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KODAI MATH. J.
14 (1991), 1—12

A NOTE ON CARATHÉODORY AND KOBAYASHI PSEUDODISTANCES

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Introduction.

Recently, Jarnicki and Pflug [7], [8] presented an effective formula for the Carathéodory pseudodistance from the origin on logarithmically coned, complete Reinhardt domains in $\boldsymbol{C}^n$. The aim of this note is to establish the wasteless formula for the pseudodistance from the origin on such domains (Theorem 2.2). We also apply this formula to the case of dimension two and represent the pseudodistance by means of the continued fraction expansion of real numbers (Theorem 4.1).

1. Preliminary.

Let D be a domain in $\boldsymbol{C}^n$. For $p, q \in D$, let

$$c_D^*(p, q)=\sup\{|f(q)|\,;\, f\in \mathrm{Hol}(D, U),\ f(p)=0\},$$

$$k_D^*(p, q)=\inf\{t\,;\, 0\leqq t<1, \text{ there exists an } f\in \mathrm{Hol}(U, D) \text{ such that } f(0)=p \text{ and } f(t)=q\},$$

and

$$g_D(p, q)=\sup\{f(q)\,;\, f \text{ is a negative plurisubharmonic function on } D \text{ such that } \limsup_{z\to p}(f(z)-\log|z-p|)<+\infty\},$$

where U is the unit disc in $\boldsymbol{C}$ and, for complex manifolds X and Y, $\mathrm{Hol}(X, Y)$ denotes the set of all holomorphic mappings from X into Y. The function $c_D=\tanh^{-1}c_D^*$ (resp. the largest pseudodistance k_D on D dominated by $k_D^{\prime}:=\tanh^{-1}k_D^*$) is called the Carathéodory (resp. Kobayashi) pseudodistance on D, and the function $g_D(p, \cdot)$ is called the pluri-complex Green function on D with pole at p (cf., e.g., [2], [3], [4], [5], [10], [11], [15]). These functions c_D^*, k_D^*, and g_D have the decreasing property for holomorphic mappings and satisfy

$$(1.1)\qquad c_D^*\leqq \exp g_D\leqq k_D^* \quad \text{on } D\times D$$

(see [10]).

Received February 13, 1990.

We shall show the following lemma, part (ii) of which is well-known ([7], [8]).

LEMMA 1.1. *Let D be a balanced domain in $\boldsymbol{C}^n$ with the Minkowski functional* $\mu_D(z)=\inf\{r>0; z\in rD\}$, *where* $rD=\{rz; z\in D\}$.

(i) $c_D^*(0, \cdot)\leqq \exp g_D(0, \cdot)\leqq k_D^*(0, \cdot)\leqq \mu_D$ *on* D.

(ii) *D is convex if and only if* $c_D^*(0, \cdot)=\mu_D$ *on* D.

(iii) *D is pseudoconvex if and only if* $\exp g_D(0, \cdot)=\mu_D$ *on* D.

Proof. (i) In view of (1.1) we have to prove only the last inequality. Let $z\in D=\{z\in C^n; \mu_D(z)<1\}$, and take r with $\mu_D(z)<r<1$. Then the function $f(\lambda)=\lambda z/r$, $\lambda\in U$ belongs to $\mathrm{Hol}(U, D)$ and satisfies $f(r)=z$, so that $k_D^*(0, z)\leqq r$. Thus, $k_D^*(0, z)\leqq \mu_D(z)$. (iii) Assume $\exp g_D(0, \cdot)=\mu_D$ on D. Then, $\log \mu_D=g_D(0, \cdot)$ is plurisubharmonic on D, and so on $\boldsymbol{C}^n$. Hence, the balanced domain D is pseudoconvex. Conversely assume that D is pseudoconvex. Then, $\log\mu_D$ is plurisubharmonic on D ([1]). Since μ_D is continuous at 0 it follows that $\limsup_{z\to 0}(\log\mu_D(z)-\log|z|)<+\infty$, so that $\log \mu_D\leqq g_D(0, \cdot)$. The proof is completed.

As an application of Lemma 1.1 we obtain Kubota's theorem on symmetric bounded domains.

COROLLARY 1.2 (Kubota [13], [12]). *Let D be a symmetric bounded domain in $\boldsymbol{C}^n$ realized as a convex balanced domain. Then,*

$$c_D^*(z, w)=\inf\{r; 0<r<1, \text{ there exists an } F\in \mathrm{Aut}(D) \text{ such that } F(w)=0 \text{ and } F(z)\in rD\} \tag{1.2}$$

for all $z, w\in D$, and

$$\{z\in D; c_D^*(0, z)<r\}=rD \tag{1.3}$$

for all $r>0$ with $r<1$, where $\mathrm{Aut}(D)$ *is the set of all holomorphic automorphisms of D.*

Proof. To prove the formula (1.2) we denote by $d_w(z)$ the right hand side of (1.2). Since D is homogeneous, $K_w:=\{F\in \mathrm{Aut}(D); F(w)=0\}$ is not empty. It follows that

$$\begin{aligned} d_w(z)&=\inf \textstyle\bigcup_{F\in K_w}\{r; 0<r<1, F(z)\in rD\} \\ &=\inf_{F\in K_w}\inf\{r; 0<r<1, F(z)\in rD\} \\ &=\inf_{F\in K_w}\mu_D(F(z)). \end{aligned}$$

Since D is convex, by Lemma 1.1 (ii) and the biholomorphic invariance of c_D^* we see that for every $F\in K_w$, $\mu_D(F(z))=c_D^*(0, F(z))=c_D^*(w, z)$; therefore, $\mu_D(F(z))$ does not depend on the choice of $F\in K_w$ and (1.2) is established. The relation

(1.3) is a direct consequence of the equality $\mu_D=c_D^*(0, \cdot)$. The proof is completed.

