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ON SURFACES OF FINITE TYPE IN EUCLIDEAN 3-SPACE

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Abstract

We prove an extension of T. Takahashi's result on minimal submanifolds in Euclidean spaces and in spheres, and as a corollary obtain support for B. Y. Chen's conjecture which claims that the round spheres are the only compact surfaces of finite type in Euclidean 3-space.

Let M^n be a (connected) n-dimensional submanifold in $\boldsymbol{E}^m$, the m-dimensional Euclidean space. Let x, H and Δ respectively be the *position vector field*, the *mean curvature field* and the *Laplace operator* of the induced metric on M^n. Then, as is well known (see e. g. [**2**]),

$$\Delta x = -nH, \tag{1.1}$$

which shows, in particular, that M^n is a *minimal submanifold in* $\boldsymbol{E}^m$ if and only if its coordinate functions are *harmonic* (i. e. they are eigenfunctions of Δ with eigenvalue 0). Moreover, in this context, T. Takahashi [**6**] proved that the submanifolds M^n for which

$$\Delta x = \lambda x, \tag{1.2}$$

i. e. for which all coordinate functions are eigenfunctions of Δ with the same eigenvalue $\lambda \in \boldsymbol{R}$, are precisely either the minimal submanifolds of $\boldsymbol{E}^m$ ($\lambda=0$) or the *minimal submanifolds* M^n *of hyperspheres* S^{m-1} in $\boldsymbol{E}^m$ (the case when $\lambda \neq 0$, actually $\lambda > 0$). In terms of B. Y. Chen's theory of submanifolds in $\boldsymbol{E}^m$ of *finite type*, condition (1.2) asserts that M^n is of *1-type* in $\boldsymbol{E}^m$. In general, a submanifold M^n in $\boldsymbol{E}^m$ is said to be of finite type if its spectral decomposition of x is finite, i. e. if

$$x = x_0 + \sum_{t=p}^{q} x_t \tag{1.3}$$

where p and q are natural numbers, such that $x_0 \in \boldsymbol{R}^m$ is a fixed vector and

$$\Delta x_t = \lambda_t x_t, \tag{1.4}$$

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where λ_t denotes an eigenvalue of Δ [**1**] [**2**]; when there are exactly k non-constant eigenvectors x_t appearing in (1.3), which all belong to different eigenvalues λ_t, then M^n is said to be of k-type in $\boldsymbol{E}^m$. Many important submanifolds in Euclidean spaces turn out to be of finite type in this sense. To find out whether or not a compact submanifold M^n in $\boldsymbol{E}^m$ is of finite type, the following result is very useful.

THEOREM A. (B. Y. Chen [**2**])

(i) *M^n is of finite type in $\boldsymbol{E}^m$ if and only if there exists a non-trivial polynomial Q (of one variable) such that $Q(\Delta)H=0$.*

(ii) *If M^n is of finite type, then there exists a unique monic polynomial P (of one variable), of least degree and such that $P(\Delta)H=0$.*

(iii) *If M^n is of finite type, then M^n is of k-type if and only if* degree $P=k$.

The same results hold if H is replaced by $x-x_0$, x_0 being the center of mass of M^n in E^m.

In [**3**], B. Y. Chen studies the following problem.

QUESTION. *Other than minimal surfaces and ordinary spheres, which surfaces in $\boldsymbol{E}^3$ are of finite type?*

Restricting attention to surfaces in $\boldsymbol{E}^3$, the above result on $\Delta x=\lambda x$, $\lambda\in\boldsymbol{R}$, can be stated as follows (which also somewhat clarifies the previous Question).

THEOREM B. (T. Takahashi [**6**])

A surface in $\boldsymbol{E}^3$ is of 1-type if and only if it is a sphere or a minimal surface.

With respect to the Question, the following result is quite interesting.

THEOREM C. (B. Y. Chen [**3**])

A tube in $\boldsymbol{E}^3$ is of finite type if and only if it is a circular cylinder (which actually is of 2-type).

As a corollary we mention the following,

COROLLARY D. (B. Y. Chen [**3**])

Every closed tube in $\boldsymbol{E}^3$ is of infinite type,

Which offers a partial solution to the following

CONJECTURE OF B. Y. CHEN.

