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ON WEAKLY STABLE YANG-MILLS FIELDS OVER POSITIVELY PINCHED MANIFOLDS AND CERTAIN SYMMETRIC SPACES

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Abstract

In this paper it is proved that for $n \geq 5$ there exists a constant $\delta(n)$ with $1/4 < \delta(n) < 1$ such that any weakly stable Yang-Mills connection over a simple connected compact Riemannian manifold M of dimension n with $\delta(n)$-pinched sectional curvatures is always flat. The pinching constants are possible to compute by elementary functions. Moreover we give some remarks on stability of Yang-Mills connections over certain symmetric spaces.

Introduction.

Let M be an n-dimensional compact Riemannian manifold with a metric g and G be a compact Lie group with the Lie algebra g. Let E be a Riemannian vector bundle over M with structure group G, and let $\mathcal{C}_E$ denote the space of G-connections on E, which is an affine space modeled on the vector space $\Omega^1(g_E)$ of smooth 1-forms with values in the adjoint bundle g_E of E. The Yang-Mills functional $\mathcal{YM}: \mathcal{C}_E \to \boldsymbol{R}$ is

$$\mathcal{YM}(\nabla)=\frac{1}{2}\int_M \|F^\nabla\|^2 dvol,$$

for each $\nabla \in \mathcal{C}_E$, where F^∇ is the curvature form of the connection ∇. Note that F^∇ is a smooth section of $\Omega^2(g_E)$. The Yang-Mills connection $\nabla \in \mathcal{C}_E$ is a critical point of $\mathcal{YM}$. A Yang-Mills connection ∇ is called *weakly stable* if, for each $\nabla^t \in \mathcal{C}_E$ with $\nabla = \nabla^0$,

$$(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0} \geqq 0 .$$

M is called *Yang-Mills unstable* (cf. [K-O-T]) if, for every vector bundle (E, G) over M, any weakly stable Yang-Mills connection on E is always fiat. First Simons proved that the Euclidean n-sphere S^n for $n \geqq 5$ is Yang-Mills unstable ([B-L]). Ever since several persons have investigated the instability of Yang-Mills fields over various Riemannian manifolds; convex hypersurfaces, submani-

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folds, compact symmetric spaces (cf. [Ka], [K-O-T], [Pal], [Sh], [Ta], [We]). In [K-O-T] it was shown that the Cayley projective plane $P_2(\mathrm{Cay})$ and the compact symmetric space of exceptional type E_6/F_4 are Yang-Mills unstable.

In this paper we first establish the instability theorem for Yang-Mills fields over a simply connected compact Riemannian manifold with sufficiently pinched sectional curvatures. Okayasu [Ok] used the construction and results of Ruh, Grove and Karcher ([Ru], [G-K-R1], [G-K-R2]) to show the instability of harmonic maps into a Riemannian manifold with sufficiently pinched sectional curvatures. By using the same idea, the second named author [Pa2] showed an instability theorem for harmonic maps from a Riemannian manifold with sufficiently pinched sectional curvatures to an arbitrary Riemannian manifold. We will also use it. Next we shall prove some results on weakly stable Yang-Mills fields over certain symmetric spaces. Some of them were stated in [K-O-T] without proof. They supplement results of Laquer [La] which determined the stablity of canonical connections over simply connected compact irreducible spaces. Moreover we prove that a weakly stable Yang-Mills field satisfying a certain condition over a quaternionic projective space $P_m(\boldsymbol{H})$ is a B_2-connection in a sense of [Ni], or equivalently a self-dual connection in a sense of [C-S], and hence it minimizes the Yang-Mills functional.

1. Preliminaries on Yang-Mills fields.

Let $\nabla\in\mathcal{C}_E$. For any $B\in\Omega^1(g_E)$, set $\nabla^t=\nabla+tB\in\mathcal{C}_E$. The second variational formula formula for the Yang-Mills functional is given as follows ([B-L]);

$$(1.1)\qquad (d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}=\mathfrak{I}^\nabla(B, B)$$
$$=\int_M(\mathcal{S}_0^\nabla(B), B)dvol$$
$$=\int_M\{(\mathcal{S}^\nabla(B), B)-(\delta^\nabla B, \delta^\nabla B)\}dvol,$$

where $\mathcal{S}_0^\nabla(B)=\delta^\nabla d^\nabla B+\mathfrak{F}^\nabla(B)$ and $\mathcal{S}^\nabla(B)=\Delta^\nabla(B)+\mathfrak{F}^\nabla(B)$. Here d^∇ and δ^∇ denote the exterior covariant differentiation induced by the connection $\nabla\in\mathcal{C}_E$ and its adjoint differential operator, and $\mathfrak{F}^\nabla$ is a symmetric bundle endomorphism of $T^*M\otimes g_E$ defined by $(\mathfrak{F}^\nabla(b))(X)=\sum_{i=1}^n[F^\nabla(e_i, X), b(e_i)]$ for $b\in T_x^*M\otimes(g_E)_x$ and $X\in T_xM$, where $\{e_i\}$ is an orthonormal basis of T_xM.

Let $\{\omega^i\}$ be the dual frame of a local orthonormal frame field $\{e_i\}$ in M, Throughout this paper we use the summation convention. Set $B=B_i\omega^i$ and $F^\nabla=(1/2)F_{ij}\omega^i\wedge\omega^j$. Then we have

$$d^\nabla B=(\nabla_iB_j-\nabla_jB_i)\omega^i\wedge\omega^j\,,$$
$$\delta^\nabla d^\nabla B=(\nabla_j\nabla_iB_j-\nabla_j\nabla_iB_i)\omega^i\,,$$
$$\mathfrak{F}^\nabla(B)=[F_{ij}, B_i]\omega^j\,,$$

$$\|F^\nabla\|^2=(F_{ij}, F_{ij})/2 .$$

And (1.1) becomes

$$(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}$$

$$=\int_M \{(\nabla_j\nabla_i B_j, B_i)-(\nabla_j\nabla_j B_i, B_i)+([F_{ij}, B_i], B_j)\}\, dvol.$$

