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ENTIRE FUNCTIONS WITH RADIALLY DISTRIBUTED ZEROS

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1. In our previous paper [1], we considered the entire functions of positive integral order and obtained the following characterization of the exponential function.

THEOREM A. *Suppose that $f(z)$ is an entire function of positive integral order p, and that $f(z)$ has no zeros in a sector $\{z; |\arg z|<\pi-\pi/2p+\eta\}$ $(\eta>0)$ and $\delta(0, f)=1$. If there exists a Jordan curve l joining $z=0$ to $z=\infty$ such that*

$$f(z)f(\omega z)\cdots f(\omega^{2p-1}z)=O(1) \qquad (z\in l)$$

where $\omega=\exp(\pi i/p)$, then $f(z)=e^{P(z)}$ where $P(z)$ is a polynomial of degree p, or else

$$\lim_{r\to\infty}\frac{|\log|f(r)||}{r^p}=+\infty .$$

In this paper, we show that we can remove the condition on the deficiency. But we confine the distribution of zeros in a sector with half opening and prove the following.

THEOREM 1. *Suppose that $f(z)$ is an entire function of positive integral order p, and that $f(z)$ has only zeros in a sector $\{z; |\arg z-\pi|\leqq\pi/4p-\eta=\alpha\}$ $(\eta>0)$. If there exists a Jordan curve l joining $z=0$ to $z=\infty$ such that*

$$f(z)f(\omega z)\cdots f(\omega^{2p-1}z)=O(1) \qquad (z\in l) \tag{1}$$

where $\omega=\exp(\pi i/p)$, then $f(z)=e^{P(z)}$ where $P(z)$ is a polynomial of degree p, or else

$$\lim_{r\to\infty}\frac{|\log|f(r)||}{r^p}=+\infty . \tag{2}$$

In our previous paper [1], we also considered the entire function of order $q=2p+1$ having only negative zeros and obtained the following characterization of the exponential function.

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THEOREM B. *Suppose that* $f(z)$ *is an entire function of order* $q=2p+1$ *where* p *is a non-negative integer, having only negative zeros and* $\delta(0, f)=1$. *Further setting* $\phi(z^2)=f(z)f(-z)$, $g(z)=\phi(-z)/\phi(0)=e^{Q(z)}g_1(z)$ *where* $Q(z)$ *is a polynomial and* $g_1(z)$ *is a canonical product, we assume that there is an arbitrarily small* $\beta>0$ *such that*

$$|\log|g(re^{i\beta})g(re^{-i\beta})|-2(\cos\beta q/2)\log|g(r)||\leqq\varepsilon(r)|\log|g(r)||$$

for all sufficiently large r *where* $0\leqq\varepsilon(r)=O(1/r^{\varepsilon_0})$, $\varepsilon_0>0$ *unless* $g(z)$ *is in case* $\deg(\mathrm{Re}\, Q(r))=0$ *and* $g_1(r)\equiv 1$. *Then* $f(z)=e^{P(z)}$ *where* $P(z)$ *is a polynomial of degree* q, *or else*

$$\lim_{r\to\infty}\frac{-\log|f(r)|}{r^q}=+\infty .$$

In our previous paper [2], we considered the entire functions with zeros distributed in a sector. But we obtained only an incomplete result there.

In this paper, we show that we can remove the condition on the deficiency and that the zeros can be distributed in a sector and prove the followings.