Under the hypothesis in Corollary 1.2 it is well-known ([11], [14]) that $c_D=k_D$. Using only the homogeneity and the convex balancedness of D, we can show this as follows: By Lemma 1.1 (i) and (ii) we have $c_D^*(0, \cdot)=k_D^*(0, \cdot)$. From the homogeneity of D and the biholomorphic invariance of c_D^* and k_D^* it follows that $c_D^*=k_D^*$; therefore, $c_D=k_D^1$ and k_D^1 is a pseudodistance, so that $c_D=k_D^1=k_D$.

2. Logarithmically coned Reinhardt domains.

Let D be a Reinhardt domain in $\boldsymbol{C}^n$ with the real representative domain $|D|=\{(|z_1|, \cdots, |z_n|)\in(\boldsymbol{R}_+)^n ; (z_1, \cdots, z_n)\in D\}$, where $\boldsymbol{R}_+=\{x\in R ; x\geqq 0\}$. Assume D is logarithmically coned, that is, the set $\log|D| :=\{x=(x_1, \cdots, x_n)\in\boldsymbol{R}^n ; e^x :=(e^{x_1}, \cdots, e^{x_n})\in|D|\}$ is a cone in $\boldsymbol{R}^n$ with vertex at the origin. Set $\boldsymbol{Z}_+=\boldsymbol{Z}\cap\boldsymbol{R}_+$ and, for a subset S of $\boldsymbol{R}^n$, set $S_*=S\setminus\{0\}$. For $z=(z_1, \cdots, z_n)\in\boldsymbol{C}^n$ and $\alpha=(\alpha_1, \cdots, \alpha_n)\in(\boldsymbol{Z}_+)^n$, set $z^\alpha=z_1{}^{\alpha_1}\cdots z_n{}^{\alpha_n}$. Let $S_D=\{\alpha\in(\boldsymbol{Z}_+)^n_* ; |z^\alpha|<1 \text{ for all } z\in D\}$, and let $S_D+S_D=\{\alpha+\beta ; \alpha, \beta\in S_D\}$. A typical example of logarithmically coned, complete Reinhardt domains is $\{z\in\boldsymbol{C}^n ; |z^\alpha|<1 \text{ for all } \alpha\in T\}$, where T is a finite subset of $(\boldsymbol{Z}_+)^n_*$.

Jarnicki and Pflug proved the following.

LEMMA 2.1 ([7; Theorem 2], [8; Theorem 2.1]). *Let D be a logarithmically coned, complete Reinhardt domain in $\boldsymbol{C}^n$ with S_D. It then holds that for $z\in D$,*

$$\begin{aligned} c_D^*(0, z)&=\sup\{|z^\alpha| ; \alpha\in S_D\} \\ &=\sup\{|z^\alpha| ; \alpha\in S_D\setminus(S_D+S_D)\}. \end{aligned}$$

The aim of this section is to prove a precision of Lemma 2.1.

Let C be a closed subset of $\boldsymbol{R}^n$. A point x in C is called a vertex of C if there exists a linear functional f on $\boldsymbol{R}^n$ and a number $c\in\boldsymbol{R}$ such that $f(x)=c$ and $f(y)<c$ for all $y\in C\setminus\{x\}$, that is, x is a vertex of the convex hull Conv C of C in the usual sense. By Vert C we denote the set of all vertices of C.

Our result is the following.

THEOREM 2.2. *Let D be a logarithmically coned, complete Reinhardt domain in $\boldsymbol{C}^n$ with S_D. Then,*

$$c_D^*(0, z)=\max\{|z^\alpha| ; \alpha\in \operatorname{Vert} S_D\}$$

for $z\in D$.

It follows from the definition that

$$S_D=\{\alpha\in(\boldsymbol{Z}_+)^n_* ; \langle\alpha, y\rangle<0 \text{ for all } y\in\log|D|\}. \tag{2.1}$$

To prove Theorem 2.2, by virtue of Lemma 2.1 we must show that if

(2.2) $$\Phi(x)=\sup\{\langle x, \alpha\rangle ; \alpha\in S_D\}$$

for $x\in\log|D|$, then

(2.3) $$\Phi(x)=\max\{\langle x, \alpha\rangle ; \alpha\in \mathrm{Vert}\, S_D\}.$$

We need a lemma.