Let D be a Riemannian connection of M and let R denote the curvature tensor field of D; $R(e_i, e_j)e_k=R_{ijkl}e_l$. The Ricci tensor field Ric of M is defined by $R_{ij}=R_{ikkj}$. The scalar curvature R of M is defined by $R=R_{ii}$. The Ricci identities are as follows:

$$D_kD_jX^i-D_jD_kX^i=R_{kjli}X^l \qquad \text{for } X=X^ie_i ,$$

$$\nabla_l\nabla_kF_{ij}-\nabla_k\nabla_lF_{ij}=-F_{mj}R_{lkim}-F_{im}R_{lkjm}+[F_{lk}, F_{ij}] ,$$

The curvature form F^∇ always satisfies the Bianchi identity $d^\nabla F^\nabla=0$, or equivalently

$$\nabla_kF_{ij}+\nabla_iF_{jk}+\nabla_jF_{ki}=0 . \tag{1.2}$$

The Yang-Mills equation is $\delta^\nabla F^\nabla=0$, namely

$$\nabla_jF_{ij}=0 . \tag{1.3}$$

Let $\nabla\in\mathcal{C}_E$. Assume that $\varphi=(1/2)\varphi_{ij}\omega^i\wedge\omega^j\in\Omega^2(g_E)$ is harmonic with respect to ∇, that is, $d^\nabla\varphi=0$ and $\delta^\nabla\varphi=0$. Note that if ∇ is a Yang-Mills connection, we can take $\varphi=F^\nabla$. Let $V\in C^\infty(TM)$ with $V=V^ie_i$. Set $B=i_V\varphi=B_i\omega_i\in\Omega^1(g_E)$. Here $B_i=V^j\varphi_{ji}$. Then by the harmonicity of φ and the Bochner-Weitzenböck formula (cf. [B-L]) we compute

$$\begin{aligned}(\mathcal{S}^\nabla(B))(X)=&\varphi(D^*DV, X)-2\sum_{i=1}^n(\nabla_{e_i}\varphi)(D_{e_i}V, X)\\ &+\varphi(V, \mathrm{Ric}(X))-\{\varphi\circ(\mathrm{Ric}\wedge I-2\mathcal{R})\}(V, X)\\ &-\sum_{i=1}^n\{[F^\nabla(e_i, V), \varphi(e_i, X)]+[F^\nabla(e_i, X), \varphi(e_i, V)]\},\end{aligned} \tag{1.4}$$

where $D^*DV=-\sum_{i=1}^n D^2V(e_i, e_i)$, and $\mathcal{R}$ denotes the curvature operator of (M, g) acting on $\wedge^2TM$. We define a quadratic form Q_φ on $C^\infty(TM)$ as

$$Q_\varphi(V)=(d^2/dt^2)\mathcal{YM}(\nabla^t)|_{t=0}=\int_M q_\varphi(V)dvol,$$

where $\nabla^t=\nabla+t(i_V\varphi)\in\mathcal{C}_E$. By straightforward computations we have

$$(1.5)\qquad q_\varphi(V)=D_jD_iV^kV^l(\varphi_{kj},\ \varphi_{li})-D_jD_jV^kV^l(\varphi_{ki},\ \varphi_{li})$$
$$+D_jV^kV^l(\nabla_i\varphi_{kj}\varphi_{li})-2D_jV^kV^l(\nabla_j\varphi_{ki},\ \varphi_{li})$$
$$+V^kV^l([F^\nabla_{jk},\varphi_{ij}]+[F^\nabla_{ji},\varphi_{kj}],\ \varphi_{li})$$
$$+V^kV^l\{R_{ikmj}(\varphi_{mj},\ \varphi_{li})-R_{jikm}(\varphi_{mj},\ \varphi_{li})+R_{km}(\varphi_{im},\ \varphi_{li})\}.$$

2. The constructionn of Ruh for a δ-pinched manifold.

We recall the idea and construction of Ruh ([Ru], [G-K-R1], [G-K-R2]). Let (M, g) be an n-dimensional simply connected compact Riemannian manifold with δ-pinched sectional curvature, namely $\delta<K\leqq 1$. We fix a normalized Riemannian metric $g_0=\{(1+\delta)/2\}g$ on M. Then we have $2\delta/(1+\delta)<K_{g_0}\leqq 2/(1+\delta)$. Consider a vector bundle $\Xi=TM\oplus\varepsilon(M)$ with a fibre metric $\langle\,,\rangle$ over M. Here $\varepsilon(M)$ is a trivial line bundle with a fiber metric and it is orthogonal to TM. Let e denote a smooth section of lengh 1 in $\varepsilon(M)$. Now we define a metric connection D'' in Ξ as follows;

$$D''_XY=D_XY-g_0(X,\ Y)e\ ,$$
$$D''_Xe=X$$

for $X, Y\in C^\infty(TM)$. It was proved that if δ is sufficiently close to 1, there exists a flat connection D' in Ξ close to D'' ([G-K-R1]). Define

$$\|D'-D''\|:=\underset{x\in M}{Max}\{\|D'_XY-D''_XY\|\ ;\ X\in T_xM,\ g_0(X,\ X)=1,\ Y\in\Xi_x,\ \|Y\|=1\}.$$

Note that it is a half of that one in [G-K-R2]. Set

$$k_1(\delta)=(4/3)(1-\delta)\delta^{-1}\{1+(\delta^{1/2}\sin(1/2)\pi\delta^{-1/2})^{-1}\},$$
$$k_2(\delta)=\{(1+\delta)/2\}^{-1}k_1(\delta),$$
$$k_3(\delta)=k_2(\delta)[1+\{1-(1/24)\pi^2k_2(\delta)^2\}^{-2}]^{1/2}.$$

[G-K-M2] proved that $\|D'-D''\|\leqq k_3(\delta)/2$. The curvature form R'' of the connection D'' is

$$(2.1)\qquad R''(X,\ Y)Z=R(X,\ Y)Z-\langle Y,\ Z\rangle X+\langle X,\ Z\rangle Y\ ,$$

$$(2.2)\qquad R''(X,\ Y)e=0$$

for $X, Y, Z\in T_xM$.

