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ON ELEMENTARY TRANSVERSELY AFFINE FOLIATIONS II

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§1. Introduction and statement of the results

In this second part of a series of papers, we study when a set of compact leaves in an elementary transversely affine foliation can be erased. Refer to [Ts] for notations and some basic properties on elementary transversely affine foliations.

Let M be a closed C^∞-manifold and $\mathcal{F}$ a codimension one, elementary transversely affie foliation on M. Let U be a connected component of $M-\cup\{K|K$ is a compact leaf of $\mathcal{F}\}$. Note that the boundary of $\bar{U}$, denoted by $\partial\bar{U}$, consists of compact leaves of $\mathcal{F}$. We say that the union of compact leaves $\partial\bar{U}$ can be *erased* if there are transversely affine foliation $\mathcal{F}'$ on M and an open neighbourhood $N(U)$ of $\bar{U}$ which satisfy following conditions:

(1) The holonomy homomorphism of F' coincides with that of $\mathcal{F}$.

(2) The restricted foliations $\mathcal{F}'_{|M-N(U)}$ and $F_{|M-N(U)}$ are diffeomorphic.

(3) The restricted foliation $\mathcal{F}'_{|N(U)}$ does not have a compact leaf.

We state the main theorem of this paper.

MAIN THEOREM. *Let M be a closed C^∞-manifold. Let $\mathcal{F}$ be a transversely oriented, codimension one, elementary transversely affine foliation on M. Let $\varphi: M\times\boldsymbol{R}\to M$ be a non-singular flow transverse to $\mathcal{F}$. Let U be a connected component of $M-\cup\{K|K$ is a compact leaf of $\mathcal{F}\}$. Assume $\partial\bar{U}\neq\emptyset$.*

Then the union of compact leaves $\partial\bar{U}$ can be erased if and only if the following condition (c) is satisfied:

(c) *There exists a C^∞-function $a:\partial\bar{U}\to\boldsymbol{R}^+$ such that the imbedding $x\mapsto\varphi(x, a(x))$ of $\partial\bar{U}$ into M is transverse to F.*

If the dimension of M is not greater than 3, it is seen that the condition (c) is always satisfied. So we get the following corollaries.

COROLLARY 1. *Let M be a closed C^∞-manifold of dimension* 2 *or* 3. *Let $\mathcal{F}$ be a transversely oriented, codimension one, elementary transversely affine foliation on M. Then there exists a foliation $\mathcal{F}'$ on M which is defined by a non-singular*

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closed one form.

COROLLARY 2. *Let $(M, \mathcal{F})$ be as in corollary* 1. *Then the manifold M fibers over the circle S^1.*

These corollaries were obtained independently by Meigniez [M]. An example due to Ghys is also stated in [M] which shows that the corollaries 1 and 2 are not true if the dimension of M is greater than 3.

In § 2, we prove the "if" part of the main theorem. In § 3, we prove the "only if" part of the main theorem and corollaries. In § 4, we remark that the main theorem and the corollaries are essentially true for topological transversely affine foliations.

§ 2. Proof of the main theorem

In this section we prove the "if" part of the main theorem. The "only if" part is proved in § 3.

Let $(M, \mathcal{F})$ and φ be as in the main theorem. Let $\pi : \tilde{M} \to M$ be the universal covering of M. Let $D : \tilde{M} \to \boldsymbol{R}$ be the developing submersion and $h : \pi_1(M) \to \mathrm{Aff}^+(\boldsymbol{R})$ the holonomy homomorphism defining the transversely affine structure of $\mathcal{F}$. Since $\partial \bar{U} \neq \emptyset$, the image $h(\pi_1(M))$ is abelian and fixes a point $t_0 \in R$. Passing to an equivalent structure if necessary, we may assume that $t_0 = 0$ and $h(\pi_1(M)) \subset GL^+(1, \boldsymbol{R})$. Then we have $D(\pi^{-1}(\partial \bar{U})) = 0$. Reversing the transverse orientation of $\mathcal{F}$ and the orientation of φ if necessary, we may assume that $D(\pi^{-1}(U)) \subset (-\infty, 0)$ and the flow φ is directed outward on all the compact leaves in $\partial \bar{U}$. Note in particular that U coincides with the interior of $\bar{U}$ (see [In] and [Ts]).