LEMMA 2.3. *Let $x\in\log|D|$. Let $(x_j)_j$ be a sequence in $\boldsymbol{R}^n$ and $(\beta_j)_j$ be a sequence in S_D such that $x_j\to x$ as $j\to\infty$ and the sequence $(\langle x_j, \beta_j\rangle)_j$ is bounded. Then, (β_j) is bounded.*

Proof. Suppose (β_j) is not bounded. We may assume that $|\beta_j|\to+\infty$ and $\beta_j/|\beta_j|\to\xi$ as $j\to\infty$ for some $\xi\in\boldsymbol{R}^n$ with $|\xi|=1$. Since $(\langle x_j, \beta_j\rangle)$ is bounded, it follows that $\langle x, \xi\rangle=\lim_{j\to\infty}\langle x_j, \beta_j\rangle/|\beta_j|=0$. Take an $\varepsilon>0$ so that $x+\varepsilon\xi\in\log|D|$. We have

$$\lim_{j\to\infty}\langle x+\varepsilon\xi, \beta_j\rangle=\lim_{j\to\infty}\langle x+\varepsilon\xi, \beta_j/|\beta_j|\rangle\cdot\lim_{j\to\infty}|\beta_j|=+\infty ;$$

therefore $\langle x+\varepsilon\xi, \beta_j\rangle>0$ for some j. This contradicts the facts $x+\varepsilon\xi\in\log|D|$ and $\beta_j\in S_D$, and completes the proof.

Proof of Theorem 2.2. To prove (2.3) fix any $x\in\log|D|$. We first note that

(2.4) $$\Phi(x)=\max\{\langle x, \alpha\rangle ; \alpha\in S_D\}.$$

Indeed, let (α_j) be a sequence in S_D such that $\Phi(x)=\lim_{j\to\infty}\langle x, \alpha_j\rangle$. By Lemma 2.3, (α_j) is bounded, so that we may assume that (α_j) converges to a point $\alpha\in S_D$; therefore $\Phi(x)=\langle x, \alpha\rangle$. Then $H:=\{y\in\boldsymbol{R}^n ; \langle x, y\rangle=\Phi(x)\}$ is a supporting hyperplane of $\mathrm{Conv}\, S_D$. By Lemma 2.3 we see that $S_D\cap H$ is bounded to the effect that $S_D\cap H$ is a finite set. Let $\alpha\in S_D\cap H$ be a vertex of $S_D\cap H$ in H. We shall show that α is also a vertex of S_D. Take $\eta\in H_*$ and $c\in\boldsymbol{R}$ such that

(2.5) $$\langle y, \eta\rangle<c \text{ for all } y\in S_D\cap H\setminus\{\alpha\} \quad\text{and}\quad \langle\alpha, \eta\rangle=c .$$

Take an $\varepsilon_0>0$ such that $x+\varepsilon\eta\in\log|D|$ for all $\varepsilon>0$ with $\varepsilon<\varepsilon_0$. We note that if $0<\varepsilon<\varepsilon_0$, then

(2.6) $$\langle x+\varepsilon\eta, \alpha\rangle=\Phi(x)+\varepsilon c .$$

In view of (2.6), to prove that $\alpha\in\mathrm{Vert}\, S_D$, it is sufficient to show the following:

(2.7) There exists an ε such that $0<\varepsilon<\varepsilon_0$ and $\langle x+\varepsilon\eta, \beta\rangle<\Phi(x)+\varepsilon c$ for all $\beta\in S_D\setminus\{\alpha\}$.

Now suppose the statement (2.7) does not hold, and take sequences (ε_j) and (α_j) such that $0<\varepsilon_j<\varepsilon_0$, $\alpha_j\in S_D\setminus\{\alpha\}$, $\langle x+\varepsilon_j\eta, \alpha_j\rangle\geqq\Phi(x)+\varepsilon_j c$ for all j, and $\lim_{j\to\infty}\varepsilon_j$

$=0$. Since $\langle x+\varepsilon_j\eta, \alpha_j\rangle<0$ for all j, by Lemma 2.3 (α_j) is bounded. We may assume that $\alpha_j\to\gamma\in S_D\setminus\{\alpha\}$. Then, $\langle x, \gamma\rangle\geqq\Phi(x)$, so that $\gamma\in H$; therefore (2.5) implies that

$$\langle\eta, \gamma\rangle<c\,. \tag{2.8}$$

On the other hand, $\alpha_j=\gamma$ for sufficiently large j, and $\langle x+\varepsilon_j\eta, \gamma\rangle\geqq\Phi(x)+\varepsilon_j c$, or $\langle\eta, \gamma\rangle\geqq c$, which contradicts (2.8). The proof is completed.

3. Half-regular continued fraction expansions.

Let $(a_j)_{j\geqq 0}$ be a sequence of integers with

$$a_j\geqq 2 \qquad (j\geqq 1)\,. \tag{3.1}$$

Consider the mappings $s_j(w)=a_j-1/w$,

$$s_0\circ\cdots\circ s_{j-1}(w)=a_0-\frac{1}{a_1-}\cdots-\frac{1}{a_{j-1}-}\frac{1}{w}$$
$$=:[a_0, a_1, \cdots, a_{j-1}, w] \qquad (j\geqq 1)\,.$$

We say that

$$[a_0, a_1, \cdots, a_{j-1}]=a_0-\frac{1}{a_1-}\cdots-\frac{1}{a_{j-1}}$$

is a half-regular continued fraction. For $j\geqq 1$, let $p_j/q_j=[a_0, \cdots, a_{j-1}]$ be the natural representation, i.e., p_j and $q_j>0$ are relatively prime integers. For convenience, let $(q_0, p_0)=(0, 1)$. It is easily seen that $(q_1, p_1)=(1, 0)$,

$$\begin{pmatrix} q_{j+1} \\ p_{j+1}\end{pmatrix}=\begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1}\end{pmatrix}\begin{pmatrix} a_j \\ -1\end{pmatrix} \qquad (j\geqq 1), \tag{3.2}$$

$$\det\begin{pmatrix} q_j & q_{j-1} \\ p_j & p_{j-1}\end{pmatrix}=1 \qquad (j\geqq 1), \tag{3.3}$$

$$[a_0, \cdots, a_{j-1}, w]=\frac{p_j w-p_{j-1}}{q_j w-q_{j-1}} \qquad (j\geqq 1) \tag{3.4}$$

(cf., e.g., [9], [16]). It follows from (3.3) and (3.4) that

(3.5) the function $[a_0, \cdots, a_{j-1}, x]$ is strictly increasing in the interval $1\leqq x<+\infty$ and has the interval $[a_0, \cdots, a_{j-1}-1]\leqq y<[a_0, \cdots, a_{j-1}]$ as its image.