Ordinary spheres are the only compact finite type surfaces in $\boldsymbol{E}^3$.

Of course, since there are no compact minimal surfaces $\boldsymbol{E}^3$, Theorem B settles the matter for 1-type surfaces.

In [**5**], O. Garay studies the hypersurfaces M^n in $\boldsymbol{E}^{n+1}$ for which

(1.5) $$\Delta x = Ax\,,$$

where A is a diagonal matrix

(1.6) $$A=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{n+1}\end{pmatrix},\qquad \lambda_i\in\boldsymbol{R},\ i\in\{1, 2, \cdots, n+1\},$$

(see also [4] for the case of surfaces of revolution M^2 in $\boldsymbol{E}^3$). This means that he imposes the condition that the coordinate functions of M^n are eigenfunctions of their Laplacian Δ with possibly distinct eigenvalues λ_i; hence, O. Garay's condition ((1.5), (1.6)) can be seen as a generalization of T. Takahashi's condition (1.2), in which case all λ_i are equal. O. Garay proved that if a hypersurface M^n of $\boldsymbol{E}^{n+1}$ satisfies his condition, it is either *minimal* in $\boldsymbol{E}^{n+1}$ or it is a sphere or it is a *spherical cylinder*. In this respect, we want to observe however that his condition is not coordinate-invariant; e. g. in $\boldsymbol{E}^3$ a circular cylinder satisfies this condition if and only if its axis of symmetry is one of the coordinate axes.

In this paper, we will study the surfaces in $\boldsymbol{E}^3$ which satisfy

(*) $$\Delta x = Ax + B\,,$$

where $A\in\boldsymbol{R}^{3\times 3}$ and $B\in\boldsymbol{R}^3$. This setting generalizes T. Takahashi's condition, following O. Garay's idea, in a way which is independent of the choice of coordinates. Our main result is the following.

THEOREM. *A surface M^2 in $\boldsymbol{E}^3$ satisfies* (*) *if and only if it is an open part of a minimal surface, a sphere or a circular cylinder.*

In particular, this yields the following

COROLLARY. *A compact surface in $\boldsymbol{E}^3$ satisfies* (*) *if and only if it is a sphere.*

We want to mention that this Corollary supports the above Conjecture of B. Y. Chen. Indeed, the compact surfaces M^2 in $\boldsymbol{E}^3$ satisfying (*) are particular surfaces of finite type ($\leqq 3$); actually, the following arguments, which will make this clear, also hold more generally for any compact submanifold M^n in $\boldsymbol{E}^m$ which satisfies a condition of the form (*). Namely, integrating (*) over M^2, and using the divergence theorem, implies that

(1.7) $$Ax_0 + B = 0\,.$$

Using this, then (*) further implies that

(1.8) $$\Delta(x - x_0) = A(x - x_0)\,,$$

and, hence, that

$$P(\Delta)(x-x_0)=P(A)(x-x_0)\,, \tag{1.9}$$

where P is any polynomial in one variable. In particular, choosing for P the characteristic polynomial of A, by the Cayley-Hamilton theorem $P(A)=0$, and thus (1.9) shows that

$$P(\Delta)(x-x_0)=0\,. \tag{1.10}$$

Finally, Theorem A then asserts that M^2 is a surface of type $\leqq 3$ in $\boldsymbol{E}^3$.

We first show that the surfaces mentioned in the theorem indeed satisfy condition (*).

Examples.

(1) *Minimal surface*

In this case we have that the mean curvature is zero, so by (1.1) a minimal surface satisfies (*) with $A=0$.

(2) *Sphere*

The sphere $S_0^2(r)$ with center 0 and radius r satisfies (*) with

$$A=\begin{pmatrix} \frac{2}{r^2} & 0 & 0 \\ 0 & \frac{2}{r^2} & 0 \\ 0 & 0 & \frac{2}{r^2} \end{pmatrix}.$$

Indeed, the sphere has mean curvature $-1/r$ and $(1/r)x$ is a unit normal on $S_0^2(r)$. So by (1.1)

$$\Delta x=\frac{2}{r^2}x\,.$$

(3) *Circular cylinder*

We consider the cylinder on the circle of radius r with center 0 lying in the $\{e_1, e_2\}$-plane. This surface has mean curvature $-1/2r$. A unit normal is given by $(1/r)\pi(x)$, where π is the projection on the $\{e_1, e_2\}$-plane. Hence by (1.1)

$$\Delta x=\frac{1}{r^2}\pi(x)\,.$$

So this cylinder satisfies (*) with

$$A=\begin{pmatrix} \frac{1}{r^2} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof of the Theorem. We consider two cases.