Let K be a compact leaf in $\partial \bar{U}$. Let $N(K)$ be the φ-saturation of K. Let $h_K : \pi_1(K) \to GL^+(1, \boldsymbol{R})$ be the composition; $h_K = h \circ \iota_{K\#}$ where $\iota_{K\#} : \pi_1(K) \to \pi_1(M)$ is the homomorphism induced by the inclusion map $\iota_K : K \to M$.

(2.1) LEMMA. *The restrictions of $\mathcal{F}$ and φ on $N(K)$ define a locally trivial foliated $\boldsymbol{R}$-bundle structure ou $N(K)$ whose total holonomy homomorphism coincides with the map h_K.*

Proof. Since φ is pointing outward on all the compact leaves in $\partial \bar{U}$, the map $\varphi_K : K \times \boldsymbol{R} \to N(K)$ obtained by restricting the flow φ is a diffeomorphism. Consider the foliation $\varphi_K^*(\mathcal{F})$ on $K \times \boldsymbol{R}$. It is transverse to the $\boldsymbol{R}$-factor and it has $K \times \{0\}$ as a compact leaf. Thus the foliation $\varphi_K^*(\mathcal{F})$ is a foliated $\boldsymbol{R}$-bundle over K. On the other hand, since $\mathcal{F}$ is elementary, it is almost without holonomy. By a theorem of Imanishi on without holonomy foliations [Im, Theorem 3.1], any path in K can be lifted to any preassigned leaf of $\varphi_K^*(\mathcal{F})$. This is equivalent to saying that the foliated $\boldsymbol{R}$-bundle $\varphi_K^*(\mathcal{F})$ is locally trivial.

Finally, it follows from the definition of transversely affine structures that the total holonomy homomorphism of $\varphi_K^*(F)$ coincides with the map h_K. The lemma is proved.

Let $\tilde{\mathfrak{F}}$ and $\tilde{\varphi}$ be the lifts of $\mathfrak{F}$ and φ to $\tilde{M}$ respectively. Let K and $N(K)$ be as in (2.1). Let $\tilde{K}$ be a connected component of $\pi^{-1}(K)$ and $N(\tilde{K})$ the $\tilde{\varphi}$-saturation of $\tilde{K}$. Note that $N(\tilde{K})$ coincides with the connected component of $\pi^{-1}(N(K))$ containing $\tilde{K}$. Since the foliation $\tilde{\mathfrak{F}}$ is defined by the developing submersion D, we get the following corollary to (2.1).

(2.2) COROLLARY. *There is a diffeomorphism $F_{\tilde{K}}: \tilde{K}\times \boldsymbol{R}\to N(\tilde{K})$ which satisfies the following conditions:*

(1) *For each $\tilde{x}\in\tilde{K}$, the image $F_{\tilde{K}}(\{\tilde{x}\}\times\boldsymbol{R})$ coincides with the flowline of $\tilde{\varphi}$ passing through $\tilde{x}$.*

(2) *For each $\tilde{x}\in\tilde{K}$ and $t\in\boldsymbol{R}$, we have $D\circ F_{\tilde{K}}(\tilde{x}, t)=t$.*

(2.3) LEMMA. *Suppose that the condition (c) of the main theorem is satisfied. Then there is a C^∞-submersion $f_{\tilde{K}}: \tilde{K}\times\boldsymbol{R}\to\boldsymbol{R}$ which satisfies the following conditions:*

(1) *The image of $f_{\tilde{K}}$ is contained in $(0, \infty)$.*

(2) *There is a smooth function $c: \tilde{K}\to\boldsymbol{R}^+$ such that $f_{\tilde{K}}(\tilde{x}, t)=|t|$ for $|t|\geqq c(\tilde{x})$.*

Proof. Let $a: \partial\bar{U}\to\boldsymbol{R}^+$ be the function given in the condition (c) of the main theorem. Let $b: \tilde{K}\to\boldsymbol{R}^+$ be the function determined by the following formula;

$$\tilde{\varphi}(\tilde{x}, a(\pi(\tilde{x})))=F_{\tilde{K}}(\tilde{x}, b(\tilde{x})), \qquad \text{where } \tilde{x}\in\tilde{K}.$$

Note that the function b is a submersion by the condition (c). Put $c(\tilde{x})=2\cdot b(\tilde{x})$.

Choose a C^∞-function $\phi: \tilde{K}\times\boldsymbol{R}\to\boldsymbol{R}$ such that

(1) $\phi(\tilde{x}, t)=\phi(\tilde{x}, -t)$, for each $\tilde{x}\in\tilde{K}$ and $t\in\boldsymbol{R}$,

(2) $\phi(\tilde{x}, t)$, is monotone increasing in t when $t\in(0, \infty)$

(3) $\phi(\tilde{x}, t)=0$, if and only if $0\leqq t\leqq b(\tilde{x})$, and

(4) $\phi(\tilde{x}, t)=1$, if and only if $c(\tilde{x})\leqq t$.