Let ω be a real number. If ω is rational (resp. irrational), then there exists a unique pair of finite (resp. infinite) sequences $(a_j)_{0\leqq j<N}$ and $(\omega_j)_{1\leqq j<N}$ with N a positive integer (resp. $N=+\infty$) such that (3.1) holds,

(3.6) $$\omega_j>1, \quad \omega=[a_0, \cdots, a_{j-1}, \omega_j] \quad (1\leqq j<N),$$

and $\omega_{N-1}=a_{N-1}$ (resp. and there exist infinitely many j with $a_j\geqq 3$). The argorithm $\omega\mapsto(a_j)_{0\leqq j<N}$, symbolically written as

$$\omega=a_0+K_{1\leqq j<N}(-1/a_j)=[a_0, a_1, \cdots],$$

is called the half-regular continued fraction expansion of ω; for every j, the integer a_j is called the j-th partial quotient, and the rational number $[a_0, \cdots, a_{j-1}]$ the j-th approximant of the expansion. Representing $[a_0, \cdots, a_{j-1}]=p_j/q_j$ naturally, we see that

$$p_j-q_j\omega=\frac{1}{q_j\omega_j-q_{j-1}}=\frac{1}{\omega_1\cdots\omega_j}$$

and $q_j\geqq j$ $(j\geqq 1)$. It follows that the sequence $(p_j/q_j)_j$ is decreasingly convergent to ω (cf., e.g., [16]). When N is finite it is convenient to set

(3.7) $$a_N=+\infty.$$

In this and subsequent sections, by $\mathrm{grad}(\alpha, \beta)$ we denote the gradient of the segment determined by two distinct points α, β in $\boldsymbol{R}^2$.

LEMMA 3.1. *Let ω be a positive real number and $a_0+K_{1\leqq j<N}(-1/a_j)$ be its half-regular continued fraction expansion with the natural representations $p_j/q_j=[a_0, \cdots, a_{j-1}]$ $(1\leqq j<N+1)$, as well as $(q_0, p_0)=(0, 1)$. Let $\alpha=(0, 1)$, $\beta=(1, \omega)$, $S=(\boldsymbol{Z}^2)_*\cap(\boldsymbol{R}_+\alpha+\boldsymbol{R}_+\beta)$, and $\alpha_j=(q_j, p_j)$ $(0\leqq j<N+1)$. It then holds that*

$$\mathrm{Vert}\, S=\{\alpha_0\}\cup\{\alpha_j\,;\, 1\leqq j<N+1,\ a_j\neq 2\}$$

(*see* (3.7)).

Proof. Let $1\leqq j<N+1$. It follows from (3.3) that the pair $\{\alpha_{j-1}, \alpha_j\}$ generates the lattice $\boldsymbol{Z}^2$, so that there is no elements of $\boldsymbol{Z}^2$ in the open triangle determined by α_{j-1}, α_j, and the origin. Furthermore, we see that $g_j:=\mathrm{grad}(\alpha_{j-1}, \alpha_j)$ is given by $g_j=[a_0, \cdots, a_{j-1}-1]=[a_0, \cdots, a_{j-1}, 1]$ (by (3.4)), so that $g_{j+1}\geqq g_j$ and that the equality holds if and only if $a_j=2$ (by (3.5)). It is trivial that $\alpha_0\in\mathrm{Vert}\, S$ and that $\alpha_N\in\mathrm{Vert}\, S$ when N is finite. Thus, the assertion follows.

Remark 3.2. Under the assumptions in Lemma 3.1 we see that $S\setminus(S+S)=\{\alpha_j\,;\,0\leqq j<N+1\}$.

LEMMA 3.3. *Let $a_i, b_j\in\boldsymbol{Z}$, $a_0\geqq 1$, $b_0\geqq 1$, $a_i\geqq 2$ $(1\leqq i\leqq k-1)$, $b_j\geqq 2$ $(1\leqq j\leqq l-1)$. If $[a_0, \cdots, a_{k-1}]=[b_0, \cdots, b_{l-1}]^{-1}$, then*

(i) *$[a_0, \cdots, a_{k-1}-1]=[b_0, \cdots, b_{l-2}]^{-1}$ provided that $a_{k-1}\geqq 2$,*

(ii) *$a_0+\cdots+a_{k-1}-k=b_0+\cdots b_{l-1}-l$, and*

(iii) *$a_{k-1}=2$ or $b_{l-1}=2$ provided that $a_{k-1}\geqq 2$.*

Proof. We first note that if $p_j/q_j=[a_0, \cdots, a_{j-1}]$ $(j=k-1, k)$ are natural representations, then

$$[a_0, \cdots, a_{k-1}, 2]=\frac{p_k+(p_k-p_{k-1})}{q_k+(q_k-q_{k-1})},$$

$$[a_0, \cdots, a_{k-1}+1]=\frac{p_k+p_{k-1}}{q_k+q_{k-1}},$$

$$[a_0, \cdots, a_{k-1}-1]=\frac{p_k-p_{k-1}}{q_k-q_{k-1}},$$

from which we see that

(3.8) if $[a_0, \cdots, a_{k-1}]=[b_0, \cdots, b_{l-1}]^{-1}$ and $[a_0, \cdots, a_{k-1}-1]$ $=[b_0, \cdots, b_{l-2}]^{-1}$ hold, then $[a_0, \cdots, a_{k-1}, 2]$ $=[b_0, \cdots, b_{l-1}+1]^{-1}$.

