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KODAI MATH. J.
13 (1990), 409—416

ON THE VALUE DISTRIBUTION OF AN ENTIRE FUNCTION OF ORDER AT MOST ONE

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§1. Introduction.

As a consequence of results on solutions to a differential equation $w''+Aw=0$, where A is entire, Shen [5] and Rossi [4] proved a curious result:

There does not exist a transcendental entire function E of order $\rho(E)<1$ such that the value of $E'(z)$ at every zero of $E(z)$ is ± 1.

On proving this, they used the lemma of Bank and Laine [1] which states that such a function E would have to be the product of two linearly independent solutions of the above second order differential equation. It follows from the counter-example given by Rossi, $E(z)=2\sqrt{z}\sin\sqrt{z}$, that the conclusion can not hold even if only one zero fails to satisfy the assumption.

In this note we prove

THEOREM. *Let $E(z)$ be a transcendental entire function of order $\rho(E)\leqq 1$ and $Q(z)\not\equiv 0$ a rational function. Suppose that $E'(z)-Q(z)$ vanishes at every zero of $E(z)$ with possibly finitely many exceptions. Then $\rho(E)=1$ and further E is of regular growth, and also the meromorphic function*

$$A(z)=\frac{E'(z)-Q(z)}{E(z)} \tag{1.1}$$

is one of the followings:

a) *A is a rational function such that for some nonzero constant a, $A(z)\to a$ as $z\to\infty$;*

b) *A is a transcendental function of regular growth with $\rho(A)=1$, and has a finite number of poles.*

This result may be read as a result on the zeros of $E'(z)$.

We can easily give examples for the case a).

Example 1. For any polynomials $p\not\equiv 0$ and q and also a nonzero constant

Received November 6, 1989; revised May 15, 1990.

a, $E(z)=p(z)(\exp(az)+q(z))$ and $Q(z)=p(z)(q'(z)-aq(z))$ satisfy the hypotheses of our theorem and then

$$A(z)=\frac{p'(z)+ap(z)}{p(z)}\longrightarrow a, \qquad \text{as } z\to\infty.$$

Example 2. For the entire function $E(z)=(e^z-1)/z$ and the rational function $Q(z)=1/z$ we have $A(z)=(z-1)/z$ given by (1.1).

Also for the case b) we have

Example 3. The entire function $E(z)=2\sqrt{z}\sin\sqrt{z}\cdot\exp(iz/\pi)$ and $Q(z)\equiv 1$ satisfy the hypotheses of our theorem. Then we have

$$A(z)=-\frac{1}{E(z)}+\frac{(\sin\sqrt{z})'}{\sin\sqrt{z}}+\frac{1}{2z}+\frac{i}{\pi},$$

and (see [2: p. 7])

$$m(|z|, A)\sim m(|z|, 1/E)\sim m(|z|, e^{iz/\pi})=|z|/\pi^2, \qquad \text{as } |z|\to\infty.$$

Example 4. The entire function $E(z)=e^{-2z}(z-e^z)$ and the rational function $Q(z)=(1-z)/z^2$ imply the meromorphic function

$$A(z)=\{(1-z)e^z+z(1-2z)\}/z^2$$

from the definition (1.1).

Of course, the theorem does not hold in general for meromorphic $E(z)$, since the function $A(z)$ as in (1.1) has always the poles possibly except for those of $Q(z)$ wherever the function $E(z)$ does. If $E(z)$ has, however, only a finite number of poles, the corresponding result to this theorem is easily obtained.

§ 2. Preliminaries.

To prove the theorem we make a direct application of a method of Rossi in [4]. It bases on the Beurling-Tsuji estimate for harmonic measure and needs the following three lemmas proved there.

LEMMA 1. *Let E be an entire function of finite order. Given $\varepsilon>0$ there exists a constant $C=C(\varepsilon)$ such that*

$$\left|\frac{E'}{E}(re^{i\theta})\right|\leqq r^C,$$

for all $r\geqq r_0\geqq 1$ and all $\theta\notin J(r)$, where the angular measure of the set $J(r)$ in $[0, 2\pi)$, $m(J(r))$ is $\leqq\varepsilon\pi$.