LEMMA. *Suppose that* $g(z)=e^{Q(z)}g_1(z)$ *is an entire function of finite order having only zeros in a sector* $\{z;|\arg z-\pi|\leqq 2\alpha<\pi/2(k+1)\}$, *where* $Q(z)$ *is a polynomial,* $g_1(z)$ *is a canonical product and* k *is the genus of* $g_1(z)$. *Then the sign of* $\log|g(r)|$ *is definite for* $r\geqq r_0$ *where* r_0 *is a positive number, unless*

$$\deg(\mathrm{Re}\, Q(r))=0 \quad\text{and}\quad g_1(z)\equiv 1 . \tag{3}$$

TEEOREM 2. *Suppose that* $f(z)$ *is an entire function of order* $q=2p+1$ *where* p *is a non-negative integer, having only zeros in a sector* $\{z;|\arg z-\pi|\leqq\alpha\}$. *Further setting* $\phi(z^2)=f(z)f(-z)$, $g(z)=\phi(-z)/\phi(0)$, *we assume that there exists a positive number* β *such that*

$$\varepsilon\log|g(re^{i\beta})g(re^{-i\beta})|\leqq 2\varepsilon(\cos\beta q/2)\log|g(r)|+\varepsilon\eta(r)\log|g(r)|, \tag{4}$$

for all sufficiently large r *where* $2\alpha+\beta<\pi/(q+1)$, $\varepsilon=\pm 1$, $\varepsilon\log|g(r)|>0$ *and* $0<\eta(r)=O(1/r^{\eta_0})$, $\eta_0>0$ *for all sufficiently large* r. *Then* $f(z)=e^{P(z)}$ *where* $P(z)$ *is a polynomial of degree* q, *or else*

$$\lim_{r\to\infty}\frac{-\log|f(r)|}{r^q}=+\infty . \tag{5}$$

Our method of proof depends heavily upon the following formula.

OZAWA FORMULA [3, p-507]. Let

$$\phi(x, y)=\frac{1}{2}\log(1+2y\cos x+y^2)+\sum_{j=1}^{k}(-1)^j\frac{y^j}{j}\cos jx .$$

Then

$$\frac{\partial\phi(x, y)}{\partial x}=\frac{(-1)^{k+1}y^{k+1}}{1+2y\cos x+y^2}(\sin(k+1)x+y\sin kx).$$

We remark that $\phi(x, y)=\log|E(-ye^{ix}, k)|$, where E is the Weierstrass primary factor.

2. *Proof of Theorem* 1. Let $f(z)$ be an entire function satisfying the hypotheses in Theorem 1. We can write

$$f(z)=e^{P(z)}f_1(z)$$

where $P(z)$ is a polynomial of degree at most p and $f_1(z)$ is a canonical product with zeros $\{a_\nu\}$. Then we suppose that $f_1^*(z)$ is a canonical product with zeros $\{-|a_\nu|\}$. If the genus of $f_1(z)$ is p, then we have from Ozawa's formula,

$$(-1)^p\log|f_1(r)|\geqq(-)^p\log|f_1^*(re^{i\alpha})|$$

$$=r^{p+1}\int_0^\infty\frac{n(x)}{x^{p+1}}\frac{x\cos(p+1)\alpha+r\cos p\alpha}{x^2+r^2+2xr\cos\alpha}dx$$

$$\geqq(r^{p+1}\cos(p+1)\alpha)\int_0^\infty\frac{n(x)}{x^{p+1}}\frac{dx}{x+r}\geqq\frac{1}{2}(r^p\cos(p+1)\alpha)\int_0^r\frac{n(x)}{x^{p+1}}dx,$$

and we have (2). Therefore, by the assumption that (2) is false, we see that the genus of $f_1(z)$ is at most $p-1$. Hence we have

$$\log M(r, f_1)=o(r^p).$$

Putting

$$\phi(\zeta)=\phi(z^{2p})=f_1(z)f_1(\omega z)\cdots f_1(\omega^{2p-1}z),$$

we have $\log M(r^{2p}, \phi)\leqq 2p\log M(r, f_1)=o(r^p)$. Therefore it follows that

$$\lim_{\rho\to\infty}\frac{\log M(\rho, \phi)}{\rho^{1/2}}=0, \qquad (\rho=|\zeta|=|z|^{2p}).$$

On the other hand, by the assumption (1) we have

$$m(\rho, \phi)\leqq K<+\infty,$$

and it follows that ϕ satisfies hypothesis in Kjellberg's Lemma [1, p-19] with $\lambda=1/2$ unless $\phi(z)$ is constant. Thus we have

$$\lim_{\rho\to\infty}\frac{\log M(\rho, \phi)}{\rho^{1/2}}=\beta, \qquad 0<\beta\leqq+\infty,$$

which is a contradiction.

