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CERTAIN ESTIMATES ON KLEINIAN GROUPS BY THE CORE OF THEIR QUOTIENT 3-MANIFOLD

Dedicated to Professor Tatsuo Fuji'i'e on his second birthday in the Chinese calendar

BY KATSUHIKO MATSUZAKI

Introduction

Kulkarni and Shalen [6] explained topologically Ahlfors' finiteness theorem and its two major supplements, the area-inequalities [2] and the finiteness of the cusps [16], considering the core of the quotient 3-manifold of finitely generated torsion-free Kleinian groups. As a consequence, they obtained the sharp inequalities on the number of cusps.

In this article we treat chiefly another result from the finiteness theorem. It is that every component subgroup of a finitely generated Kleinian group is also finitely generated. This assertion is qualitative. So we will estimate quantitatively the minimal number of generators of component subgroups. Our consequence (§ 2) is;

THEOREM. *Let G be a torsion-free Kleinian group with r generators. Then for any component subgroup H of G, the minimal number of generators of H is not more than* $2r-1$.

Moreover we construct examples of Kleinian groups which attain the equalities in those estimates along the line from the finiteness theorem (§ 3).

Another part of this article is concerned with Ahlfors' measure zero problem (§ 4). Bonahon's theorem [3] shows us a sufficient condition for the limit set to be null on the 2-dimensional Lebesgue measure. With the aid of this result, we have;

THEOREM. *Let G be a torsion-free Kleinian group with r generators. If* {the hyperbolic area of $\Omega(G)/G\}\geqq 4\pi(r-2)$, *then the limit set $\Lambda(G)$ has measure zero.*

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§ 1. Definitions.

In this paper, *Kleinian groups are finitely generated and torsion-free discrete*

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subgroups of Möbius transformations of the second kind. Let G be a Kleinian group. We denote the region of discontinuity by $\Omega(G)$ and the limit set by $\Lambda(G)$. Its action on the Riemann sphere can be naturally extended to the upper half space H^3. Its quotient bordered 3-manifold is denoted by $M(G)$; that is, $M(G)=H^3\cup(\Omega)/G$. Let Δ be a connected component of $\Omega(G)$. Then we call the stabilizer of Δ in G the component subgroup for Δ. Consider the Riemann surface $\Delta/\mathrm{stab}(\Delta)$. It is a component of $\partial M(G)$. Ahlfors' finiteness theorem asserts that $\partial M(G)$ consists of a finite number of components and each of them is analytically finite.

Although $M(G)$ may not be compact in general, by Scott's theorem, there is a compact submanifold of $M(G)$ which is homotopy equivalent to $M(G)$. We call such a compact submanifold a core of $M(G)$. Kulkarni and Shalen's work was dene by constructing a core which contains the compact part of $\partial M(G)$ under some assumptions. Their argument was refined by McCullough to the following form.

PROPOSITION 1.1 ([14]). *Let Y be a 3-manifold with finitely generated fundamental group. Let C be a compact submanifold of ∂Y. Then there is a core X of Y such that $X\cap\partial Y=C$.*

A cusp in $M(G)$ is the projection of some region in $H^3\cup\Omega(G)$ associated with a parabolic fixed point (cf. [8, § 2]). It is homeomorphic to (cylinder)$\times$(0, 1) or (torus)$\times$(0, 1), according as the parabolic subgroup is isomorphic to $\boldsymbol{Z}$ or $\boldsymbol{Z}+\boldsymbol{Z}$. We call its border in $M(G)$ a cusp cylinder or cusp torus respectively.

By Proposition 1.1, we can take a core of $M(G)-\{\text{cuspidal parts}\}$, which is also a core of $M(G)$ so that it may contain all the $\partial M(G)$ except for small neighborhoods of punctures on $\partial M(G)$, all the cusp cylinders except for open ends, and all the cusp tori. The core which has such a property is denoted by $N=N(G)$. For the detailed construction, see [12]. The cuspidal parts in ∂N is denoted by $\partial_p N$. Therefore $\partial N-\partial_p N$ consists of topologically finite surfaces in ∂M and "degenerate" surfaces.