3. Trace formula for second variations of Yang-Mills fields over a δ-pinched manifold.

Assume that M is a simply connected compact Riemannian manifold with δ-pinched sectional curvatures. Let $P=\{v\in C^\infty(\Xi)\,;\, D'v=0\}$, which is linerly isometric to $\boldsymbol{R}^{n+1}$. For each $v\in P$, we denote by $V=v^T$ the TM-component of v in Ξ. Set $\mathcal{V}=\{V\in C^\infty(TM)\,;\, V=v^T$ for some $v\in P\}$, which has a natural inner product so that it is linearly isometric to P. Choose an orthonormal basis $\{V_\alpha\}_{\alpha=0,\dots,n}$ of $\mathcal{V}$. Set $V_\alpha=(v_\alpha)^T$. Then $\sum_{\alpha=0}^n V_\alpha^k V_\alpha^l=\delta^{kl}$. In this section we compute the trace $\mathrm{Tr}_{\mathcal{V}}Q_\varphi=\sum_{\alpha=0}^n Q_\varphi(V_\alpha)$ of Q_φ on $\mathcal{V}$ relative to the inner product.

A straightforward computation shows

LEMMA 3.1.

$$D_jV^k=\langle D''_{e_j}v,\, e_k\rangle-\langle v,\, e\rangle\delta_{jk}\,. \tag{3.1}$$

$$D_jD_iV^k=\langle (D''^2v)(e_i,\, e_j),\, e_k\rangle-\delta_{jk}\langle D''_{e_i}v,\, e\rangle-\delta_{ik}\langle D''_{e_j}v,\, e\rangle-\delta_{ik}\langle v,\, e_j\rangle\,. \tag{3.2}$$

LEMMA 3.2.

$$\int_M\{D_jD_iV^kV^l(\varphi_{kj},\,\varphi_{li})+D_jV^kV^l(\nabla_i\varphi_{kj},\,\varphi_{li})\}dvol \tag{3.3}$$

$$=\int_M\{R_{jimk}V^mV_l(\varphi_{kj},\,\varphi_{li})-D_jV^kD_iV^l((\varphi_{kj},\,\varphi_{li})\}dvol.$$

$$\int_M -2D_jV_\alpha^kV_\alpha^l(\nabla_j\varphi_{ki},\,\varphi_{li})dvol \tag{3.4}$$

$$=\int_M\{-2D_kD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\,\varphi_{li})-2D_jV_\alpha^kD_kV_\alpha^l(\varphi_{ij},\,\varphi_{li})$$

$$-D_iD_jV_\alpha^kV_\alpha^l(\varphi_{ij},\,\varphi_{kl})-D_jV_\alpha^kD_iV_\alpha^l(\varphi_{ij},\,\varphi_{kl})$$

$$-2D_iD_jV_\alpha^kV_\alpha^l(\varphi_{jk},\,\varphi_{li})-2D_jV_\alpha^kD_iV_\alpha^l(\varphi_{jk},\,\varphi_{li})\}dvol.$$

Proof. (3.3) is due to the Ricci identity and the divergence theorem. We show (3.4). By $d^\nabla\varphi=0$, we have

$$-2D_jV_\alpha^kV_\alpha^l(\nabla_j\varphi_{ki},\,\varphi_{li}) \tag{3.5}$$

$$=2D_jV_\alpha^kV_\alpha^l(\nabla_k\varphi_{ij},\,\varphi_{li})+2D_jV_\alpha^kV_\alpha^l(\nabla_i\varphi_{jk},\,\varphi_{li}),$$

By using the divergence theorem, we get

$$\int_M 2D_jV_\alpha^kV_\alpha^l(\nabla_i\varphi_{jk},\,\varphi_{li})dvol$$

$$=\int_M \{-2D_iD_jV^k_\alpha V^l_\alpha(\varphi_{jk}, \varphi_{li})-2D_jV^k_\alpha D_iV^l_\alpha(\varphi_{jk}, \varphi_{li})\}dvol.$$

We compute

$$2D_jV^k_\alpha V^l_\alpha(\nabla_k\varphi_{ij}, \varphi_{li})$$
$$=2D_k\{D_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{li})\}-2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$-2D_jV^k_\alpha D_kV^l_\alpha(\varphi_{ij}, \varphi_{li})-2D_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \nabla_k\varphi_{li}).$$

Since

$$D_jV^k_\alpha V^l_\alpha=-V^k_\alpha D_jV^l_\alpha, \tag{3.6}$$

we have

$$D_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \nabla_k\varphi_{li})=D_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \nabla_l\varphi_{ik}).$$

Hence by Bianchi identity we get

$$-2D_jV^k_\alpha V^k_\alpha(\varphi_{ij}, \nabla_k\varphi_{li})=D_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \nabla_i\varphi_{kl}).$$

Thus by using the divergence theorem we obtain

$$\int_M 2D_jV^k_\alpha V^l_\alpha(\nabla_k\varphi_{ij}, \varphi_{li})dvol$$
$$=\int_M\{-2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{li})-2D_jV^k_\alpha D_kV^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$-D_iD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{kl})-D_jV^k_\alpha D_iV^l_\alpha(\varphi_{ij}, \varphi_{kl})\}dvol.$$

q. e. d.

By (1.5), (3.3) and (3.4), we get

$$\mathrm{Tr}_{\mathcal{V}}Q_\varphi=\int_M\{-D_jV^k_\alpha D_iV^l_\alpha(\varphi_{kj}, \varphi_{li})-D_jD_jV^k_\alpha V^l_\alpha(\varphi_{ki}, \varphi_{li})$$
$$-2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{li})-2D_jV^k_\alpha D_kV^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$-D_iD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{kl})-D_jV^k_\alpha D_iV^l_\alpha(\varphi_{ij}, \varphi_{kl})$$
$$-2D_iD_jV^k_\alpha V^l_\alpha(\varphi_{jk}, \varphi_{li})-2D_jV^k_\alpha D_iV^l_\alpha(\varphi_{jk}, \varphi_{li})$$
$$+R_{jilk}(\varphi_{kj}, \varphi_{li})+R_{ikmj}(\varphi_{mj}, \varphi_{ki})$$
$$-R_{jikm}(\varphi_{mj}, \varphi_{ki})+R_{km}(\varphi_{im}, \varphi_{ki})\}dvol. \tag{3.7}$$

LEMMA 3.3.

$$-2D_iD_jV^k_\alpha V^l_\alpha(\varphi_{jk}, \varphi_{li}) \tag{3.8}$$
$$=D_jV^k_\alpha D_iV^l_\alpha(\varphi_{jk}, \varphi_{li})+D_iV^k_\alpha D_jV^l_\alpha(\varphi_{jk}, \varphi_{li})$$
$$+R_{jimk}V^m_\alpha V^l_\alpha(\varphi_{jk}, \varphi_{li}),$$

(3.9) $$-D_iD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{kl}) = -(1/2)R_{ijmk}V^m_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{kl}).$$

Proof. (3.9) is due to the Ricci identity. We show (3.8). Differentiating covariantly (3.6), we have

(3.10) $$D_iD_jV^k_\alpha V^l_\alpha + V^k_\alpha D_iD_jV^l_\alpha + D_jV^k_\alpha D_iV^l_\alpha + D_iV^k_\alpha D_jV^l_\alpha = 0.$$

(3.8) follows from (3.10) and the Ricci identity. q.e.d.