We shall prove the assertions (i), (ii), and (iii) by induction on the number $\mu=a_0+\cdots+a_{k-1}-k$. If $\mu=1$, then $(k, a_0)=(1, 2)$ or $(k, a_0, a_1)=(2, 1, 2)$. We then have $[2]=[1, 2]^{-1}$ or $[1, 2]=[2]^{-1}$, respectively. In these cases the assertions hold trivially. Assume that the assertions are true for the case when $a_0+\cdots+a_{k-1}-k$ is at most μ $(\geqq 1)$, and let $a_0+\cdots a_{k-1}-k=\mu+1$ and $[a_0, \cdots, a_{k-1}]=[b_0, \cdots, b_{l-1}]^{-1}$. First, assume $a_{k-1}\geqq 3$, and let $[c_0, \cdots, c_{l'-1}]$ be the half-regular continued fraction expansion of the number $[a_0, \cdots, a_{k-1}-1]^{-1}$. By the induction hypothesis we see that $c_0+\cdots+c_{l'-1}-l'=\mu$ and $[a_0, \cdots, a_{k-2}]=[c_0, \cdots, c_{l'-1}+1]^{-1}$. By (3.8) we see $[c_0, \cdots, c_{l'-1}, 2]=[a_0, \cdots, a_{l-1}]^{-1}$. By the uniqueness of the expansion we have $(c_0, \cdots, c_{l'-1}, 2)=(b_0, \cdots, b_{l-1})$, so that $[a_0, \cdots, a_{k-1}-1]=[b_0, \cdots, b_{l-2}]^{-1}$ and $b_{l-1}=2$. Furthermore, $b_0+\cdots+b_{l-1}-l=c_0+\cdots+c_{l'-1}+2-(l'+1)=\mu+1$. Thus, (i), (ii), and (iii) hold. Finally, assume $a_{k-1}=2$, and let $[c_0, \cdots c_{l'-1}]$ be the half-regular continued fraction expansion of $[a_0, \cdots, a_{k-2}]^{-1}$. By the induction hypothesis we see that $c_0+\cdots+c_{l'-1}-l'=a_0+\cdots+a_{k-2}-(m-1)=\mu$, $[a_0, \cdots, a_{k-2}-1]=[c_0, \cdots, c_{l'-2}]^{-1}$ and $c_{l'-2}\geqq 2$ even if $l'=1$ because of the fact $\mu\geqq 1$. It follows from (3.8) that $[a_0, \cdots, a_{k-2}, 2]=[c_0, \cdots, c_{l'-1}+1]^{-1}$; therefore $(c_0, \cdots, c_{l'-1}+1)=(b_0, \cdots, b_{l-1})$ so that $l'=l$, $[a_0, \cdots, a_{k-2}]=[b_0, \cdots, b_{l-1}-1]^{-1}$ and $b_{l-1}\geqq 2$. Let $s=b_{l-1}-2$. By the induction hypothesis we have $a_j=2$ $(k-1\leqq j\leqq k-s)$, and $[a_0, \cdots, a_{k-s-1}]=[b_0, \cdots, b_{l-2}, 2]^{-1}$. Again by the induction hypothesis we get $[a_0, \cdots, a_{k-s-1}-1]=[b_0, \cdots, b_{l-2}]^{-1}$. Since

$$\begin{aligned}[a_0, \cdots, a_{k-s-1}-1]&=[a_0, \cdots, a_{k-s-1}, 1]\\ &=[a_0, \cdots, a_{k-s-1}, 2, 2, \cdots, 2, 1]\\ &=[a_0, \cdots, a_{k-s-1}, a_{k-s}, \cdots, a_{k-2}, a_{k-1}-1],\end{aligned}$$

we have $[a_0, \cdots, a_{k-1}-1]=[b_0, \cdots, b_{l-2}]^{-1}$. Furthermore, $b_0+\cdots+b_{l-1}-l=$

$c_0+\cdots+(c_{l'-1}+1)-l'=\mu+1$. We have obtained all the assertions (i), (ii), (iii) and proved the lemma.