If ϕ is constant, then we see that $f(z)=e^{P(z)}$, where $P(z)$ is a polynomial of degree p.

3. *Proof of Lemma.* If $g_1(z)$ has zeros $\{b_\nu\}$, then we suppose that $g_1^*(z)$ is a canonical product with zeros $\{-|b_\nu|\}$. From Ozawa's formula, we have

$$(-1)^k \log|g_1(r)| \geqq (-1)^k \log|g_1^*(re^{2i\alpha})|$$

$$= r^{k+1}\int_0^\infty \frac{n(x)}{x^{k+1}} \frac{x\cos 2\alpha(k+1)+r\cos 2\alpha k}{x^2+r^2+2rx\cos 2\alpha}dx$$

$$\geqq r^{k+1}\cos 2\alpha(k+1)\int_0^\infty \frac{n(x)}{x^{k+1}}\frac{dx}{x+1} > \frac{1}{2}r^k\cos 2\alpha(k+1)\int_0^r \frac{n(x)}{x^{k+1}}dx\,.$$

Since $2\alpha(k+1)<\pi/2$, it follows that

$$(-1)^k\frac{\log|g_1(r)|}{r^k} \geqq \frac{1}{2}\cos 2\alpha(k+1)\int_0^r \frac{n(x)}{x^{k+1}}dx \longrightarrow +\infty \qquad (r\to+\infty),$$

unless case (3). Hence, if $k \geqq l = \deg(\operatorname{Re} Q(r))$, then $\operatorname{sign}(\log|g(r)| = \operatorname{sign}(\log|g_1(r)|)$ and the sign of $\log|g(r)|$ is definite for $r \geqq r_0$ where r_0 is a positive number.

If $k<l$, then we have from Ozawa's formula,

$$(-1)^k \log|g_1(r)| \leqq (-1)^k \log|g_1^*(r)| = r^{k+1}\int_0^\infty \frac{n(x)}{x^{k+1}}\frac{dx}{x+r}$$

$$\leqq r^k\int_0^r \frac{n(x)}{x^{k+1}}dx + r^{k+1}\int_r^\infty \frac{n(x)}{x^{k+2}}dx = o(\operatorname{Re} Q(r))\,.$$

Hence $\operatorname{sign}(\log|g(r)|) = \operatorname{sign}(\operatorname{Re} Q(r))$ and the sign of $\log|g(r)|$ is definite for $r \geqq r_0$ where r_0 is a positive number.

4. *Proof of Theorem* 2. Let $f(z)$ be an entire function satisfying the hypotheses in Theorem 2. We can write

$$f(z) = e^{P(z)}f_1(z)$$

where $P(z)$ is a polynomial of degree at most q and $f_1(z)$ is a canonical product. We suppose that (5) is false. Then, proceeding as in § 2 we see that the genus of $f_1(z)$ is at most $q-1=2p$. Hence we have

$$\lim_{r\to\infty}\frac{\log M(r, g)}{r^{q/2}} = 0\,. \tag{6}$$

Now we can write

$$g(z) = e^{Q(z)}g_1(z)\,,$$

where $Q(z)$ is a polynomial of degree at most p and the genus of the canonical product $g_1(z)$ is not greater than p.

We can easily deal with case (3). In this case we have

$$g(z) = \phi(-z)/\phi(0) = \exp\{i(\alpha_{k'}z^{k'} + \cdots + \alpha_1 z)\},$$

where α_j $(j=1, \cdots, k')$ are all real. Hence we have $f(z) = \exp(P(z))$ where $P(z)$

is a polynomial of degree q, which is the desired result.

Now we consider the other cases than (3).