LEMMA 3.4.

(3.11) $$\begin{aligned}-D_jD_jV^k_\alpha V^l_\alpha(\varphi_{ki}, \varphi_{li}) &= \langle D''_{e_j}v_\alpha, D''_{e_i}v_\beta\rangle V^k_\beta V^l_\alpha(\varphi_{ki}, \varphi_{li}) \\ &\quad + \{2\langle D''_{e_k}v_\alpha, e\rangle + \langle v_\alpha, e_k\rangle\} V^l_\alpha(\varphi_{ki}, \varphi_{li}).\end{aligned}$$

Proof. From $\langle v_\alpha, v_\beta\rangle = \delta_{\alpha\beta}$, we have

(3.12) $$\begin{aligned}&\langle (D''^2 v_\alpha)(e_i, e_j), v_\beta\rangle + \langle (D''^2 v_\beta)(e_i, e_j), v_\alpha\rangle \\ &= -\langle D''_{e_i}v_\alpha, D''_{e_j}v_\beta\rangle - \langle D''_{e_j}v_\alpha, D''_{e_i}v_\beta\rangle.\end{aligned}$$

Using (3.2) and (3.12), we obtain (3.11). q.e.d.

LEMMA 3.5.

(3.13) $$\begin{aligned}&\int_M -2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{li})dvol \\ &= \int_M [2\langle D''_{e_j}v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li}) \\ &\quad + 2\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_j}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{li}) \\ &\quad + 2\{(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle - (1/4)\langle R''(e_l, e_k)e_k, e_l\rangle \\ &\qquad - (1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle \\ &\qquad - (1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle \\ &\qquad - (1/2)\langle D''_{e_k}v_\alpha, e\rangle V^k_\alpha + (1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\}\|\varphi\|^2 \\ &\quad - 2\langle R''(e_k, e_j)e_l, e_k\rangle(\varphi_{ij}, \varphi_{li}) + 2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li}) \\ &\quad + 2\langle v_\alpha, e_j\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})]dvol.\end{aligned}$$

Proof. By (3.2), we have

(3.14) $$\begin{aligned}&-2D_kD_jV^k_\alpha V^l_\alpha(\varphi_{ij}, \varphi_{li}) \\ &= -2\{\langle (D''^2 v_\alpha)(e_j, e_k), e_k\rangle - (n+1)\langle D''_{e_j}v_\alpha, e\rangle \\ &\quad - \langle v_\alpha, e_j\rangle\} V^l_\alpha(\varphi_{ij}, \varphi_{li}).\end{aligned}$$

By using the Ricci identity we get

$$(3.15)\qquad \langle (D''^2 v_\alpha)(e_j, e_k), e_k\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$=\{\langle (D''^2 v_\alpha)(e_k, e_j), e_k\rangle + \langle R''(e_k, e_j)v_v, e_k\rangle\} V^l_\alpha(\varphi_{ij}, \varphi_{li}).$$

We compute

$$(3.16)\qquad \langle (D''^2 v_\alpha)(e_k, e_j), e_k\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$=D_j\{\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})\} - \langle D''_{e_j} v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{li})$$
$$-\langle D''_{e_k} v^v, e_k\rangle \langle D''_{e_l} v_\alpha, e_l\rangle (\varphi_{ij}, \varphi_{li}) - \langle D''_{e_k} v_\alpha, e_k\rangle \langle v_\alpha, e\rangle (\varphi_{ij}, \varphi_{ij})$$
$$-\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha(\varphi_{ij}, \nabla_j \varphi_{li}).$$

By the Bianchi identity we get

$$(3.17)\qquad -\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha(\varphi_{ij}, \nabla_j \varphi_{li}) = (1/4)\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha D_l \|\varphi\|^2.$$

We compute

$$(3.18)\qquad \langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha D_l \|\varphi\|^2$$
$$=D_l\{\langle D''_{e_k} v_\alpha, e_k\rangle V^l_\alpha \|\varphi\|^2\} - \langle (D''^2 v_\alpha)(e_k, e_l), e_k\rangle V^l_\alpha \|\varphi\|^2$$
$$+\langle D''_{e_k} v_\alpha, e\rangle V^k_\alpha \|\varphi\|^2 - \langle D''_{e_k} v_\alpha, e_k\rangle \langle D''_{e_l} v_\alpha, e_l\rangle \|\varphi\|^2$$
$$+n\langle D''_{e_k} v_\alpha, e_k\rangle \langle v_\alpha, e\rangle \|\varphi\|^2.$$

By using (3.12) and the Ricci identity we get

$$(3.19)\qquad \langle (D''^2 v_\alpha)(e_k, e_l), e_k\rangle V^l_\alpha$$
$$=-(1/2)\{\langle R''(e_l, e_k)e_k, e_l\rangle + \langle D''_{e_k} v_\alpha, D''_{e_l} v_\beta\rangle V^k_\alpha V^l_\alpha$$
$$+\langle D''_{e_k} v_\alpha, D''_{e_l} v_\beta\rangle V^l_\beta V^k_\alpha\}.$$

Hence, by the divergence theorem, (3.13) follows from (3.14), (3.15), (3.16), (3.17), (3.16) and (3.19). q.e.d.