To state the second lemma we need some notation. Let D be a region in the complex plane $\boldsymbol{C}$. We denote by $\theta_D(r)$ the measure of all θ in $[0, 2\pi)$ such that $re^{i\theta}\in D$. To each $r\geqq 1$, if the entire circle $|z|=r$ lies in D, set $\theta_D^*(r)=+\infty$; otherwise $\theta_D^*(r)=\theta_D(r)$. As usual, the order $\rho(u)$ and the lower order $\mu(u)$ of a function $u(z)$ subharmonic in $\boldsymbol{C}$ are given by

$$\rho(u)=\varlimsup_{r\to\infty}\frac{\log M(r, u)}{\log r} \quad \text{and} \quad \mu(u)=\varliminf_{r\to\infty}\frac{\log M(r, u)}{\log r},$$

where $M(r, u)=\sup_{|z|=r}u(z)$. Also for an entire function $E(z)$, they are given by $\rho(E)=\rho(\log|E|)$ and $\mu(E)=\mu(\log|E|)$. Then we have

LEMMA 2. *Let u be a subharmonic function in $\boldsymbol{C}$ and let D be an open component of $\{z : u(z)>0\}$. Then*

$$\begin{matrix}\rho(u)\\ \mu(u)\end{matrix}\geqq \varlimsup_{R\to\infty}(\log R)^{-1}\pi\int_1^R\frac{dt}{t\theta_D^*(t)}. \tag{2.1}$$

Furthermore, given $\varepsilon>0$, define $F=\{r : \theta_D^*(r)\leqq\varepsilon\pi\}$. Then

$$\varlimsup_{R\to\infty}(\log R)^{-1}\int_{F\cap[1,R]}dt/t\leqq\varepsilon\rho(u). \tag{2.2}$$

In this lemma we shall make a minor modification to $\theta_D^*(r)$ when $\theta_D(t)=0$, $1\leqq t\leqq t_0$.

LEMMA 3. *Let $l_1(t)$ and $l_2(t)$ be two positive and measurable functions on $[1, \infty)$ with $l_1(t)+l_2(t)\leqq(2+\varepsilon)\pi$, where $\varepsilon>0$. If $G\subset[1, \infty)$ is any measurable set and*

$$\pi\int\frac{dt}{tl_2(t)}\leqq\alpha\int_G dt/t, \qquad \alpha\geqq 1/2,$$

then

$$\pi\int\frac{dt}{tl_1(t)}\geqq\frac{\alpha}{(2+\varepsilon)\alpha-1}\int_G dt/t.$$

§ 3. Proof of Theorem.

Now the function $A(z)$ given by (1.1) is regular at every simple zero of $E(z)$ possibly with finitely many exceptions. The poles of $A(z)$ may therefore occur only at multiple zeros of $E(z)$ or poles of $Q(z)$. The number of these points is however at most finite since our assumption requires the rational function $Q(z)$ should vanish at each multiple zero of $E(z)$ except for finitely many. Thus $A(z)$ is a meromorphic function having only a finite number of poles.

We now distinguish the cases whether A is rational or transcendental.

CASE 1; in which A is rational. The entire function $E(z)$ considered here is a solution to the non-homogeneous differential equation

(3.1) $$w'=A(z)w+Q(z)$$

with the rational coefficients. We apply the Wiman-Valiron theory (see [6: p. 105, Theorem 30]) to the equation with $w=E(z)$ and note that $E(z)$ is now transcendental. Then the central index $n(r)$ satisfies the relation

(3.2) $$n(r)=r|A(z_r)|\{1+o(1)\}$$

as r tends to infinity outside a set Δ of finite logarithmic measure, $m_l(\Delta)$, where z_r is a point at which $|E(z_r)|=\max_{|z|=r}|E(z)|$ and $|z_r|=r$. The rational function $A(z)$ has the asymptotic representation

(3.3) $$A(z)=az^m\{1+O(1/|z|)\}, \qquad (z\to\infty)$$

for some nonzero constant a and an integer m. Thus (3.2) gives

$$\rho(E)\geqq \varlimsup_{\substack{r\to\infty \\ (r\notin\Delta)}} \frac{\log n(r)}{\log r}=m+1 .$$

Since the right-hand side is non-negative and $\rho(E)\leqq 1$, we have $m=0$ or $m=-1$. The former implies $\rho(E)=1$, and further we can see $\mu(E)=1$. In fact, if $r\in\Delta$ and $\log r>m_l(\Delta)+1$, then there exists a $\tau=\tau(r)$ with $\exp(-m_l(\Delta)-1)\leqq\tau<1$ such that $\tau r\notin\Delta$. Thus it follows from monotonicity of $n(r)$ (see [6]) that $n(r)\geqq n(\tau r)\geqq n(\exp(-m_l(\Delta)-1)r)$. Hence we have

$$\log n(r)=(1+o(1))\log r, \qquad \text{as } r\to\infty .$$

It is easy to see that $m(r, 1/E)\leqq m(r, 1/Q)+m(r, E'/E)+m(r, A)+\log 2=O(\log r)$. These results give the case a) as in our theorem. Next we suppose $m=-1$. Then we transform (3.1) into the equation

