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ON A PROBLEM OF HAYMAN

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I. Introduction

Let $f(z)$ be meromorphic in the complex plane. We will use the following standard notations of Nevanlinna theory,

$$T(r, f),\quad m(r, f),\quad N(r, f),\quad \bar{N}(r, f),\quad S(r, f), \cdots$$

(see Hayman [3]).

A meromorphic function $a(z)$ is said to be a small function related to f if

$$T(r, a)=S(r, f).$$

Hayman [2] proved the following result:

THEOREM A. *If k is a positive integer and $f(z)$ is a transcendental meromorphic function in the complex plane, then*

$$T(r, f)<\left(2+\frac{1}{k}\right)N\left(r, \frac{1}{f}\right)+\left(2+\frac{2}{k}\right)\bar{N}\left(r, \frac{1}{f^{(k)}-1}\right)+S(r, f).$$

Hayman [3, p. 73] asked whether the coefficients of $N(r, 1/f)$ and $\bar{N}(r, 1/f^{(k)}-1)$ are best possible, where $\bar{N}(r, 1/f^{(k)}-1)$ is the counting function of the roots of $f^{(k)}-1=0$ in $|z|\leqq r$, multiple roots been counted once.

Concerning this problem, Frank and Hennekemper [1] proved the following:

THEOREM B. *Let $f(z)$ be a meromorphic function which has only simple poles $k\geqq 2$, $c\in \mathbf{C}\setminus\{0\}$, $f\not\equiv$constant and $f^{(k)}-c\not\equiv 0$. Then*

$$T(r, f)\leqq N\left(r, \frac{1}{f}\right)+\left(1+\frac{2}{k-1}\right)\bar{N}\left(r, \frac{1}{f^{(k)}-c}\right)+S(r, f).$$

In this paper, we shall prove tho following result:

THEOREM 1. *Suppose that $f(z)$ is transcendental and meromorphic in the complex plane, and that k is a positive integer. If $p(z)$ is a nonzero polynomial or a nonzero constant, then for any $\varepsilon>0$, we have*

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$$T(r, f) \leqq \left(1+\frac{1}{k}+\varepsilon\right)\left\{N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f^{(k)}-p}\right)\right\}+S(r, f).$$

Remark. Here $S(r, f)$ depends on $\varepsilon>0$, but the associated exceptional set is independent of ε.

THEOREM 2. *Let $f(z)$ be a nonconstant rational function, and let k be a positive integer. If $c \in \boldsymbol{C} \setminus \{0\}$ and $f^{(k)}-c \not\equiv 0$, then we have*

$$T(r, f) \leqq N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f^{(k)}-c}\right)+O(1).$$

2. Some lemmas

In our first lemma we recall some of the basic relations of Nevanlinna theory.

LEMMA 1 ([3]). *Suppose that f and g are nonzero meromorphic functions in the plane. Then for any positive integer i, we have*

$$m(r, f^{(i)}/f)=S(r, f), \tag{1}$$

$$T(r, f^{(i)}) \leqq (i+1)T(r, f)+S(r, f). \tag{2}$$

In addition

$$N\left(r, \frac{f}{g}\right)-N\left(r, \frac{g}{f}\right)=N(r, f)+N\left(r, \frac{1}{g}\right)-N(r, g)-N\left(r, \frac{1}{f}\right). \tag{3}$$

LEMMA 2 (Steinmetz [4, Theorem 1]). *Let the linear differential operator*

$$L(y)=y^{(q)}+a_{q-1}(z)y^{(q-1)}+ \cdots a_0(z)y$$

have rational coefficients $a_0, \cdots, a_{q-1}$ and let f be a transcendental meromorphic function in the plane. Then either f is a rational function of a (local) fundamental set $y_1, \cdots, y_q$ of the differential equation $L(y)=0$ or inequality

$$m\left(r, \frac{1}{L(f)}\right) \leqq m(r, L(f))+(1+\eta)N(r, f)+S(r, f)$$

holds for every $\eta>0$.

LEMMA 3. *Suppose that $f(z)$ is meromorphic in $\boldsymbol{C}$, and that $f^{(k)}(z)$ is nonconstant. Then for any small function $a(z)$ related to f ($a \not\equiv 0, \infty$), we have*

$$T(r, f) \leqq \bar{N}(r, f)+N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f^{(k)}-a}\right)$$
$$-N\left(r, \frac{1}{af^{(k+1)}-a'f^{(k)}}\right)+S(r, f).$$

Proof. From the identity

$$\frac{1}{f}=\frac{1}{a}\left\{\frac{f^{(k)}}{f}-\left(a\frac{f^{(k+1)}}{f}-a'\frac{f^{(k)}}{f}\right)\frac{f^{(k)}-a}{af^{(k+1)}-a'f^{(k)}}\right\}$$

and Lemma 1 and $T(r, a)=S(r, f)$ it follows that

$$\begin{aligned}
m\left(r, \frac{1}{f}\right) &\leqq m\left(r, \frac{f^{(k)}-a}{af^{(k+1)}-a'f^{(k)}}\right)+S(r, f) \\
&= m\left(r, \frac{af^{(k+1)}-a'f^{(k)}}{f^{(k)}-a}\right)+N\left(r, \frac{af^{(k+1)}-a'f^{(k)}}{f^{(k)}-a}\right) \\
&\quad -N\left(r, \frac{f^{(k)}-a}{af^{(k+1)}-a'f^{(k)}}\right)+S(r, f) \\
&= m\left(r, a\left(\frac{f^{(k)}}{a}-1\right)' \Big/ \left(\frac{f^{(k)}}{a}-1\right)\right)+N(r, af^{(k+1)}-a'f^{(k)}) \\
&\quad +N\left(r, \frac{1}{f^{(k)}-a}\right)-N(r, f^{(k)}-a)-N\left(r, \frac{1}{af^{(k+1)}-a'f^{(k)}}\right) \\
&\quad +S(r, f) \\
&\leqq \overline{N}(r, f)+N\left(r, \frac{1}{f^{(k)}-a}\right)-N\left(r, \frac{1}{af^{(k+1)}-a'f^{(k)}}\right) \\
&\quad +S(r, f^{(k)})+S(r, f).
\end{aligned} \tag{4}$$