Therefore, by (2.1), (3.8), (3.11) and (3.13), (3.7) reduces to the following trace formula.

$$(3.20)\qquad \mathrm{Tr}_{\mathcal{V}}\, Q_\varphi$$
$$=\int_M [2\{5-2n+(n(n-1)-R)/4\}\|\varphi\|^2 + R_{jl}(\varphi_{ij}, \varphi_{il})$$
$$+\langle D''_{e_i} v_\alpha, D''_{e_i} v_\beta\rangle V^k_\beta V^l_\alpha(\varphi_{ki}, \varphi_{li}) - 2\langle D''_{e_k} v_\alpha, e_k\rangle \langle D''_{e_j} v_\alpha, e_l\rangle (\varphi_{ij}, \varphi_{il})$$

$$+2\{(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle$$
$$-(1/2)\langle D''_{e_k}v_\alpha, e\rangle V^k_\alpha+(1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\}\|\varphi\|^2$$
$$-2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{il})-8\langle D''_{e_j}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij}, \varphi_{ik})$$
$$+2\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_k}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il})-\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_i}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{kl})$$
$$+\langle D''_{e_l}v_\alpha, e_j\rangle\langle D''_{e_i}v_\alpha, e_k\rangle(\varphi_{ij}, \varphi_{kl})]dvol.$$

4. Instability theorem for Yang-Millf fields over a δ-pinched Riemannian manifold.

Note that if $\delta=1$, then $D'=D''$, hence (3.20) becomes

$$\mathrm{Tr}_{\mathcal{V}}\, Q_\varphi=2(4-n)\int_M\|\varphi\|^2.$$

Since the sectional curvatures of M are δ-pinched, we have

$$2\{5-2n+(1/4)(n(n-1)-R)\}\|\varphi\|^2+R_{jl}(\varphi_{ij}, \varphi_{il})$$
$$\leqq 2[5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta)]\|\varphi\|^2,$$

We can make estimates for each other term of (3.20) as follows:

$$\langle D''_{e_i}v_\alpha, D''_{e_i}v_\beta\rangle V^l_\beta V^l_\alpha(\varphi_{kl}, \varphi_{li})\leqq(n/2)k_3(\delta)^2\|\varphi\|^2,$$
$$-2\langle D''_{e_l}v_\alpha, e_k\rangle\langle D''_{e_j}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il})\leqq n(n+1)k_3(\delta)^2\|\varphi\|^2,$$
$$(2-(n/2))\langle D''_{e_k}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle\leqq n(n/4-1)k_3(\delta),$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_k\rangle\langle v_\alpha, e_l\rangle\leqq(n^2/16)k_3(\delta)^2,$$
$$-(1/4)\langle D''_{e_k}v_\alpha, D''_{e_l}v_\beta\rangle\langle v_\beta, e_l\rangle\langle v_\alpha, e_k\rangle\leqq(n^2/16)k_3(\delta)^2,$$
$$-(1/2)\langle D''_{e_k}v_\alpha, e\rangle V^k_\alpha\leqq(n/4)k_3(\delta),$$
$$(1/2)\langle D''_{e_k}v_\alpha, e_k\rangle\langle D''_{e_l}v_\alpha, e_l\rangle\leqq(n^2/8)k_3(\delta)^2,$$
$$-2(n+1)\langle D''_{e_j}v_\alpha, e\rangle V^l_\alpha(\varphi_{ij}, \varphi_{il})\leqq 2(n+1)k_3(\delta)\|\varphi\|^2,$$
$$-8\langle D''_{e_j}v_\alpha, e_k\rangle\langle v_\alpha, e\rangle(\varphi_{ij}, \varphi_{ik})\leqq 8k_3(\delta)\|\varphi\|^2,$$
$$2\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_k}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{il})\leqq nk_3(\delta)\|\varphi\|^2,$$
$$\langle D''_{e_l}v_\alpha, e_j\rangle\langle D''_{e_i}v_\alpha, e_k\rangle(\varphi_{ij}, \varphi_{kl})$$
$$-\langle D''_{e_j}v_\alpha, e_k\rangle\langle D''_{e_i}v_\alpha, e_l\rangle(\varphi_{ij}, \varphi_{kl})\leqq k_3(\delta)\|\varphi\|^2.$$

Hence we get

$$\mathrm{Tr}_{\mathcal{V}} Q_\varphi \leqq 2[5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta) + (1/4)(n^2+n+20)k_3(\delta)+(1/4)(3n^2+5n+2)k_3(\delta)^2]\int_M \|\varphi\|^2 . \tag{4.1}$$

Therefore we obtain

THEOREM 4.1. *If* $n \geqq 5$ *and*

$$5-2n+(1/4)n(n-1)\{1-2\delta/(1+\delta)\}+2(n-1)/(1+\delta) + (1/4)(n^2+n+20)k_3(\delta)+(1/4)(3n^2+5n+2)k_3(\delta)^2<0 , \tag{4.2}$$

then M is Yang-Mills unstable.

COROLLARY 4.2. *For $n \geqq 5$, there exists a constant $\delta(n)$, which depends only on n, with $1/4<\delta(n)<1$ such that any n-dimensional simply connected compact Riemannian manifold M with $\delta(n)$-pinched sectional curvatures is Yang-Mills unstable.*

Remark. As n tends to the infinity, the right hand side of (4.2) divided by $(1/4)(3n^2+5n+2)$ tends to $(1/3)\{1-2\delta/(1+\delta)\}+(1/3)k_3(\delta)+k_3(\delta)^2>0$. In our argument it is not possible to find a pinching constant δ independent of the dimension of the base manifold M such that M is Yang-Mills unstable.