Remark 3.4. Let $\omega=c_0+K_{1\leq j<M}(1/c_j)$ be the regular continued fraction expansion of a real number ω, where c_j are integers with $c_j\geqq 1$ $(j\geqq 1)$ (cf., e.g., [6], [9]). Let r_j/s_j be the natural representation of the rational number

$$c_0+\frac{1}{c_1+}\cdots+\frac{1}{c_{j-1}}=c_0+K_{1\leq k<j}(1/c_k)$$

and set $\gamma_j=(s_j, r_j)$ $(j\geqq 1)$, $\gamma_0=(0, 1)$. To get the unique expansion for rational numbers we place the following restriction: the length M of the expansion must be even whenever M is finite. For example, if $\omega=4/5$ (resp. $16/9$), we expand it as

$$\frac{4}{5}=0+\frac{1}{1+}\frac{1}{3+}\frac{1}{1}\qquad\left(\text{resp. } \frac{16}{9}=1+\frac{1}{1+}\frac{1}{3+}\frac{1}{2}\right),$$

not

$$\frac{4}{5}=0+\frac{1}{1+}\frac{1}{4}\qquad\left(\text{resp. } \frac{16}{9}=1+\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{1}\right).$$

We then have the following relationship between the regular continued fraction expansion $c_0+K_{1\leq j<M}(1/c_j)$ of any non-integral real number ω and the half-regular continued fraction expansion $a_0+K_{1\leq j<N}(-1/a_j)$ of ω with the notation in Lemma 3.1 as well as the convension $a(j)=a_j$, $\alpha(j)=\alpha_j$:

$$\begin{cases} c_0=a(0)-1, \quad c_{2j}=a(c_1+c_3+\cdots+c_{2j-1})-2 & (j\geqq 1) \\ c_{2j+1}=\max\{k\geqq 1;\ a(c_1+c_3+\cdots+c_{2j-1}+k)\neq 2\} & (j\geqq 0), \end{cases}$$

$$\begin{cases} \gamma_0=\alpha(0), \quad \gamma_{2j}=\alpha(c_1+c_3+\cdots+c_{2j-1}) \quad (j\geqq 1) \\ \gamma_{2j+1}=\alpha(c_1+c_3+\cdots+c_{2j-1}+k)-\alpha(c_1+c_3+\cdots+c_{2j-1}+k-1) \\ \qquad\qquad (j\geqq 0,\ 1\leqq k\leqq c_{2j+1}). \end{cases}$$

Conversely,

$$\begin{cases} a(0)=c_0+1, \quad a(c_1+c_3+\cdots+c_{2j-1})=c_{2j}+2 \quad (j\geqq 1) \\ a(c_1+c_3+\cdots+c_{2j-1}+k)=2 \quad (1\leqq k\leqq c_{2j+1}-1,\ j\geqq 0), \end{cases}$$

$$\alpha(c_1+c_3+\cdots+c_{2j-1}+k)=\gamma_{2j}+k\gamma_{2j+1} \quad (0\leqq k\leqq c_{2j+1}-1,\ j\geqq 0).$$

In terms of the regular continued fraction expansion of ω, the assertions in Remark 3.2 and Lemma 3.1 are written as follows:

$$S\setminus(S+S)=\{\gamma_0\}\cup\{\gamma_{2j}+k\gamma_{2j+1};\ 0\leqq j<M/2,\ 1\leqq k\leqq c_{2j+1}\},$$

$$\operatorname{Vert} S=\{\gamma_{2j};\ 0\leqq j<M/2+1\}.$$

4. Two dimensional case.

Let D be a logarithmically coned, complete Reinhardt domain in $\boldsymbol{C}^2$. Assume $D\neq\boldsymbol{C}^2$. There then exist real numbers γ, δ with $(1/2)\pi\leqq\gamma\leqq\pi$, $(3/2)\pi\leqq\delta\leqq 2\pi$ such that

$$\log|D|=\{(r\cos\theta, r\sin\theta)\in\boldsymbol{R}^2; r>0, \gamma<\theta<\delta\}.$$

If $\delta>\gamma+\pi$, then

$$D=\{z\in\boldsymbol{C}^2; |z_1|^{-\tan\gamma}|z_2|<1\}\cup\{z\in\boldsymbol{C}^2; |z_1|^{-\tan\delta}|z_2|<1\},$$

and $S_D=\phi$, so that $c_D^*(0, z)=0$. Since D is not pseudoconvex, by Lemma 1.1 we see that $\exp g_D(0, z)<\mu_D(z)$ for some $z\in D$, with

$$\mu_D(z)=\min\{(|z_1|^{-\tan\gamma}|z_2|)^{1/(1-\tan\gamma)}, \quad (|z_1|^{-\tan\delta}|z_2|)^{1/(1-\tan\delta)}\}.$$

Next, assume $\delta=\gamma+\pi$. Then, $D=\{z\in\boldsymbol{C}^2; |z_1|^{-\tan\gamma}|z_2|<1\}$, and

$$\exp g_D(0, z)=k_D^*(0, z)=\mu_D(z)=(|z_1|^{-\tan\gamma}|z_2|)^{1/(1-\tan\gamma)}.$$

If $\tan\gamma$ is an irrational number, then $S_D=\phi$ and $c_D^*(0, z)=0$; while if $\tan\gamma$ is a rational number $-q/p$ with natural representation, then $S_D=(\boldsymbol{Z}_+)_*(q, p)$, and $S_D\setminus(S_D+S_D)=\mathrm{Vert}\,S_D=\{(q, p)\}$, so that $c_D^*(0, z)=|z_1|^q|z_2|^p$ for $z\in D$, while $k_D^*(0, z)=(|z_1|^q|z_2|^p)^{1/(q+p)}$.