Define $f_{\tilde{K}}: \tilde{K}\times\boldsymbol{R}\to\boldsymbol{R}$ by

$$f_{\tilde{K}}(\tilde{x}, t)=|t|\cdot\phi(\tilde{x}, t)+b(\tilde{x})\cdot\{1-\phi(\tilde{x}, t)\}.$$

Then it is obvious that the map $f_{\tilde{K}}$ satisfies the conditions (1) and (2) of the lemma.

We shall check that the map $f_{\tilde{K}}$ is a submersion.

Let $(\tilde{x}, t)\in\tilde{K}\times\boldsymbol{R}$ be such that $|t|\geqq c(\tilde{x})$. Then we have $f_{\tilde{K}}(\tilde{x}, t)=|t|$, and the differential $Df_{\tilde{K}}$ of the map $f_{\tilde{K}}$ is surjective at $(\tilde{x}, t)$.

Let $(\tilde{x}, t)\in\tilde{K}\times\boldsymbol{R}$ be such that $|t|\leqq b(\tilde{x})$. Then we have $f_{\tilde{K}}(\tilde{x}, t)=b(\tilde{x})$, and the differential $Df_{\tilde{K}}$ is surjective at $(\tilde{x}, t)$.

Let $(\tilde{x}, t)\in\tilde{K}\times\boldsymbol{R}$ be such that $b(\tilde{x})<t<c(\tilde{x})$. Then we have

$$\frac{\partial f_{\tilde{K}}}{\partial t}(\tilde{x}, t)=\phi(\tilde{x}, t)-\{b(\tilde{x})-t\}\frac{\partial\phi}{\partial t}(\tilde{x}, t)>0\,.$$

Thus the differential $Df_{\tilde{K}}$ is surjective at $(\tilde{x}, t)$.

Finally, let $(\tilde{x}, t)\in\tilde{K}\times R$ be such that $-c(\tilde{x})<t<-b(\tilde{x})$. Then we have

$$\frac{\partial f_{\tilde{K}}}{\partial t}(\tilde{x}, t)=-\phi(\tilde{x}, t)-\{b(\tilde{x})-t\}\frac{\partial\phi}{\partial t}(\tilde{x}, t)<0\,.$$

Thus the differential $Df_{\tilde{K}}$ is surjective at $(\tilde{x}, t)$.

We have proved the lemma.

Now we can prove the "if" part of the main theorem.

Proof of the "if" part of the main theorem: For a compact leaf K in $\partial\bar{U}$, let $N(K)$ be the φ-saturation of K. We shall modify the developing submersion $D:\tilde{M}\to\boldsymbol{R}$ in $\cup\{\pi^{-1}((N(K)))\,|\,K$ is a compact leaf in $\partial\bar{U}\}$ and get a new developing map D'.

A connected component of $\pi^{-1}((N(K)))$ has the form $N(\tilde{K})$ where $\tilde{K}$ is a connected component of $\pi^{-1}(K)$ and $N(\tilde{K})$ is the $\tilde{\varphi}$-saturation of $\tilde{K}$. Define $D'_{\tilde{K}}: N(\tilde{K})\to\boldsymbol{R}$ by $D'_{\tilde{K}}=f_{\tilde{K}}\circ F_{\tilde{K}}^{-1}$ where $f_{\tilde{K}}$ and $F_{\tilde{K}}$ are maps obtained in (2.2) and (2.3).

Define $D': \tilde{M}\to\boldsymbol{R}$ by

$$D'=\begin{cases} D & \text{on } \tilde{M}-\cup\{\pi^{-1}((N(K)))\,|\,K \text{ is a compact leaf in } \partial\bar{U}\}-\pi^{-1}(U) \\ -D & \text{on } \pi^{-1}(U)-\cup\{\pi^{-1}((N(K)))\,|\,K \text{ is a compact leaf in } \partial\bar{U}\} \\ D'_{\tilde{K}} & \text{on } N(\tilde{K}), \text{ where } K \text{ is a compact leaf in } \partial\bar{U} \text{ and } \tilde{K} \text{ is a lift of } K. \end{cases}$$

By (2.2) (2) and (2.3) (2), the map D' is a well-defined smooth map. By (2.3), the map D' is a submersion. By the definition, the map D' is h-equivariant.