Case (1). $\log|g(r)|>0$ and

$$\log|g(re^{i\beta})g(re^{-i\beta})|-2(\cos\beta q/2)\log|g(r)|\leqq\eta(r)\log|g(r)|$$

for all sufficiently large r.

We set

$$Q(z)=a_{k'}z^{k'}+\cdots a_1z\,,\qquad \deg(\mathrm{Re}\,Q(r))=l\quad(\leqq k')$$

and

$$\arg a_j=\theta_j\qquad(j=1,\cdots,k').$$

We define a harmonic function $H(re^{i\theta})$ in $D=\{z\,;\,0<|z|<R,\,0<\arg z<\beta\}$ as follows,

$$H(re^{i\theta})=\int_{-\theta}^{\theta}\log|g(re^{i\phi})|\,d\phi$$

$$=\frac{2}{l}|a_l|r^l\sin l\theta\cos\theta_l+\cdots+2|a_1|r\sin\theta\cos\theta_1$$

$$+\int_{-\theta}^{\theta}\log|g_1(re^{i\phi})|\,d\phi\,.$$

Furthermore we consider the subcases, denoting the genus of $g_1(z)$ by k.

Case (1-1). $k\geqq l$. In this case the sign of $\log|g(r)|$ coincides with the one of $\log|g_1(r)|$ for all sufficiently large r.

Setting $I_1=[0,\pi/2)\cup(3\pi/2,2\pi]$, $I_2=(\pi/2,3\pi/2)$ we define

$$\begin{aligned}&H_1(re^{i\theta})=\sum_{\theta_j\in I_1}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j+H_3(re^{i\theta}),\\ (7)\qquad &H_2(re^{i\theta})=\sum_{\theta_j\in I_2}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j\end{aligned}$$

where

$$H_3(re^{i\theta})=\int_{-\theta}^{\theta}\log|g_1(re^{i\phi})|\,d\phi=\int_0^{\theta}\log|g_1(re^{i\phi})g_1(re^{-i\phi})|\,d\phi\,.$$

If $g_1(z)$ has zeros $\{b_\nu\}$, then we have from Ozawa's formula

$$\left.\frac{\partial^2H_3}{\partial\theta^2}\right|_{\theta=\beta}=\frac{\partial}{\partial\theta}(\log|g_1(re^{i\theta})g_1(re^{-i\theta})|)|_{\theta=\beta}$$

$$=\sum_{\nu=1}^{\infty}\frac{\partial}{\partial\theta}\left(\log\left|E\Big(\frac{re^{i\theta}}{b_\nu},k\Big)E\Big(\frac{re^{i\theta}}{\bar{b}_\nu},k\Big)\right|\right)\Big|_{\theta=\beta}$$

$$\leqq-(\sin k\beta)^2\sum_{\nu=1}^{\infty}\frac{(r/|b_\nu|)^{k+2}}{(r/b_\nu|+1)^2}\leqq-\frac{k}{4}(\sin k\beta)^2r^k\int_0^r\frac{n(t)}{t^{k+1}}dt\,.$$

Hence we have

$$\left.\frac{\partial^2 H}{\partial \theta^2}\right|_{\theta=\beta}<0,$$

for all sufficiently large r and therefore we have from the harmonicity of $H(re^{i\theta})$,

$$\frac{\partial^2 H}{\partial(\log r)^2}=r^2\left(\frac{\partial^2 H}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial H}{\partial r}\right)>0$$

with $\theta=\beta$, for all sufficiently large r. From Ozawa's formula again we have

$$H_3(re^{i\beta})\geqq\beta r^k(\cos(k+1)(2\alpha+\beta))\int_0^r\frac{n(x)}{x^{k+1}}dx,$$

and $H(re^{i\beta})$ is unbounded. Thus we see that $H(re^{i\beta})$ is an increasing convex function of $\log r$ for all sufficiently large r.