Finally we assume that $\delta<\gamma+\pi$. Then,

$$D=\{z\in\boldsymbol{C}^2; |z_1|^{-\tan\gamma}|z_2|<1, |z_1|^{-\tan\delta}|z_2|<1\}.$$

Setting $\tau=-\tan\gamma$, $\omega=-1/\tan\delta$, we have

(4.1) $$0\leqq\omega<1/\tau\leqq+\infty,$$

and

(4.2) $$D=\{z\in\boldsymbol{C}^2; |z_1||z_2|^{\omega}<1, |z_1|^{\tau}|z_2|<1\}.$$

Setting

(4.3) $$\alpha_\infty=(1, \omega), \qquad \beta_\infty=(\tau, 1),$$

we have

(4.4) $$S_D=(\boldsymbol{Z}^2)_*\cap(\boldsymbol{R}_+\alpha_\infty+\boldsymbol{R}_+\beta^\infty).$$

By Lemma 1.1 (iii) we have

$$\exp g_D(0, z)=k_D^*(0, z)=\mu_D(z)$$
$$=\max\{(|z_1||z_2|^{\omega})^{1/(1+\omega)}, (|z_1|^{\tau}|z_2|)^{1/(\tau+1)}\}.$$

For the Carathéodory pseudodistance we have the following.

THEOREM 4.1. *Let* ω *and* τ *be two real numbers satisfying* (4.1), *and let* $\omega=a_0+K_{1\leq m<M}(-1/a_m)$, $\tau=b_0+K_{1\leq n<N}(-1/b_n)$ *be the half-regular continued fraction expansions. Let* $p_m/q_m=[a_0, \cdots, a_{m-1}]$, $s_n/r_n=[b_0, \cdots, b_{n-1}]$ *be the natural representations, and set* $\alpha_m=(q_m, p_m)$, $\beta_n=(s_n, r_n)$ *with* $\alpha_0=(0, 1)$, $\beta_0=(1, 0)$.

(i) *There exist unique integers* $m_0\geqq 0$, $n_0\geqq 0$ *such that* $\alpha_{m_0}=\beta_{n_0}=:\gamma$.

(ii) *If* D *is the domain defined by* (4.2), *and if* $E_D=\{\gamma\}\cup\{\alpha_m; m_0<m<M+1, a_m\neq 2\}\cup\{\beta_n; n_0<n<N+1, b_n\neq 2\}$, *then* $\operatorname{Vert} S_D=E_D$ *(see* (3.7)); *therefore* $c_D^*(0, z)=\max\{|z^{\alpha}|; \alpha\in E_D\}$ *for all* $z\in D$.

Proof. To prove (i), assume $1/\tau=+\infty$, or $\tau=0$. Then, $b_0=0$, and $\beta_1=(0, 1)$, so that $\alpha_0=\beta_1$. Next, assume $0\leqq\omega<1/\tau<+\infty$. Since the sequence $([a_0, \cdots, a_{m-1}])_m$ is strictly decreasing and converges to ω, there exists an integer $m_0\geqq 0$ such that $[a_0, \cdots, a_{m_0}]<1/\tau\leqq[a_0, \cdots, a_{m_0-1}]$, or $[a_0, \cdots, a_{m_0-1}]^{-1}\leqq\tau<[a_0, \cdots, a_{m_0}]^{-1}$. Let $[c_0, \cdots, c_{j-1}]$ be the half-regular continued fraction expansion of the number $[a_0, \cdots, a_{m_0}]^{-1}$. Then, by Lemma 3.2 we have $[c_0, \cdots, c_{j-1}-1]=[a_0, \cdots, a_{m_0-1}]^{-1}$. If $[a_0, \cdots, a_{m_0-1}]^{-1}=\tau$, then $\tau=[c_0, \cdots, c_{j-1}-1]=[c_0, \cdots, c_{n_0-1}-1]$, where $n_0=\max\Delta$ if $\Delta:=\{l\in\{1, \cdots, j\}; c_{l-1}\neq 2\}\neq\phi$ and $n_0=1$ if $\Delta=\phi$. Thus, $[b_0, \cdots, b_{N-1}]=[c_0, \cdots, c_{n_0-1}-1]$ and $n_0=N$; therefore $\alpha_{m_0}=\beta_N$. Next, assume $[a_0, \cdots, a_{m_0-1}]^{-1}<\tau$. Since the interval $[c_0, \cdots, c_{j-1}-1]<t<[c_0, \cdots, c_{j-1}]$ coincides with $\{[c^0, \cdots, c_{j-1}, w]; w>1\}$, there exists a real number $w>1$ such that $\tau=[c_0, \cdots, c_{j-1}, w]$. Since $\tau=[b_0, \cdots, b_{j-1}, \omega_j]$ for some $\omega_j>1$, we see that $(c_0, \cdots, c_{j-1})=(b_0, \cdots, b_{j-1})$, so that $[a_0, \cdots, a_{m_0}]^{-1}=[b_0, \cdots, b_{j-1}]$, or $\alpha_{m_0}=\beta_j$. We thus proved the assertion (i).