5. Trace formula for second variations of Yang-Mills fields over submanifolds in Euclidean space.

Assume that M is isometrically immersed in a Euclidean space $\boldsymbol{R}^N$. Let Φ denote the immersion. We may assume that $\Phi(M)$ is not contained in any hyperplane of $\boldsymbol{R}^N$. Set $\mathcal{V}=\{U\in C^\infty(TM)\,;\,U=\mathrm{grad}\,f_u$ for some $u\in\boldsymbol{R}^N\}$. Here f_u denotes the hight function on M defined by $f_u(u)=\langle\Phi(x), u\rangle$. Suppose that ∇ is a connection on a Riemannian vector bundle (E, G) over M and $\varphi\in\Omega^2(g_E)$ is harmonic with respect to ∇. Then we recall

PROPOSITION 5.1 ([K-O-T]). *For* $U=\mathrm{grad}\,f_u\in\mathcal{V}$,

$$\begin{aligned}\mathcal{S}^\nabla(i_U\varphi)(X) = & -\{\varphi\circ(\mathrm{Ric}\wedge I-2R)\}(U, X) \\ & +n\varphi(A_\eta(U), X)+\varphi(U, \mathrm{Ric}\,(X))-\varphi(\mathrm{Ric}\,(U), X) \\ & -\sum_{i=1}^n \{[F^\nabla(e_i, U), \varphi(e_i, X)]+[F^\nabla(e_i, X), \varphi(e_i, U)]\} \\ & -2\sum_{i,j=1}^n \langle B(e_i, e_j), u\rangle(\nabla_{e_j}\varphi)(e_i, X)-n\sum_{i=1}^n\langle D^\perp{}_{e_i}\eta, \mu\rangle\varphi(e_i, X).\end{aligned} \tag{5.1}$$

$$\mathrm{tr}_{\mathcal{U}}\, Q_{\varphi}=2\int_{M}(\varphi\circ\{(n/2)(A_{\eta}\wedge I)-\mathrm{Ric}\wedge I+2\mathcal{R}\},\ \varphi)dvol, \tag{5.2}$$

where $\mathcal{R}$, B, A, η and $D^{\perp}$ denote the curvature operator of M acting on $\wedge^{2}TM$, the second fundamental form, the shape operator, the mean curvature and the normal connection of Φ, respectively.

Consider a compact Riemannian homogeneous space with irreducible isotropy representation M.

LEMMA 5.2. *If ∇ is a weakly stable Yang-Mills connection, then we have*

$$\sum_{i=1}^{n}\{[F^{\nabla}(e_i, Y),\ \varphi(e_i, X)]+[F^{\nabla}(e_i, X),\ \varphi(e_i, Y)]\}=0 \tag{5.3}$$

for every $X, Y\in T_{x}M$.

Proof. Let K be the group of isometries of M and let k be its Lie algebra of Killing vector fields on M. Since M has irreducible isotropy representation, we can fix a K-invariant inner product on k which induces the K-invariant Riemannian metric of M. By [B-L, (10.4) Lemma], for each $V\in k$

$$\mathcal{S}_{0}^{\nabla}(i_{V}\varphi)(X)=-\sum_{i=1}^{n}\{[F^{\nabla}(e_i, V),\ \varphi(e_i, X)]+[F^{\nabla}(e_i, X),\ \varphi(e_i, V)]\}.$$

Hence $\mathrm{tr}_{k}Q_{\varphi}=0$. Since ∇ is weakly stable, we have $\mathfrak{I}^{\nabla}(i_{V}\varphi, i_{V}\varphi)=0$ for all $V\in k$. For any $B\in\Omega^{1}(g_{E})$,

$$0\leqq\mathfrak{I}^{\nabla}(i_{V}\varphi+tB,\ i_{V}\varphi+tB)=2t\mathfrak{I}^{\nabla}(i_{V}\varphi,\ B)+t^{2}\mathfrak{I}^{\nabla}(B,\ B),$$

hence $\mathfrak{I}^{\nabla}(i_{V}\varphi, B)=0$. Thus $\mathcal{S}_{0}^{\nabla}(i_{V}\varphi)=0$ for all $V\in k$. q.e.d.

Consider $\Phi: M\to S^{N-1}(\sqrt{n/\lambda_1})\subset \boldsymbol{R}^{N}$ be the first standard minimal immersion of M (cf. [K-O-T]). Since M is an Einstein manifold and Φ is a minimal immersion onto a sphere of radius $\sqrt{n/\lambda_1}$, if $\varphi=F^{\nabla}$, then (5.1) becomes

$$\begin{aligned}\mathcal{S}^{\nabla}(i_{U}\varphi)(X)=&[\varphi\circ\{(\lambda_1-2c)I+2\mathcal{R}\}](U,\ X)\\ &-2\sum_{i,j=1}^{n}\langle B(e_i,\ e_j),\ u\rangle(\nabla_{e_j}\varphi)(e_i,\ X),\end{aligned} \tag{5.4}$$

where c and λ_1 denote the Einstein constant of M and the first eigenvalue of the Laplace-Beltrami operator of M acting on functions, respectively.

Assume that M is a compact irreducible symmetric space. Let

$$\overset{2}{\wedge}T_{x}M=h_0+h_1+\cdots+h_p \tag{5.5}$$

be the orthogonal decomposition into eigenspaces of $\mathcal{R}$, where h_0 is the eigenspace with eigenvalue 0 and h_s is the eigenspace with eigenvalue $\mu_s>0$. We decompose $\varphi=\varphi_0+\varphi_1+\cdots+\varphi_p$ along (5.5). Note that $\nabla\varphi=0$ if and only if $\nabla\varphi_s=0$ for each $s=0, \cdots, p$. Assume that $\nabla\varphi=0$. If ∇ is weakly stable Yang-Mills field, then by (5.3) we have

$$S^{\nabla}(i_V\varphi_s)=(\lambda_1-2c+2\mu_s)(i_V\varphi_s) \quad \text{for each } s=0, \cdots, p. \tag{5.6}$$

6. Remarks on Yang-Mills fields over compact symmetric spaces.

First we remark on the stability of the canonical connections over compact globally Riemannian symmetric spaces. Laquer [La] determined the indices and nullities of the canonical connection on the standard principal bundle of each simply connected compact irreducible symmetric spaces. We denote by $i(\nabla)$ and $n(\nabla)$ the index and nullity of a Yang-Mills connection ∇ (cf. [B-L] for their definitions).