Let $N(U)=\cup\{N(K)\,|\,K$ is a compact leaf in $\partial\bar{U}\}\cup U$. Then, by (2.2) and (2.3), the map D' satisfies the following conditions:

(1) $D'=D$ on $\pi^{-1}(M-N(U))$, and

(2) $D'(\pi^{-1}(N(U)))\subset(0, \infty)$.

Let $\mathcal{F}'$ be the underlying foliation of the transversely affine structure (D', h). It is easy to see from (1) and (2) above that $\mathcal{F}'$ satisfies the conditians (1), (2) and (3) in § 1. Thus we have proved the "if" part of the main theorem.

§ 3. Proof of corollaries

Let $(M, \mathcal{F})$ and U be as in the main theorem. As in the first paragraph of § 2, we may and do assume that $h(\pi_1(M))\subset GL^+(1, \boldsymbol{R})$ and $D(\pi^{-1}(\partial\bar{U}))=0$. Let K be a compact leaf in $\partial\bar{U}$. Let $h_K: \pi_1(K)\to GL^+(1, \boldsymbol{R})$ be the composition $h_K = h\circ\iota_{K\#}$ where $\iota_{K\#}: \pi_1(K)\to\pi_1(M)$ is the homomorphism induced by the inclusion map $\iota_K: K\to M$. Let $(E_{h_K}, \mathcal{F}_{h_K})$ be the foliated $\boldsymbol{R}$-bundle over K defined by the representation h_K.

(3.1) LEMMA. *The condition* (*c*) *of the main theorem is equivalent to either of the following conditions* (*c'*) *or* (*c''*).

(*c'*) *For each compact leaf K in $\partial\bar{U}$, there exists a non-zero section* s_K *of* $(E_{h_K}, \mathcal{F}_{h_K})$ *transverse to* $\mathcal{F}_{h_K}$.

(*c''*) *For each compact leaf K in $\partial\bar{U}$, the cohomology class ρ_K of the map* $\log\circ h_K: \pi_1(K)\to\boldsymbol{R}$ *is represented by a non-singular closed one form on K.*

Proof. By (2.1), a neighbournood of K is foliation preservingly diffeomorphic to E_{h_K}. The imbedding $x\mapsto\varphi(x, a(x))$ of K into M as in (c) corresponds to a non-zero transverse section to E_{h_K} via this diffeomorphism. This proves the equivalence of (c) and (c').

We prove the equivalence of (c') and (c''). Let ω_1 be a closed one form on K which represents the cohomology class ρ_K. Let $\pi: \tilde{K}\to K$ be the universal covering of K. Since $\pi^*(\omega_1)$ is exact, there is a function $g: \tilde{K}\to\boldsymbol{R}^+$ such that $\pi^*(\omega_1)=d\log(g)$. Consider the function $\tilde{f}: \tilde{K}\times\boldsymbol{R}\to\boldsymbol{R}$ defined by $\tilde{f}(x, t)=t/g(x)$. Since g is h-equivariant, the function $\tilde{f}$ induces a function $f: E_{h_K}\to\boldsymbol{R}$. Similarly, the non-singular one form $\tilde{\Omega}=dt/g$ on $\tilde{K}\times\boldsymbol{R}$ induces a non-singular one form Ω on E_{h_K}. We have the relation $\Omega=df+f\cdot\omega_1$.

Now assume (c'). Let s_K be a non-zero transverse section of the foliated bundle E_{h_K}. Then the form $\omega=\dfrac{s_K^*\Omega}{f\circ s_K}=\dfrac{d(f\circ s_K)}{f\circ s_K}+\omega_1$ is a non-singular closed one form on K which belongs to the class ρ_K. Thus we have (c'').

Conversely, assume (c'') and choose a non-singular closed one form ω whose cohomology class is ρ_K. Define a function $r: \tilde{K}\to\boldsymbol{R}^+$ by $r(\tilde{x})=\exp\left(\int_{\tilde{x}_0}^{\tilde{x}}\pi^*\omega\right)$ where $\tilde{x}_0$ is a base point of $\tilde{K}$ and the integral is taken along some path connecting $\tilde{x}_0$ to $\tilde{x}$. Since the function r is h-equivariant, the map $\tilde{s}: \tilde{K}\to\tilde{K}\times\boldsymbol{R}$, $s(\tilde{x})=(\tilde{x}, r(\tilde{x}))$ projects down to a section s of E_{h_K}. Clearly, the section s is non-zero and is transverse to F_{h_K}. So we get (c'). The lemma is proved.