Notations.

$r:=$the rank of a Kleinian group, i.e. the minimal number of generators
$b:=\#\{\text{components of } \partial M \text{ or } \partial N-\partial_p N\}$
$\chi:=(-1)\times\{\text{Euler characteristic}\}$
$\beta_i:=$the i-th Betti number
$m:=$Lebesgue measure on $\boldsymbol{C}$.

§ 2. The ranks of component subgroups

We begin with starting a useful proposition for our estimates (cf. [4]).

PROPOSITION 2.1. *Let G be a Kleinian group with $r(G)=r$. Then the fol-*

lowings are equivalent:

(a) *G is free.*

(b) $\beta_1(M(G))=r$ *and* $\beta_2(M(G))=0$.

Proof. (a)→(b): Since $\pi_1(N(G))$ is free, we known $N(G)$ is a handlebody of genus r [5, Ch. 5]. Then $\beta_1(M(G))=\beta_1(N(G))=r$ and $\beta_2(M(G))=\beta_2(N(G))=0$. (b)→(a): Because $M(G)$ $(N(G))$ is aspherical [8, p. 388], the homology groups of G of coefficient $\boldsymbol{Z}$ with trivial action are isomorphic to the ordinary homology groups of $M(G)$ (cf. [5, p. 75]). That is, $H_i(G, \boldsymbol{Z})\cong H_i(M(G), \boldsymbol{Z})$ $(i=1, 2, \cdots)$. Since $M(G)$ is homotopy equivalent to a 2-dimensional CW-complex, $\beta_2(M(G))=0$ implies that $H_2(G, \boldsymbol{Z})\cong H_2(M(G), \boldsymbol{Z})=0$. Here we use the following lemma [15, p. 179]. "*Let G be a group such that* $H_2(G, \boldsymbol{Z})=0$. *Let* $\{x_j\}$ *be a set of elements of G whose images under the canonical epimorphism* $\varphi\colon G\to H_1(G, Z)$ *are linearly independant over* $\boldsymbol{Z}$. *Then* $\{x_j\}$ *is a basis of a free subgroup of G*". Let $\{x_j\}=\{x_1, \cdots, x_r\}$ be a minimal system of generators of G. To apply the lemma to our case, we have only to check that $\varphi(x_1), \cdots, \varphi(x_r)$ are linearly independent over $\boldsymbol{Z}$. It is easy: They generate $H_1(G, \boldsymbol{Z})$, for φ is surjective. And rank $H_1(G, \boldsymbol{Z})=\beta_1(M(G))=r$ means that they are linearly independent. □

Bers [2] proved that for a finitely generated Kleinian group G which may contain elliptic elements, the hyperbolic area of $\Omega(G)/G$ is bounded by $4\pi(r(G)-1)$. In the case without torsion, this theorem can be explained simply by using the core $N(G)$. Further, the condition which attain the equality is obtained. The result itself is originally due to Abikoff [1]:

PROPOSITION 2.2 (Area-inequalities). *Let G be a Kleinian group. Then,* $\chi(\partial M(G))\leqq\chi(\partial N(G))\leqq 2(r(G)-1)$. *The first equality holds if and only if G is geometrically finite. The second equality holds if and only if G is free.*

Proof. By Poincaré's duality theorem, we know $\chi(\partial N(G))=2\chi(N(G))$. Let $\beta_i=\beta_i(N(G))=\beta_i(M(G))$. Since $\beta_0=1$, $\beta_3=0$, $\beta_1\leqq r$ and $\beta_2\geqq 0$, it follows that $\chi(\partial M)\leqq\chi(\partial N)=2(\beta_1-\beta_2-1)\leqq 2(r-1)$. By Marden's characterization of geometric finiteness [8, Prop. 4.2], $\chi(\partial M(G))=\chi(\partial N(G))$ if and only if G is geometrically finite. $2(\beta_1-\beta_2-1)=2(r-1)$ if and only if $\beta_1=r$ and $\beta_2=0$. By Proposition 2.1, it is that G is free.

Now we prove our main theorem mentioned in the introduction.