(3.4) $$y'+\frac{1}{\zeta^2}A(1/\zeta)y+\frac{1}{\zeta^2}Q(1/\zeta)=0 ,$$

by setting $y(\zeta)=w(1/\zeta)$. Fix $r>0$ sufficiently small and let D be the simply connected domain $\{\zeta: 0<|\zeta|<r, 0<arg\,\zeta<2\pi\}$. Then a solution to (3.4) in D is given by

$$y(\zeta; \zeta_0, y_0)=\exp\Big(-\int_{\zeta_0}^{\zeta}\frac{1}{s^2}A(1/s)ds\Big)\Big(y_0-\int_{\zeta_0}^{\zeta}\frac{1}{t^2}Q(1/t)\exp\Big\{\int_{\zeta_0}^{t}\frac{1}{s^2}A(1/s)ds\Big\}dt\Big)$$

where $\zeta_0\in D$ and $y_0\in \boldsymbol{C}$ (see [3]). In the domain D, we may write

$$\zeta^{-2}A(\zeta^{-1})=a\zeta^{-1}\{1+O(|\zeta|)\},$$

$$\zeta^{-2}Q(\zeta^{-1})=b\zeta^{-(k+2)}\{1+O(|\zeta|)\}, \qquad b\in\boldsymbol{C}-\{0\},\ k \text{ integer},$$

and thus for a constant d_0

(3.5) $$y(\zeta; \zeta_0, y_0)=e^{-d_0}\zeta^{-a}\{1+O(|\zeta|)\}\Big(y_0-e^{d_0}b\{1+O(|\zeta|)\}\int_{\zeta_0}^{\zeta}t^{-(k-a+2)}dt\Big),$$

as $|\zeta|\to 0$ there. The function $E(1/\zeta)$ posesses the above representation and by its monodromy property about the origin then it follows that either the constant a must be an integer or the coefficient of ζ^{-a} in (3.5) must vanish identically in a neighborhood of the origin. Then $E(1/\zeta)$ has the origin as possibly a pole, which implies that the entire function $E(z)$ cannot be transcendental. This is a contradiction and the completes the proof in this case.

CASE 2; in which A is transcendental. From the reason given at the beginning of this section, we may now write $A(z)=B(z)/P(z)$ with an entire function $B(z)$ and a polynomial $P(z)$. Then we have

$$B(z)=P(z)\Big(\frac{E'(z)}{E(z)}-\frac{Q(z)}{E(z)}\Big). \tag{3.6}$$

Let k and l be the degrees of the rational function $Q(z)$ and the polynomial $P(z)$, respectively. After the manner of Rossi, fix $\varepsilon>0$ and choose an integer N such that

$$N>\max(C,\ k)+l\,, \tag{3.7}$$

where C is the constant as in Lemma 1 and

$$\log M(2,\ B)<N\log 2\,.$$

Since B is transcendental there exists a point z_0, $|z_0|>2$, such that $\log|B(z_0)|>N\cdot\log|z_0|$. Let D_1 be the connected component of the set

$$\{z:\log|B(z)|-N\cdot\log|z|>0\},$$

containing z_0. By the choice of N, $\log|B(z)|-N\cdot\log|z|$ is harmonic in D_1 and identically zero on the boundary. Thus the function u defined by

$$u(z)=\begin{cases}\log|B(z)|-N\log|z| & (z\in D_1),\\ 0 & (z\in \boldsymbol{C}-D_1),\end{cases}$$

is subharmonic in $\boldsymbol{C}$ with the lower order and the order

$$\mu(u)\leqq\mu(B) \quad\text{and}\quad \rho(u)\leqq\rho(B)\,. \tag{3.8}$$

It is easily shown that $\rho(B)=\rho(A)\leqq\rho(E)$, which will be mentioned later.

Let D_2 be any connected component of $\{z:\log|E(z)|>0\}$ and let $D_3=\{re^{i\theta}:\theta\in J(r)\}$ where $J(r)$ is as in Lemma 1. If the set $(D_1\cap D_2)-D_3$ contains an unbounded sequence $r_ne^{i\theta_n}$, $n=1, 2, \cdots$, we obtain from the definitions of D_1, D_2, and D_3, Lemma 1 and also (3.6)

$$r_n{}^N\leqq|B(r_ne^{i\theta_n})|\leqq|P(r_ne^{i\theta_n})|\{r_n{}^C+|Q(r_ne^{i\theta_n})|\},\qquad r_n\geqq r_0\,.$$

This clearly contradicts (3.7) for n large enough.

