$\geqq \pi \int_{[2r_1, (1/2)R_n]-G(\varepsilon^2)-\Delta(\varepsilon)} \frac{dt}{2\varepsilon\pi t}$$

$$\geqq \frac{1}{2\varepsilon}\left(\int_{2r_1}^{(1/2)R_n} \frac{dt}{t} - \int_{[2r_1, R_n]\cap G(\varepsilon^2)} \frac{dt}{t} - \int_{[2r_1, (1/2)R_n]\cap\Delta(\varepsilon)} \frac{dt}{t}\right). \tag{4.15}$$

From (3.7), (4.12) and (4.15), we have

$$\varliminf_{n\to\infty} \frac{\log\log M(R_n, E)}{\log R_n} \geqq \frac{1}{2\varepsilon}(1-\varepsilon^2). \tag{4.16}$$

Since ε is arbitrary, we can make the right-hand side of (4.16) larger than μ_1, by choosing a small ε at the beginning. This contradicts (4.3). Hence $\mu(E)=+\infty$.

For any $\alpha>1$, we have by (4.2)

$$1/2T(r, E) \leqq N(\alpha r, 1/E) + 1/2T(\alpha r, A), \quad (r \text{ large enough}). \tag{4.17}$$

We note that $\rho(A)<+\infty$, then (4.17) and $\mu(E)=+\infty$ give

$$\varliminf_{r\to\infty} \frac{\log N(r, 1/E)}{\log r} = +\infty,$$

which implies $\lambda_*(f_1 f_2)=+\infty$. Theorem 2 is proved.

5. Proof of Theorem 1

Properties (A) and (B) together with (4.1) give

$$T_+(r, E) = N_+\left(r, \frac{1}{E}\right) + \frac{1}{2}T_+(r, A) + O(\log T_+(r, E) + \log r), \quad \text{n.e.} \tag{5.1}$$

We assume that

$$\lambda_1(E) < +\infty \tag{5.2}$$

and will arrive at a contradiction from this assumption.

From (5.1) and (5.2), we have

$$T_+(r, E) = O(r^\alpha), \quad (0<\alpha<\infty), \tag{5.3}$$

similary

$$T_-(r, E) = O(r^\alpha), \quad (0<\alpha<\infty). \tag{5.4}$$

Since $\rho(A)<+\infty$, we can choose λ such that $\rho(A)<(1/2)\lambda<+\infty$. Theorem 2 gives that $\mu(E)=+\infty$. Applying Lemma 2 to E, we have

$$m(r, E)=O(r^{-\lambda} \log M(r, E)) \quad \text{n.e.} \tag{5.5}$$

By a theorem of Hayman and Stewart [3, Theorem 6], for any constant $K>1$, we have

$$\log M(r, E) \leqq m(r, E)[\log m(r, E)]^{K}, \qquad r \in G, \tag{5.6}$$

where

$$\varliminf_{r\to\infty} (\log R)^{-1} \int_{G\cap[1,R]} \frac{dt}{t} > 0. \tag{5.7}$$

From (5.5), (5.6), (5.7) and Lemma 3 with $\rho_1=(1/2)\lambda$, there exist a sequence $r_n \to \infty$ and a constant c such that

$$1 \leqq c r_n^{-\lambda}[\log\log M(r_n, E) - \lambda \log r_n + \log c]^{K} \leqq c r_n^{-\lambda+(1/2)K\lambda}. \tag{5.8}$$

If we choose $K<2$, (5.8) gives a contradiction. The proof of Theorem 1 is complete.

Remark. Indeed, we have proved the following slightly stronger results.

THEOREM 3. *Let A be the same as in Theorem* 1 *and P a non-constant polynomial. If f_1 and f_2 are linearly independent solutions of the differential equation*

$$f'' + APf = 0,$$

then

$$\lambda_1(f_1 f_2) = +\infty.$$

THEOREM 4. *Under the hypothesis of Theorem* 3,

$$\lambda_*(f_1 f_2) = +\infty.$$

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