THEOREM 2.3. *Let G be a non-elementary Kleinian group with* $r(G)=r$. *Then for any component subgroup H of G,* $r(H)\leqq 2r-1$. *The equality holds if and only if* (1) *G is geomertically finite and* $\partial M(G)$ *is connected,* (2) *G is free, and* (3) $\partial M(G)$ *is incompressible in* $M(G)$. *Further, there are examples for every* $r\geqq 2$, *which attain the equality.*

Remark. (3) is equivalent to the fact that all the components of $\Omega(G)$ are simply connected. $\partial M(G)$ is connected if and only if all the components of $\Omega(G)$ are conjugate to one another by G.

Proof. Let $H=\mathrm{stab}(\Delta)$ and $S=\Delta/H$. We consider two cases; S is not closed, or S is closed. (I) If S is not closed, then $r(H)\leqq r(\pi_1(S))=\chi(S)+1$. Since $\chi(S)\leqq\chi(\partial N)$ and $\chi(\partial N)\leqq 2(r-1)$ (Proposition 2.2), we see $r(H)\leqq 2r-1$. (II) If S is closed, then $r(H)\leqq r(\pi_1(S))=\chi(S)+2$. Here we may assume that G is not free, for if G is free and $\partial M(G)$ has a closed surface, G must be a Schottky group, hence $G=H$ and $r(H)=r$. By Proposition 2.1, it implies either $\beta_1<r$ or $\beta_2>0$ that G is not free, thus $\chi(\partial N)\leqq 2(r-2)$. In this case, $r(H)\leqq\chi(S)+2\leqq\chi(\partial N)+2\leqq 2r-2<2r-1$. Note that in (II), the equality $r(H)=2r-1$ cannot occur.

We investigate the condition for the equality. From Proposition 2.2, (1) is equivalent to the fact that $\chi(S)=\chi(\partial N)$, and (2) is $\chi(\partial N)=2(r-1)$. (3) is a sufficient condition for $r(\pi_1(S))=r(H)$, Thus the equality holds if (1), (2) and (3) are satisfied. Conversely if the equality holds, then by the above notice, S is not closed, i.e. only (I) is in our attention. For the equality, it is necessary that $r(H)=r(\pi_1(S))$, $\chi(S)=\chi(\partial N)$ and $\chi(\partial N)=2(r-1)$ are satisfied. From the last two equalities, (1) and (2) are derived. We have only to show (3). Assume that S is compressible. Then there are some relations in $\pi_1(S)$ by which H is the quotient group of $\pi_1(S)$. Here H is free, because G is free by (2) which has been already obtained. So H has less rank than $\pi_1(S)$ [7, Prop. 2.7]. From this argument, we know S must be incompressible for the equality.

The examples of sharpness for $r\geqq 2$ will be discussed in the following section (Example 2).

§ 3. Other inequalities and examples of sharpness

In this section we exhibit the examples of Kleinian groups which attain the equality in Theorem 2.3. Incidentally we also show that the area-inequalities (§ 2.) and the following two estimates are sharp. They are corresponding respectively to Theorems 2, 3 and 4 of Abikoff [1], which are shown under some assumption. (See also Marden [8, § 7].) The assumption can be removed by McCullough's work. For the convenience of the readers, we now sketch their proofs here.

PROPOSITION 3.1. *Let G be a non-elementary Kleinian group. Then, $b(\partial M(G))\leqq 2(r(G)-1)$. The equality holds if and only if G is geometrically finite, free and all the components of $\partial M(G)$ are thrice punctured spheres or once punctured tori.*

Proof. For each component S of $\partial M(G)$, $\chi(S)\geqq 1$. It follows that $b(\partial M(G))\leqq\chi(\partial M(G))$. The equality holds if and only if $\chi(S)=1$ for any S. Combining with Proposition 2.2, we have the assertion. □

PROPOSITION 3.2 (Finiteness of Cusps). *Let G be a non-elementary Kleinian group. Then,* # {conjugacy classes of parabolic fixed points}$\leqq 3(r(G)-1)$. (We count the fixed point of a parabolic abelian subgroup of rank 2 twice.) *The equality holds if and only if G is geometrically finite, free, and all the components of $\partial M(G)$ are thrice punctured spheres.*