Then for arbitrary fixed $\varepsilon>0$, we may assume that $(D_1\cap D_2)-D_3$ is bounded. This implies that for $r\geqq r_1\geqq r_0$

$$K(r)=\{\theta : re^{i\theta}\in D_1\cap D_2\}\subset J(r),$$

and thus by Lemma 1 the angular measure of $K(r)$ satisfies

$$m(K(r))\leqq\varepsilon\pi . \tag{3.9}$$

Setting $l_j(r)=\theta_{D_j}(r)$ given in § 2, we have $l_j(r)>0$ for r sufficiently large since each D_j is an unbounded domain, $j=1, 2$. Also (3.9) gives $l_1(r)+l_2(r)\leqq (2+\varepsilon)\pi$ $(r\geqq r_1)$. If need were, by putting $\theta_{D_j}(r)\equiv\pi$ $(j=1, 2)$ for $r<r_1$, we could assume each $l_j(r)>0$ $(j=1, 2)$ and this inequality to be true for any $r\geqq 1$. Now let us set

$$\overline{\lim_{R\to\infty}}(\log R)^{-1}\pi\int_1^R\frac{dt}{tl_2(t)}=\alpha . \tag{3.10}$$

By the definition of the l_2, $\alpha\geqq 1/2$. Since l_1 and l_2 satisfy the hypotheses of Lemma 3, we obtain

$$\underline{\lim_{R\to\infty}}(\log R)^{-1}\pi\int_1^R\frac{dt}{tl_1(t)}\geqq\frac{\alpha}{(2+\varepsilon)\alpha-1}. \tag{3.11}$$

Define $B_j=\{r : \theta^*_{D_j}(r)=\infty\}$ and $\tilde{B}_j=[1, \infty)-B_j$, $j=1, 2$. If $r\in B_1$, we have $\theta^*_{D_2}(r)\leqq\varepsilon\pi$ by (3.9). Thus $B_1\subset\{r : \theta^*_{D_2}(r)\leqq\varepsilon\pi\}$ and similarly $B_2\subset\{r : \theta^*_{D_1}(r)\leqq\varepsilon\pi\}$. By (2.2) we have

$$\overline{\lim_{R\to\infty}}(\log R)^{-1}\int_{B_2\cap[1, R]}dt/t\leqq\varepsilon\rho(u). \tag{3.12}$$

Then (2.1), (3.10) and (3.12) give

$$\begin{aligned}\rho(E)&\geqq\overline{\lim_{R\to\infty}}(\log R)^{-1}\pi\int_1^R\frac{dt}{t\theta^*_{D_2}(t)}\\&=\overline{\lim_{R\to\infty}}(\log R)^{-1}\pi\int_{\tilde{B}_2\cap[1, R]}\frac{dt}{tl_2(t)}\\&=\overline{\lim_{R\to\infty}}(\log R)^{-1}\left(\pi\int_1^R\frac{dt}{tl_2(t)}-\frac{1}{2}\int_{B_2\cap[1, R]}dt/t\right)\\&\geqq\alpha-(\varepsilon/2)\rho(u).\end{aligned} \tag{3.13}$$

While by (2.2) we have

$$\overline{\lim_{R\to\infty}}(\log R)^{-1}\int_{B_1\cap[1, R]}dt/t\leqq\varepsilon\rho(E), \tag{3.14}$$

and by (3.8), (2.1), (3.11) and (3.14) we obtain

(3.15)
$$\mu(A)=\mu(B)\geqq\mu(u)\geqq\varliminf_{R\to\infty}(\log R)^{-1}\pi\int_1^R\frac{dt}{t\theta^*_{\mathfrak{D}_1}(t)}$$
$$=\varliminf_{R\to\infty}(\log R)^{-1}\left(\pi\int_1^R\frac{dt}{tl_1(t)}-\frac{1}{2}\int_{B_1\cap[1,R]}dt/t\right)$$
$$\geqq\frac{\alpha}{(2+\varepsilon)\alpha-1}-(\varepsilon/2)\rho(E).$$

Inequalities (3.13) and (3.15) give then

$$\mu(A)\geqq\frac{\rho(E)+(\varepsilon/2)\rho(u)}{(2+\varepsilon)\{\rho(E)+(\varepsilon/2)\rho(u)\}-1}-(\varepsilon/2)\rho(E).$$

Since ε was arbitrarily chosen and $\rho(u)\leqq\rho(E)$ we obtain the inequality

$$\mu(A)\geqq\frac{\rho(E)}{2\rho(E)-1}$$

and thus

$$\mu(A)^{-1}+\rho(E)^{-1}\leqq 2. \tag{3.16}$$

On the other hand, (3.6) implies easily

$$m(r, A)=m(r, 1/E)+O(\log r) \qquad (r\to\infty)$$

and therefore

$$T(r, A)+N(r, 1/E)=T(r, E)+O(\log r) \qquad (r\to\infty).$$

This shows $\rho(A)\leqq\rho(E)$ as previously mentioned. Since $\rho(E)\leqq 1$, (3.16) implies $\mu(A)\geqq 1$ and thus $\mu(A)=\rho(A)=\rho(E)=1$. By interchanging the roles of E and (u rather than) A the above discussion yields

$$\mu(E)^{-1}+\rho(A)^{-1}\leqq 2,$$

and thus $\mu(E)=1$. Hence we obtain the case b) and the proof of Theorem is completed.

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