First case: M^2 is a cylinder.

In this case, the position vector x of M^2 can be given by

$$x=\gamma(s)+t\xi$$

where s, t are parameters, ξ is a constant vector and $\gamma(s)$ is a curve, with arc-length parametrization, in a plane orthogonal to ξ.

From the definition of the Laplacian, one checks that

$$\Delta x=\gamma''$$

where γ'' is the acceleration vector of γ.

Without loss of generality we may suppose that $\xi=(0, 0, 1)$ and that $\gamma(s)=(\gamma_1(s), \gamma_2(s), 0)$. If we write

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

then equation (*) becomes

$$\begin{aligned} \gamma_1''&=a_{11}\gamma_1+a_{12}\gamma_2+a_{13}t+b_1\,, \\ \gamma_2''&=a_{21}\gamma_1+a_{22}\gamma_2+a_{23}t+b_2\,, \\ 0&=a_{31}\gamma_1+a_{32}\gamma_2+a_{33}t+b_2\,. \end{aligned} \tag{2.1}$$

Since γ_1'', γ_2'' do not depend on t, we find that $a_{13}=a_{23}=a_{33}=0$.

If $a_{31}\neq 0$ or $a_{32}\neq 0$, the curve γ is a line, so M^2 will be part of a plane and hence minimal. So we suppose further that $a_{31}=a_{32}=0$ and that γ isn't a line. This implies that $b_3=0$. System (2.1) reduces to

$$\begin{aligned} \gamma_1''&=a_{11}\gamma_1+a_{12}\gamma_2+b_1\,, \\ \gamma_2''&=a_{21}\gamma_1+a_{22}\gamma_2+b_2\,, \end{aligned}$$

or, in vector notation

$$\gamma''=\tilde{A}\gamma+\tilde{B}\,, \tag{2.2}$$

where

$$\tilde{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \tilde{B}=\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We now use the Frenet frame $\{T, N\}$ of the curve γ. The curve has arc-length parametrization, so $T=\gamma'$, the velocity vector of γ.

Equation (2.2) becomes

$$T'=\tilde{A}\gamma+\tilde{B}\,.$$

Using the Frenet formula $T'=\kappa N$ where κ is the curvature function of γ, we get

$$\kappa N=\tilde{A}\gamma+\tilde{B}\,.$$

Derivation of this equation gives

$$\kappa' N+\kappa N'=\tilde{A}T\,.$$

From the second Frenet formula $N'=-\kappa T$ we obtain

$$\kappa' N-\kappa^2 T=\tilde{A}T\,. \tag{2.3}$$

We derive again to obtain

$$\kappa'' N+\kappa' N'-2\kappa\kappa' T-\kappa^2 T'=\tilde{A}T'$$

or

$$(\kappa''-\kappa^3)N-3\kappa\kappa' T=\kappa\tilde{A}N\,. \tag{2.4}$$

From (2.3) and (2.4) we can compute the entries of the matrix $\tilde{A}$ with respect to the frame $\{T, N\}$

$$\tilde{A}T.T=-\kappa^2\,,$$

$$\tilde{A}T.N=\kappa'\,,$$

$$\tilde{A}N.T=-3\kappa'\,,$$

$$AN.N=\frac{1}{\kappa}(\kappa''-\kappa^3)\,.$$

The determinant $(\tilde{A}T.T)(\tilde{A}N.N)-(\tilde{A}T.N)(\tilde{A}N.T)$ and the trace $(\tilde{A}T.T)+(\tilde{A}N.N)$ of the matrix $\tilde{A}$ are constant, so there exist constants c and d such that