Proof. Let $\{S_j\}$ $(j=1, 2, \cdots, b)$ be the components of $\partial N(G)-\partial_p N(G)$. Let g_j and n_j be the numbers of handles and holes of S_j respectively. Then we have $2(\beta_1-\beta_2-1)=\chi(\partial N)=\sum_{j=1}^{b}\chi(S_j)=\sum(2g_j-2+n_j)$, where $\beta_i=\beta_i(N(G))$. Therefore #{cusp cylinders}$=(1/2)\sum n_j=\beta_1-\beta_2-1-\sum(g_j-1)$. Here $-\sum(g_j-1)\leqq b\leqq 2(\beta_1-\beta_2-1)$ by the same reason as Proposition 3.1. So #{cusp cylinders}$\leqq 3(\beta_1-\beta_2-1)$. The equality is attained if and only if every S_j is a sphere with three holes.

On the other hand #{cusp tori}$\leqq\beta_2$, because they generate linearly independent elements in $H_2(N, \boldsymbol{Z})$. Hence, #{conjugacy classes of parabolic fixed points} = #{cusp cylinders}+2#{cusp tori} $\leqq 3(\beta_1-\beta_2-1)+2\beta_2=3(\beta_1-1)-\beta_2\leqq 3(r-1)$.

It is now clear that it attains the equality if the conditions stated in this proposition are satisfied. For the converse, we have only to remark that a sphere with three holes cannot be a degenerate surface. It is because a Fuchsian group of signature (0, 3) has rigidity [12, §5]. If every S_j is such a surface, then G is geometrically finite. □

All our examples are on boundaries of Schottky spaces. The theorem of Maskit on parabolic elements [11] guarantees that such groups are obtained by squeezing some simple disjoint primitive loops on the closed Riemann surfaces which bound the handlebodies.

Example 1. For every $r\geqq 2$, we construct a geometrically finite free Kleinian group G with $r(G)=r$ such that all the components of $\partial M(G)$ are thrice punctured spheres. This is a sharp example for Propositions 2.2, 3.1 and 3.2.

Take a Fuchsian group of signature $(0, r+1)$. If $r=2$, it is the example. So we assume $r\geqq 3$. On the boundary of its Teichmüller space, there is a terminal regular b-group which has $r-2$ accidental parabolic transformations (A.P.T.) [9]. Then on the quotient surface of its invariant component, we can choose $r-2$ non-trivial, non-mutually-homotopic disjoint simple primitive loops, which correspond to loxodromic elements (see Fig. 1). Then by Maskit's theorem, there is a geometrically finite Kleinian group on the boundary of the deformation space, such that all those loops become representing parabolic elements. It has $(r+1)+(r-2)+(r-2)=3r-3$ cusps, and all the components are thrice punctured spheres.

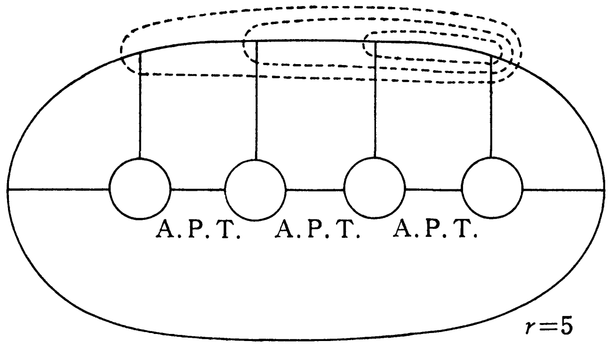

Fig. 1.