To prove (ii), let α_∞, β_∞ be as in (4.3), and let $S_1=(\boldsymbol{Z}^2)_*\cap(\boldsymbol{R}_+\alpha_0+\boldsymbol{R}_+\alpha_\infty)$, $S_2=(\boldsymbol{Z}^2)_*\cap(\boldsymbol{R}_+\beta_0+\boldsymbol{R}_+\beta_\infty)$. It follows from (4.4) that $S_D=S_1\cap S_2$. By Lemma 3.1 we see that

$$\{\alpha_m; m_0<m<M+1, a_m\neq 2\}\subset(\operatorname{Vert} S_1)\cap S_2\subset\operatorname{Vert} S_D,$$

$$\{\beta_n; n_0<n<N+1, b_n\neq 2\}\subset(\operatorname{Vert} S_2)\cap S_1\subset\operatorname{Vert} S_D.$$

Furthermore, since $\operatorname{grad}(\gamma, \alpha_{m_0+1})\leqq\omega<1/\tau\leqq\operatorname{grad}(\gamma, \beta_{n_0+1})$, we see that $\gamma\in\operatorname{Vert} S_D$. Thus, $E_D\subset\operatorname{Vert} S_D$. Conversely, let $\alpha\in\operatorname{Vert} S_D$ and $\alpha\neq\gamma$. Then $\operatorname{grad}(0, \alpha)\neq\operatorname{grad}(0, \gamma)$. Assume $\operatorname{grad}(0, \alpha)<\operatorname{grad}(0, \gamma)$. There then exists a linear functional f on $\boldsymbol{R}^2$ and $c\in\boldsymbol{R}$ such that $f(\alpha)=c$, $f(x)<c$ for all $x\in S_D\setminus\{\alpha\}$. Let g be the gradient of the line $f^{-1}(c)$. Then, $\operatorname{grad}(\gamma, \alpha_{m_0+1})\leqq\operatorname{grad}(\gamma, \alpha)\leqq g\leqq\operatorname{grad}(0, \alpha_\infty)\leqq\operatorname{grad}(0, \gamma)$. It follows that the set $S_1\setminus S_2$ is contained in the cone $\gamma+\boldsymbol{R}_+(\gamma-\alpha_{m_0+1})+\boldsymbol{R}_+\gamma$, and that $f(x)<c$ for all $x\in S_1\setminus\{\alpha\}$; therefore, $\alpha\in\operatorname{Vert} S_1$, and $\alpha=\alpha_m$ for some m with $m_0<m<M+1$ and $a_m\neq 2$ (by Lemma 3.1). Similarly, if $\operatorname{grad}(0, \alpha)>\operatorname{grad}(0, \gamma)$, then $\alpha\in\operatorname{Vert} S_2$, and $\alpha=\beta_n$ for some n with

$n_0 < n < N+1$ and $b_n \neq 2$. We thus have proved Vert $S_D \subset E_D$ and the theorem.

Remark 4.2. Let ω, τ, D, and E_D be as in Theorem 4.1. Let $\omega = a_0 + K_{1 \leq m < M}(1/a_m)$ and $\tau = b_0 + K_{1 \leq n < N}(1/b_n)$ be the regular continued fraction expansions of ω and τ (see Remark 3.4). Represent $p_m/q_m = a_0 + K_{1 \leq j < m}(1/a_j)$ and $s_n/r_n = b_0 + K_{1 \leq j < n}(1/b_j)$ naturally, and set $\alpha_m = (q_m, p_m)$, $\beta_n = (s_n, r_n)$ with $\alpha_0 = (0, 1)$, $\beta_0 = (1, 0)$. There then exist non-negative integers m_0, n_0, j_0, and k_0 such that $\alpha_{2m_0} + j_0 \alpha_{2m_0+1} = \beta_{2n_0} + k_0 \beta_{2n_0+1} := \gamma$, and it holds that $E_D = \{\gamma\} \cup \{\alpha_{2m};\ m_0 < m < M/2+1\} \cup \{\beta_{2n};\ n_0 < n < N/2+1\}$.

Finally we present some examples which are applicable to Theorem 4.1.

Example 1 ([7], [8]). Let $D = \{z \in \boldsymbol{C}^2;\ |z_1| < 1,\ |z_1|^{345}|z_2|^{128} < 1\}$. In this case we have $\omega = 0$, $\tau = 345/128 = [3, 4, 2, 2, 3, 6]$, and the following table:

b_n	3	4	2	2	3	6	$+\infty$
s_n	1	3	11	19	27	62	345
r_n	0	1	4	7	10	23	128

It follows that $E_D = \{(1, 0), (3, 1), (27, 10), (62, 23), (345, 128)\}$. Thus $c_D^*(0, z) = \max\{|z^\alpha|;\ \alpha \in E_D\}$.

Example 2. Let $D = \{z \in \boldsymbol{C}^2;\ |z_1|^2|z_2|^7 < 1,\ |z_1|^{\sqrt{2}}|z_2| < 1\}$. Noting $0 < 1/\sqrt{2} < 7/2$, we have $\omega = 1/\sqrt{2} = [1, 4, 2, 4, 2, 4, \cdots]$, $\tau = 2/7 = [1, 2, 2, 3]$, and the tables:

a_m	1	4	2	4	2	4	$\cdots$
q_m	0	1	4	7	24	41	$\cdots$
p_m	1	1	3	5	17	29	$\cdots$

b_n	1	2	2	3	$+\infty$
s_n	1	1	1	1	2
r_n	0	1	2	3	7

It follows that $\alpha_1 = \beta_1$, and $E_D = \{(1, 1)\} \cup \{\alpha_{2m+1};\ m \geqq 1\} \cup \{(1, 3), (2, 7)\}$, where $\alpha_{2m+1} = 6\alpha_{2m-1} - \alpha_{2m-3}$ $(m \geqq 2)$, $\alpha_1 = (1, 1)$, $\alpha_3 = (7, 5)$. Thus $c_D^*(0, z) = \max\{|z^\alpha|:\ \alpha \in E_D\}$.

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