Now from (2) we have

$$S(r, f^{(k)})=S(r, f). \tag{5}$$

The conclusion follows from (4), (5) and $T(r, f)=m(r, 1/f)+N(r, 1/f)+O(1)$.

LEMMA 4. *Let $t\geqq 2$ be an arbitrary integer. Suppose that $f(z)$ is transcendental and meromorphic in the complex plane, and that $q(z)$ is a nonzero polynomial. Then for any $\eta>0$ we have*

$$t\overline{N}(r, f)\leqq N\left(r, \frac{1}{qf^{(t)}-q'f^{(t-1)}}\right)+(1+\eta)N(r, f)+S(r, f).$$

Proof. Let $h(z)$ be a solution of the linear differential equation

$$L(y)=0, \tag{6}$$

where

$$L(y)=qy^{(t)}-q'y^{(t-1)}. \tag{7}$$

If $h^{(t-1)}\not\equiv 0$, then from (6) and (7) we deduce that

$$h^{(t)}/h^{(t-1)}=q'/q.$$

Thus there exists a nonzero constant c such that

$$h^{(t-1)}=cq\,,$$

which gives

$$h(z)=q^*(z)\,,$$

where $q^*(z)$ is a polynomial of degree $\deg(q)+t-1$.

If $h^{(t-1)}\equiv 0$, then $h(z)$ is a polynomial of degree $t-2$ or less. Let

$$h_t(z)=q^*(z)\,,\qquad h_j(z)=z^{j-1}\qquad (j=1,\cdots,t-1)\,.$$

Then $\{h_1(z),\cdots,h_t(z)\}$ is a (local) fundamental solution set of $L(y)=0$. Since $f(z)$ is transcendental, the solutions $h_i(z)$ $(i=1,\cdots,t)$ are small functions related to f. Thus, by Lemma 2, $L(f)\not\equiv 0$ and

$$m\Big(r,\frac{1}{L(f)}\Big)\leqq m(r,\,L(f))+(2+\eta)N(r,\,f)+S(r,\,f) \tag{8}$$

It follows form (8) and the first fundamental theorem [3, p. 5] that

$$\begin{aligned}N(r,\,L(f))&=T\Big(r,\frac{1}{L(f)}\Big)-m(r,\,L(f))+O(1)\\&\leqq N\Big(r,\frac{1}{L(f)}\Big)+(2+\eta)N(r,\,f)+S(r,\,f)\,.\end{aligned} \tag{9}$$

It is easy to verify that

$$\begin{aligned}N(r,\,L(f))&=N(r,\,qf^{(t)}-q'f^{(t-1)})\\&\geqq N(r,\,f)+t\overline{N}(r,\,f)-O(\log r)\,.\end{aligned}$$

Lemma 3 follows from this and (9).

3. Proof of Theorem 1

Applying Lemma 4 to $t=k+1$, $\eta=(\varepsilon k^2/k+\varepsilon k+1)$ and $q=p$ we have

$$\begin{aligned}\overline{N}(r,\,f)&\leqq\frac{1}{k+1}N\Big(r,\frac{1}{pf^{(k+1)}-p'f^{(k)}}\Big)+\Big(1-\frac{k}{k+\varepsilon k+1}\Big)N(r,\,f)+S(r,\,f)\\&\leqq\frac{1}{k+1}N\Big(r,\frac{1}{pf^{(k+1)}-p'f^{(k)}}\Big)+\frac{\varepsilon k+1}{k+\varepsilon k+1}T(r,\,f)+S(r,\,f)\,.\end{aligned} \tag{10}$$

On the other hand, Lemma 3 gives

$$\begin{aligned}T(r,\,f)<\overline{N}(r,\,f)&+N\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{f^{(k)}-p}\Big)\\&-N\Big(r,\frac{1}{pf^{(k+1)}-p'f^{(k)}}\Big)+S(r,\,f)\,.\end{aligned}$$

Combining this with (10) we derive that

$$T(r, f) \leqq \left(1+\frac{1}{k}+\varepsilon\right)\left\{N\left(r, \frac{1}{f}\right)+N\left(r, \frac{1}{f^{(k)}-p}\right)\right\}$$

$$-\left(1-\frac{\varepsilon}{k+1}+\varepsilon\right)N\left(r, \frac{1}{pf^{(k+1)}-p'f^{(k)}}\right)+S(r, f).$$

This is what we need.

Remark 1. By a simple calculation and using Example (i) in [3, p. 6] we can prove Theorem 2.

Remark 2. Since writing this paper I have learned (through Professor Yuzan He and correspondence) of progress made by Lo Yang, where Yang proved a result which is similar to Theorem 1 for constant p. I wish to thank both for their comments.

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