Example 2. First we consider the case $r=2$. Take two simple primitive loops on the border of a handlebody N of genus 2 as in Fig. 2. When we denote two standard generators of $\pi_1(N)$ by f and g, those two loops correspond to f and fg^2. We can easily see that they don't divide ∂N. Moreover $\partial N-\{\text{two loops}\}$ is incompressible in N. In fact, if not, there is a properly embedded disk in $N-\{\text{two loops}\}$ which induces a non-trivial free product decomposition $\pi_1(N)\cong G=\langle p\rangle*\langle q\rangle$, such that all the parabolic elements of G are in a conjugate of $\langle p\rangle$ or $\langle q\rangle$. This means that we can choose some conjugates of f and fg^2 as generators of G. But it is impossible, because every word constructed by the generators must have an even integer as the sum of powers of g, so they cannot generate g itself. For $r>2$, add appropriate $r-2$ cusp cylinders as in Fig. 2, so that $\partial N-\{\text{loops}\}$ may be connected and incompressible. Then by Maskit's theorem, we obtain the Kleinian groups which are examples of sharpness in Theorem 2.3 for $r\geqq 2$.

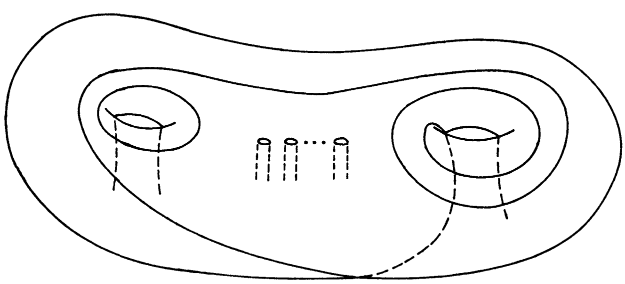

Fig. 2.

§ 4. The measure zero problem

The remainder of this paper is concerned with Ahlfors' conjecture; that is, finitely generated Kleinian groups have the measure zero limit set.

PROPOSITION 4.1. *Let G be a Kleinian group which is constructed from Kleinian groups satisfying the condition* (*) *by Klein's combination theorem. Then* $m(\Lambda(G))=0$. *The condition* (*) *is as follows:*

(*) *For every non-trivial free product decomposition $G=A*B$, there is a parabolic element of G which is not in a conjugate of A or B.*

Proof. This is a consequence from Bonahon's theorem [3] and the combination theorem [10]. □

LEMMA 4.2. *Let G be a free Kleinian group. Let S be a component of $\partial N(G)-\partial_p N(G)$ which is compressible in $N(G)$. Then $\chi(S)\geqq 3$, or G is geometrically finite.*

Proof. Since S is compressible, there is a properly embedded disk D in N which induces a non-trivial free product decomposition of G with $\partial D\subset S$. Let $S-\partial D$ be S_1 and S_2 if ∂D divides S, or only S_0 if ∂D does not divide S. In the former case, we denote $S_i\cup\bar{D}$ by $\hat{S}_i$ $(i=1, 2)$. In the latter case, S_0 has topologically two boundary components corresponding to ∂D. S_0 to which two copies of $\bar{D}$ are attached is regarded as $\hat{S}_0$. We have $\chi(S)=\chi(\hat{S}_1)+\chi(\hat{S}_2)+2$ if ∂D divides S, or $\chi(S)=\chi(\hat{S}_0)+2$ if ∂D does not divide S.

For $i=0, 1, 2$, $\chi(\hat{S}_i)=-2$ does not occur since N is irreducible, and neither does $\chi(\hat{S}_i)=-1$ since a loop round the hole is not homotopically trivial. Hence $\chi(\hat{S}_i)\geqq 0$. Assume that $\chi(S)\leqq 2$. It follows that $\chi(\hat{S}_i)=0$ for $i=0, 1, 2$. Then $\hat{S}_i$ is either a torus or a sphere with two holes. In the latter case, a loop round a hole is freely homotopic in N to a loop round the other hole. Therefore they correspond to the same conjugacy class of primitive parabolic elements. It means that these two holes are connected with a cusp cylinder.