THEOREM 6.1 ([La]). *Let $M=K/H$ be a simply connected compact irreducible symmetric space associated with a symmetric pair (K, H) and let ∇ the canonical connection of the principal bundle $K\to K/H$.*

(1) *If M is a group manifold, then $i(\nabla)=1$ and $n(\nabla)=0$.*

(2) *If $M=S^n$ $(n\geqq 5)$, $P_2(\mathrm{Cay})$, E_6/F_4, then $i(\nabla)=n+1$, 26, 54 and $n(\nabla)=0$, respectively.*

(3) *If $M=P_m(H)$ $(m\geqq 1)$, then $i(\nabla)=0$, $n(\nabla)=10$ $(m=1)$ or $m(2m+3)$ $(m\geqq 2)$.*

(4) *If M is otherwise, then $i(\nabla)=n(\nabla)=0$.*

We should note that the values $i(\nabla)$ for $M=S^n$ $(n\geqq 5)$, $P_2(\mathrm{Cay})$, E_6/F_4 and $n(\nabla)$ for $M=P_m(H)$ $(m\geqq 2)$ are equal to the dimension of the first eigenspace of the Laplace-Beltrami operator of M acting on functions, and $n(\nabla)$ for $M=P_1(H)=S^4$ is equal to its twice. It is known that, in the cases of $M=S^n$, $P_m(H)$, $P_2(\mathrm{Cay})$, the space of all gradient vector fields for the first eigenfunctions on M coincides with the space of all proper infinitesimal conformal transformations or projective transformations on M.

We observe the case when M is a non-simply connected, compact irreducible symmetric space. From [La] we see that if M is a group manifold, then $i(\nabla)=1$, $n(\nabla)=0$. Suppose that M is not a group manifold. We easily check that if the canonical connection of the universal covering $\tilde{M}$ of M has $i(\nabla)=n(\nabla)=0$, then the canonical connection of M also has $i(\nabla)=n(\nabla)=0$. When $\tilde{M}=S^n$, by virtue of [B-L, (9.1) Theorem], we have $i(\nabla)=n(\nabla)=0$. From the theory of symmetric spaces (cf. [He]) we know that if $\tilde{M}=P_n(H)$ or $P_2(\mathrm{Cay})$, then $\tilde{M}=M$, and if $\tilde{M}=E_6/F_4$, then $M=E_6/F_4\cdot Z_3$. We show that the canonical connection of $M=E_6/F_4\cdot \boldsymbol{Z}_3$ has $i(\nabla)=n(\nabla)=0$. From Theorem 6.1 we see $n(\nabla)=0$. First we recall the realization of E_6/F_4 and $E_6/F_4\cdot \boldsymbol{Z}_3$ (cf. [Yo]). Consider the Jordan algebra $\mathfrak{X}=\{u\in M(3, \mathrm{Cay}); u^*=u\}$ of (real) dimension 27. Let $R^{54}=C^{27}=\mathfrak{X}^C$ be the complexification of $\mathfrak{X}$ with a natural real inner product $\langle\, ,\rangle$. Let $S^{53}=\{u\in R^{54}; \langle u, u\rangle=3\}$, a hypersphere of $\mathfrak{X}^C$. Set $\tilde{M}=\{u\in S^{53}; \det(u)=1\}$ and let

Φ denote the embedding $\tilde{M}\to S^{53}\subset\boldsymbol{R}^{54}$.

PROPOSITION 6.2. (1) *$\tilde{M}$ is isometric to a simply connected compact irreducible symmetric space E_6/F_4* (cf. [Yo]).

(2) *The embedding Φ is the first standard minimal immersion of $\tilde{M}=E_6/F_4$* (cf. [Oh]).

Now we define a finite group Γ acting freely and isometrically on $\boldsymbol{R}^{54}-\{0\}$ and $\tilde{M}$ by

$$\Gamma=\{1, \sigma, \sigma^2\}\cong\boldsymbol{Z}_3,$$

$$\sigma(u)=e^{(2/3)\pi\sqrt{-1}}u \qquad \text{for each } u\in\boldsymbol{R}^{54}.$$

Then the quotient $M=\tilde{M}/\Gamma$ is isometric to the symmetric space $E_6/F_4\cdot\boldsymbol{Z}_3$.

Set $K=E_6$, $H=F_4$ and $N=54$. Let R^{∇} be the curvature form of the cononical connection ∇ for (K, H). Then we have

$$\overset{2}{\wedge} T_x\tilde{M}=\mathrm{so}(T_x\tilde{M})=h_0+h_1,$$

where h_1 is isometric to the Lie algebra of F_4, which is the holonomy algebra of $\tilde{M}$. Since $\lambda_1-2c+2\mu_1<0$ by virtue of the result of [K-O-T], from (5.4) we see that

$$\Theta=\{i_U R^{\nabla}\,;\, U=\mathrm{grad}\, f_u \quad \text{for some } u\in\boldsymbol{R}^N\}$$

is an eigenspaces of $\mathcal{S}^{\nabla}$ of dimension 54 with a negative eigenvalue. From Theorem 6.1 we see $i(\nabla)=\dim\Theta$. In order to show that the canonical connection of M has $i(\nabla)=0$, it suffices to show that if $i_U R^{\nabla}\in\Theta$ is invariant by Γ, then $U=0$. It follows from the following two lemmas.

LEMMA 6.3. *Let $V\in C^{\infty}(TM)$. If*

$$\gamma(i_V R^{\nabla})=i_V R^{\nabla} \qquad \textit{for each } \gamma\in\Gamma,$$

then $\gamma_* V=V$ for eoch $\gamma\in\Gamma$.

Proof. For any $X\in T_xM$,

$$R^{\nabla}(V_x, X)=\gamma(i_V R^{\nabla})(X)=\gamma(R^{\nabla}(V_{\gamma^{-1}(x)}, \gamma_*^{-1}X))$$
$$=R^{\nabla}(\gamma_* V_{\gamma^{-1}(x)}, X),$$

hence $R^{\nabla}(\gamma_* V_{\gamma^{-1}(x)}-V_x, X)=0$. If we let the canonical decomposition $k=h+m$ at $x\in\tilde{M}$ and we use the identification $m=T_xM$, then $R^{\nabla}(X, Y)=-\mathrm{ad}_m[X, Y]$ (cf. [K-N]). Thus $\mathrm{ad}_m[\gamma_* V_{\gamma^{-1}(x)}-V_x, X]=0$ for each $X\in m$. Since $h=[m, m]$ and k is semisimple, $\gamma_* V_{\gamma^{-1}(x)}-V_x=0$. q.e.d.

LEMMA 6.4. *Let $U=\mathrm{grad}\, f_u\in C^{\infty}(TM)$ for some $u\in\boldsymbol{R}^N$. If $\gamma\in\Gamma-\{1\}$ and*

$\gamma_* U=U$, *then* $u=0$.