Now we can prove the "only if" part of the main theorem.

Proof of the "only if" part of the main theorem: Let $(M, \mathcal{F})$ and U be as in the main theorem. As usual, we assume $D(\pi^{-1}(U))=0$ and $h(\pi_1(M))\subset GL^+(1, \boldsymbol{R})$. Suppose $\partial\bar{U}$ can be erased. Then there exists an h-equivariant submersion $D': \tilde{M}\to\boldsymbol{R}$ such that $D'=D$ on $\pi^{-1}(M-N(U))$ and $D'(\pi^{-1}(N(U)\subset \boldsymbol{R}-\{0\}$, where $N(U)$ is an open neighbourhood of $\bar{U}$. Let K be a compact leaf in $\partial\bar{U}$ and $\tilde{K}$ its lift to $\tilde{M}$. We use the notations of (3.1). As in the last paragraph of the proof of (3.1), the map $s: \tilde{K}\to\tilde{K}\times\boldsymbol{R}$, $s(\tilde{x})=(\tilde{x}, D'(\tilde{x}))$ defines a non-zero section of E_{h_K} transverse to $\mathcal{F}_{h_K}$. So the condition (c') is satisfied. By (3.1), the condition (c) is satisfied. This completes the proof of the main theorem.

(3.2) LEMMA. *Let M be a closed 3-manifold and F a codimension-one, transversely oriented, elementary transversely affine foliation on M. Then each compact leaf of F is diffeomorphic to the torus T^2.*

Proof. Let U be a connected component of $M-\cup\{K|K$ is a compact leaf of $\mathcal{F}\}$. Then the transverse orientation of F is directed either simultaneously outward on all the compact leaves in $\partial\bar{U}$ or simultaneously inward on them [In]. Then, from a theorem of Goodman [G], it follows that each compact leaf in $\partial\bar{U}$ is diffeomorphic to a torus. The lemma is proved.

With these preliminaries, we can prove the corollaries.

Proof of Corollary 1: Let $(M, \mathcal{F})$ be as in Corollary 1. If the dimension of M is 2, then each compact leaf of $\mathcal{F}$ is a circle. If the dimension of M is 3, then, by (3.2), each compact leaf of $\mathcal{F}$ is diffeomorphic to a torus. In either case, each cohomology class of a compact leaf is represented by a harmonic one form. Thus the condition (c'') of (3.1) is always satisfied. So, by (3.1), the condition (c) of the main theorem is satisfied. By a repeated application of the main theorem, we can erase all the isolated compact leaves of $\mathcal{F}$ to get an elementary transversely affine foliation $\mathcal{F}'$ on M which is either without compact leaves or is a bundle foliation. The foliation $\mathcal{F}'$ is defined by a non-singular closed one form (see [B]). The corollary is proved.

Proof of Corollary 2: This follows directly from Corollary 1 and the Tischler fibration theorem [Ti].

§ 4. Remark

Although we have treated smooth transversely affine foliations, the main theorem and the corollary 2 are literally true for *topological* transversely affine foliations (see [Ts]). The Corollary 1 is true in the following sense.

COROLLARY 1′. *Let M be a closed C^∞-manifold of dimension* 2 *or* 3. *Let* $\mathcal{F}$

be a transversely oriented, codimension one, elementary topological transversely affine foliation on M. Then there is a foliation $\mathcal{F}'$ *on M which is without holonomy.*

REFERENCES

[Bo] BOBO SEKE, Sur les structures transversalment affines des feuilletages des codimension un, Ann. Inst. Fourier, Grenoble **30** (1980), 1-29.

[G] S. GOODMAN, Closed leaves in foliations of codimension one, Comment. Math. Helvetici **50** (1975), 383-388.

[Im] H. IMANISHI, On the theorem of Denjoy-Sacksteder for codimension one foliations without holonomy, J. Math. Kyoto Univ. **14** (1974), 607-634.

[In] T. INABA, Resilient leaves in transversely affine foliation, Tôhoku Math. J. **41** (1989), 625-631.

[M] G. MEIGNIEZ, Actions de groupes sur la droite et feuilletage de codimension un, Thèse, l'Université Claude Bernard-Lyon 1, 1988.

[Ti] D. TISCHLER, On fibering certain manifolds over S^1, Topology **9** (1970), 153-154.

[Ts] N. TSUCHIYA, On elementary transversely affine foliations I, to appear in Kodai Math. J.

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