Proceeding as in [1, p-26, 27], from (4) and (6), we find a sequence of $r=\{r_n\}$ tending to infinity with n such that

$$C_1\frac{\log|g(r)|}{r^{q/2}}-C_2\frac{\log M(2s,g)}{(2s)^{q/2}}\leqq C\frac{\log|g(r)|}{r^{q/2+\eta_0}},\tag{8}$$

where C_1, C_2 and C are positive constants which do not depend on r and s $(>r)$. For each fixed r, if s tends to ∞, then we arrive at an impossible inequality from $\eta_0>0$.

Case (1-2). $l>k$. In this case, since $Re(Q(r))$ is positive for all sufficiently large r, θ_l lies in $I_1=[0,\pi/2)\cup(3\pi/2,2\pi]$.

Firstly we assume that k is even. In this case, we use the functions H, H_1, H_2 and H_3 defind by (7). $H_1(re^{i\beta})$ is a nondecreasing convex function of $\log r$ on $(0,\infty)$ and $H_1(0)=H_1(0+)=0$. Since the degree of $H_1(re^{i\beta})-H_3(re^{i\beta})$ is higher than one of $H_2(re^{i\beta})$, $H(re^{i\beta})$ is a nondecreasing convex function of $\log r$ for all sufficiently large r. Hence arguments similar to those in case (1-1) lead to a contradiction.

Secondly we assume that k is odd. In this case we define

$$H_1(re^{i\theta})=\sum_{\theta_j\in I_1}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j,$$

$$H_2(re^{i\theta})=\sum_{\theta_j\in I_2}\frac{2}{j}|a_j|r^j\sin j\theta\cos\theta_j+H_3(re^{i\theta}),$$

where

$$H_3(re^{i\theta})=\int_0^\theta\log|g_1(re^{i\phi})g_1(re^{-i\phi})|\,d\phi.$$

Then we have $H(re^{i\theta})=H_1(re^{i\theta})+H_2(re^{i\theta})$.

It is trivial that $H_1(re^{i\beta})$ is a nondecreasing convex function of $\log r$ on $(0,\infty)$ with $H_1(0)=H_1(0+)=0$.

Now we show that $H(re^{i\beta})$ is a nondecreasing convex function of $\log r$ for all sufficiently large r.

We have from Ozawa's formula

$$-H_3(re^{i\beta})\leqq 2\beta r^{k+1}\int_0^\infty \frac{n(x)}{x^{k+1}}\frac{dx}{x+r}$$

$$\leqq 2\beta r^k\int_0^r \frac{n(x)}{x^{k+1}}dx+2\beta r^{k+1}\int_r^\infty \frac{n(x)}{x^{k+2}}dx=o(r^l) \qquad (r\to\infty).$$

Hence $|H_3(re^{i\beta})|/r^l\to 0$ as $r\to+\infty$ and $H(re^{i\beta})$ is unbounded.

If $g_1(z)$ has zeros $\{b_\nu\}$, then we have from Ozawa's formula again,

$$\left.\frac{\partial^2 H_3}{\partial\theta^2}\right|_{\theta=\beta}\leqq\frac{2}{\cos(\beta+2\alpha)}\sum_{\nu=1}^\infty\frac{(r/|b_\nu|)^{k+1}}{(1+r/|b_\nu|)^2}\left\{1+\frac{r}{|b_\nu|}\right\}$$

$$\leqq\frac{2(k+1)r^{k+1}}{\cos(\beta+2\alpha)}\int_0^\infty\frac{n(t)}{t^{k+1}}\frac{dt}{1+r/t}$$

$$\leqq\frac{2(k+1)r^{k+1}}{\cos(\beta+2\alpha)}\frac{1}{r}\int_0^r\frac{n(t)}{t^{k+1}}dt+\frac{2(k+1)r^{k+1}}{\cos(\beta+2\alpha)}\int_r^\infty\frac{n(t)}{t^{k+2}}dt.$$

Hence $(\partial^2 H/\partial\theta^2)_{\theta=\beta}$ is negative and $(\partial^2 H/\partial(\log r)^2)_{\theta=\beta}$ is positive for all sufficiently large r. Therefore $H(re^{i\beta})$ is a nondecreasing convex function of $\log r$ for all sufficiently large r. Thus arguments similar to those in case (1-1) lead to a contradiction.