Proof. For each $x\in\tilde{M}$ and $X\in T_x M$,

$$\langle\gamma_* U,\ X\rangle=\langle U,\ \gamma_*^{-1}X\rangle=\langle\gamma^{-1}(X),\ u\rangle=\langle X,\ \gamma(u)\rangle=\langle U,\ X\rangle=\langle X,\ u\rangle\,,$$

hence $\langle X, \gamma(u)-u\rangle=0$. Thus $\langle x, \gamma(u)-u\rangle$ is constant in $x\in\tilde{M}$. Since $\Phi(\tilde{M})$ is not contained in any hyperplane of $\boldsymbol{R}^N$, we have $\gamma(u)=u$. Since Γ acts freely on $\boldsymbol{R}^N-\{0\}$, we get $u=0$. q.e.d.

Next we remark on weakly stable Yang-Mills fields over a quaternionic projective space $M=P_m(H)$. Generally let M be a quaternionic Kähler manifold. The $Sp(m)\cdot Sp(1)$-structure induces the orthogonal decomposition

$$\overset{2}{\wedge} T^*M=W_0+W_1+W_2\,,$$

where $(W_0)_x$, $(W_1)_x\cong sp(1)$, $(W_2)_x\cong sp(m)$ are irreducible $Sp(m)\cdot Sp(1)$-modules. The curvature form $F^\nabla=F_0^\nabla+F_1^\nabla+F_2^\nabla$ of a connection ∇ on the vector bundle E over M splits into components F_i^∇ to $End(E)\otimes W_i$ at each point. A connection ∇ with $F^\nabla=F_2^\nabla$ (resp. $F^\nabla=F_1^\nabla$) is called a B_2-connection (resp. A_1'-connection) as in [Ni], or a *self-dual* connection (resp. an *anti-self-dual* connection) as in [C-S]. They are Yang-Mills connections which minimizes the Yang-Mills functional ([C-S], [Ni]).

PROPOSITION 6.5. *Let E be a Riemannian vector bundle over $P_m(\boldsymbol{H})$. If ∇ is a weakly stable Yang-Mills connection on E satisfying $F_1^\nabla=0$, then ∇ is a B_2-connection (self-dual).*

Proof. We may suppose that g is an $Sp(m+1)$-invariant Riemannian metric on $P_m(\boldsymbol{H})=Sp(m+1)/Sp(m)\times Sp(1)$ induced by the Killing form of the Lie algebra of $Sp(m+1)$. From [K-O-T] we know

$$\begin{aligned}&\mathfrak{R}=\mathfrak{R}_0+\mathfrak{R}_1+\mathfrak{R}_2\,,\\ &\mathfrak{R}_0=0\,,\\ &\mathfrak{R}_1=(m/2(m+2))I\,,\\ &\mathfrak{R}_2=(1/2(m+2))I\,.\end{aligned}\tag{6.1}$$

Hence by virtue of (5.2), we get

$$\begin{aligned}&\mathrm{Tr}_{\mathcal{V}}\, Q_{F^\nabla}\\ &=2\int_M (F^\nabla\circ\{2\mathfrak{R}-(1/(m+2))I\},\ F^\nabla)dvol\\ &=2\Big\{-1/(m+2)\int_M (F_0^\nabla,\ F_0^\nabla)dvol+(m-1)/(m+2)\int_M (F_1^\nabla,\ F_1^\nabla)dvol\Big\}\,,\end{aligned}$$

Proposition 6.5 follows from this equation. q.e.d.

From the proof of Proposition 6.5, we see that if ∇ satisfies the assumption, then

$$\sum_{i,j=1}^{n} \langle B(e_i, e_j), u\rangle (\nabla_{e_j} F^{\nabla})(e_i, X)=0\,, \tag{6.2}$$

for all $u \in \boldsymbol{R}^N$ and all $X \in T_x M$. Using the properties of the second fundamental form of Φ and the curvature tensor field of $P_m(\boldsymbol{H})$, we can check that (6.2) implies that the restriction of F^{∇} to every quaternionic projective line $P_1(\boldsymbol{H}) \subset P_m(\boldsymbol{H})$ is always a Yang-Mills field. Hence by (5.6) and (6.1) we obtain that, for any B_2-connection ∇ over $P_m(\boldsymbol{H})$ and any infinitesimal projective transformation U on $P_m(\boldsymbol{H})$, we have $\mathcal{S}^{\nabla}(i_U F^{\nabla})=0$. This means the existence of an infinitesimal action of the projective transformation group of $P_m(\boldsymbol{H})$ on the space of all B_2-connections over $P_m(\boldsymbol{H})$. In fact, it is known that the projective transformation group of $P_m(\boldsymbol{H})$ acts on the moduli space of all B_2-connections on E.

By (5.4), (5.6) and (6.1) we obtain that the indices $i(\nabla)$ and the nullity $n(\nabla)$ of the canonical connection of $M=S^n$ $(n \geqq 5)$, $P_2(\mathrm{Cay})$ and E_6/F_4 come from $\mathrm{span}_{\boldsymbol{R}}\{i_U R^{\nabla}\,;\, U \in \mathfrak{U}\}$, and the nullities for $M=P_1(\boldsymbol{H})=S^4$ and $P_m(\boldsymbol{H})$ $(m \geqq 2)$ come from $\mathrm{span}_{\boldsymbol{R}}\{i_U R_1^{\nabla}, i_U R_2^{\nabla}\,;\, U \in \mathfrak{U}\}$ and $\mathrm{span}_{\boldsymbol{R}}\{i_U R_2^{\nabla}\,;\, U \in \mathfrak{U}\}$, respectively. We do not know whether each weakly stable canonical connection over a compact symmetric space minimizes the Yang-Mills functional. And it is interesting to investigate relationships of Yang-Mills fields with holonomy groups and the classification of vector bundles with Yang-Mills connections satisfying $\nabla F^{\nabla}=0$ over compact symmetric spaces. From results of [B-L, p. 211] and [K-O-T] we can find gap phenomena for Yang-Mills fields over every compact irreducible symmetric space which is not locally Hermitian symmetric. The classification of such Yang-Mills connections may also be useful to establish accurately isolation theorems for Yang-Mills fields over compact symmetric spaces.

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