If necessary, by jointing such a cusp cylinder to $\hat{S}_i$, we may assume that $\hat{S}_i$ is a torus. Since G is free, it bounds a solid torus. Hence N is a handlebody of genus 2, and S is the only component of $\partial N-\partial_p N$. Further $S\subset\partial M(G)$, for G has the region of discontinuity. Therefore G is geometrically finite by Marden's characterization. □

THEOREM 4.3. *Let G be a Kleinian group with $r(G)=r$. If $\chi(\partial M(G))\geqq 2(r-2)$, then $m(\Lambda(G))=0$.*

Proof. Consider components of $\partial N-\partial_p N$. Bonahon's condition (*) is equivalent to the fact that all the components of $\partial N-\partial_p N$ are incompressible in N

(see [3, p. 72]). Then we know G does not satisfy the assumption of Proposition 4.1 if and only if there exists a compressible component of $\partial N-\partial_p N$ which is not contained in ∂M.

In the case G is free, by Lemma 4.2, such a compressible component S has $\chi(S)\geqq 3$. Then $\chi(\partial M(G))\leqq\chi(\partial N)-\chi(S)\leqq 2(r-1)-3<2(r-2)$. In the case G is not free, by Proposition 2.1, $\beta_1<r$ or $\beta_2>0$. It follows that $\chi(\partial N)\leqq 2(r-2)$. It is clear that $\chi(S)\geqq 1$. Hence $\chi(\partial M(G))\leqq\chi(\partial N)-\chi(S)\leqq 2(r-2)-1<2(r-2)$. Therefore if $\chi(\partial M(G))\geqq 2(r-2)$, degenerate compressible surfaces can not exist.

COROLLARY 4.4 ([13]). *If* $r(G)=2$, *then* $m(\Lambda(G))=0$.

Corollary 4.4 means that if $r(G)=2$, then $\partial N(G)-\partial_p N(G)$ has no degenerate compressible surface. But if $r(G)=3$, it may consist of two surfaces S_1 and S_2, S_1 is not degenerate with $\chi(S_1)=1$ and S_2 is degenerate with $\chi(S_2)=3$. The following theorem asserts that such a group actually exists.

THEOREM 4.5. *There is an example of the three generator Kleinian group* G *which is geometrically infinite, does not satisfy the condition* (*), *and whose* $\partial M(G)$ *consists of one incompressible surface.*

Proof. Let H be a Fuchsian group of the first kind with two free generators, acting on the unit disk D. Let f_0 be a loxodromic element such that the isometric circles of f_0 and f_0^{-1} are contained in D and the distance of two fixed points of f_0 is small enough. Then by the combination theorem [10], $G_0=\langle H, f_0\rangle$ are geometrically finite Kleinian group and $G_0=H*\langle f_0\rangle$. Let $f_t=g_t f_0 g_t^{-1}$, where $\{g_t\}$ $(t\geqq 0)$ is a continuous path starting at *id.* in $PSL(2, \boldsymbol{C})$. We choose $\{g_t\}$ so that the Euclidian distance between the fixed points of f_t may be larger to infinity as t increases, and all the parabolic elements of $G_t=\langle H, f_t\rangle$ are conjugate to elements in H. This can be done because only a countable number of points in $PSL(2, \boldsymbol{C})$ violate our second requirement.

Then G_t is a quasiconformal (QC) deformation of G_0 when t is small, for G_0 is QC stable [8, Prop. 9.1]. But there exists

$$T=\sup\{\, t \mid \text{for all } s\in[0, t),\ G_s \text{ is q.c. deformation of } G_0\}$$

before a fixed point of f_t hits at ∂D. Let f_T be f and G_T be G.

Since G_t $(t<T)$ has $\Delta=\{|z|>1\}\cup\{\infty\}$ as its component, it is easy to see that Δ is a component of the region of discontinuity of G, particularly G is Kleinian. Moreover since G_t is $H*\langle f_t\rangle$ when $t<T$ and G_t converges algebraically to G, we know $G=H*\langle f\rangle$. Therefore G does not satisfy the condition (*), for any parabolic element of G is in a conjugate of H. Further G is geometrically infinite, for if G were geometrically finite, then G would be QC stable: it contradicts our definition of T. $S=\Delta/H$ is an incompressible components of $\partial M(G)$ with $\chi(S)=1$. By Lemma 4.2, we can see that $\partial M(G)$ has no other components than S, because $\chi(\partial N(G))=4$. □

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