Case (2). $\log|g(r)|<0$ and $\log|g(re^{i\beta})g(re^{-i\beta})|-2(\cos\beta q/2)\log|g(r)|\geqq \eta(r)\log|g(r)|$ for all sufficiently large r.

Put $\tilde{Q}(z)=-Q(z)$, $\tilde{g}_1(z)=g_1(z)^{-1}$ and $\tilde{g}(z)=e^{\tilde{Q}(z)}\tilde{g}_1(z)$. Then (4) is equivalent to

$$\log|\tilde{g}(re^{i\beta})\tilde{g}(re^{-i\beta})|-2(\cos\beta q/2)\log|\tilde{g}(r)|\leqq\eta(r)\log|\tilde{g}(r)|.$$

Thus our case is handled in a fashion almost similar to case (1).

We only show how to handle the inequality corresponding to (8). Proceeding as in case (1-1), we have

$$C_1\frac{\log|\tilde{g}(r)|}{r^{q/2}}-C_2\frac{\log M_\beta(2s,\tilde{g})}{(2s)^{q/2}}\leqq C\frac{\log|\tilde{g}(r)|}{r^{q/2+\eta_0}}, \tag{9}$$

where $M_\beta(2s,\tilde{g})=\sup_{|\theta|<\beta}|\tilde{g}(2se^{i\theta})|$. In this inequality we must show that

$$\lim_{r\to\infty}\frac{\log M_\beta(r,\tilde{g})}{r^{q/2}}=0.$$

Since $\log M_\beta(r,\tilde{g})\leqq\sup_{|\theta|<\beta}\mathrm{Re}(\tilde{Q}(re^{i\theta}))+\log M_\beta(r,\tilde{g}_1)$ and $\lim_{r\to\infty}\{\sup_{|\theta|<\beta}\mathrm{Re}(R(\tilde{Q}(re^{i\theta}))\}/r^{q/2}=0$, it is sufficient to show that

$$\lim_{r\to\infty}\frac{\log M_\beta(r,\tilde{g}_1)}{r^{q/2}}=0, \tag{10}$$

in the case that the genus of $g_1(z)$ is not smaller than the degree of $\mathrm{Re}(Q(r))$.

Since (6) implies $\lim_{r\to\infty}\{\log M(r, g_1)\}/r^{q/2}=0$, we have $m_\beta(r, \tilde{g}_1)/r^{q/2}\to 0$ as $r\to\infty$, where

$$m_\beta(r, \tilde{g}_1)=\frac{1}{2\pi}\int_{-\beta}^{\beta}\log^+|\tilde{g}_1(re^{i\theta})|\,d\theta=\frac{1}{2\pi}\int_{-\beta}^{\beta}(-\log|g_1(re^{i\theta})|)^+d\theta\,.$$

Now in this case, we have from Ozawa's formula for θ ($|\theta|\leqq\beta$),

$$\begin{aligned}-\log|g_1(re^{i\theta})|&\geqq r^{k+1}\int_0^\infty\frac{n(x)}{x^{k+1}}\frac{x\cos(k+1)(\beta+2\alpha)+r\cos k(\beta+2\alpha)}{x^2+r^2+2xr\cos(\beta+2\alpha)}dx\\&\geqq(\cos(k+1)(\beta+2\alpha))r^{k+1}\int_0^\infty\frac{n(x)}{x^{k+1}}\frac{dx}{x+r}\\&\geqq(\cos(k+1)(\beta+2\alpha))\log M_\beta(r, \tilde{g}_1)\,.\end{aligned}$$

Hence we obtain (10).

Proceeding as in case (1), we have a contradiction from (9).

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