$$-\kappa\kappa''+\kappa^4+3(\kappa')^2=c\,, \tag{2.5}$$

$$\frac{\kappa''}{\kappa}-2\kappa^2=d\,. \tag{2.6}$$

Eliminating κ'' from these two equations we find that

$$(\kappa')^2=\frac{1}{3}(c+d\kappa^2+\kappa^4)\,.$$

Deriving this last equation gives

$$\kappa'\kappa''=\frac{1}{3}(d\kappa\kappa'+2\kappa^3\kappa')\,.$$

If we suppose that $\kappa' \not\equiv 0$, then we have

$$\kappa'' = \frac{1}{3}(d\kappa + 2\kappa^3).$$

Substitution in (2.6) gives

$$\kappa(d + 2\kappa^2) = 0$$

which contradicts the assumption that κ' wasn't identically zero. Hence the only solution to the system (2.2) is that κ is a constant and that γ is a circle. So the only cylinder which satisfies (*) is a circular cylinder.

Second case: M^2 is not a cylinder.

(1) *Rank of A is* 3.

In this case we may suppose that $B=0$. Indeed, let $C \in \boldsymbol{R}^{3\times 1}$ be a solution of $A.C+B=0$. Define new coordinates x' by $x=x'+C$. Then equation (*) becomes

$$\Delta x' = Ax'.$$

Suppose now that M^2 is given locally as the graph of a function f, this is

$$x = (x_1, x_2, f(x_1, x_2)).$$

From (1.1) we see that $\Delta x = Ax$ is normal to the surface, so

$$\begin{aligned} &Ax.\left(1, 0, \frac{\partial f}{\partial x_1}\right) = 0, \\ (3.1) \qquad &\\ &Ax.\left(0, 1, \frac{\partial f}{\partial x_2}\right) = 0, \end{aligned}$$

since $(1, 0, \partial f/\partial x_1)$ and $(0, 1, \partial f/\partial x_2)$ are tangent vectors.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix};$$

then system (3.1) becomes

$$\frac{\partial f}{\partial x_1} = -\frac{a_{11}x_1 + a_{12}x_2 + a_{13}f}{a_{31}x_1 + a_{32}x_2 + a_{33}f},$$

$$\frac{\partial f}{\partial x_2} = -\frac{a_{21}x_1 + a_{22}x_2 + a_{23}f}{a_{31}x_1 + a_{32}x_2 + a_{33}f}.$$

Since the function f satisfies

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}=\frac{\partial^2 f}{\partial x_2 \partial x_1},$$

the two above equations imply that

$$\begin{aligned}&(a_{21}-a_{12})(a_{31}x_1+a_{32}x_2+a_{33}f)\\&\quad+(a_{32}-a_{23})(a_{11}x_1+a_{12}x_2+a_{13}f)\\&\quad+(a_{13}-a_{31})(a_{21}x_1+a_{22}x_2+a_{23}f)=0\,.\end{aligned}$$

We may suppose that x_1, x_2 and f are linearly independent, and so we get

$$(a_{21}-a_{12})a_{31}+(a_{32}-a_{23})a_{11}+(a_{13}-a_{31})a_{21}=0\,,$$

$$(a_{21}-a_{12})a_{32}+(a_{32}-a_{23})a_{12}+(a_{13}-a_{31})a_{22}=0\,,$$

$$(a_{21}-a_{12})a_{33}+(a_{32}-a_{23})a_{13}+(a_{13}-a_{31})a_{23}=0\,.$$

If we denote the cofactor of the entry a_{ij} in the matrix A by A_{ij}, this system reduces to

$$A_{23}=A_{32}\,,$$

$$A_{13}=A_{31}\,,$$

$$A_{12}=A_{21}\,,$$

i. e. the matrix A^{cof} of cofactors of A is symmetric. Since

$$A^{-1}=\frac{1}{\det A}\cdot A^{\mathrm{cof}}\,,$$

we find that A^{-1} is symmetric. Hence A is also a symmetric matrix.

After a coordinate transformation we may suppose that A is a diagonal matrix

$$A=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}$$

with $\lambda_1.\lambda_2.\lambda_3\neq 0$.

Suppose now that $(x_1(u, v), x_2(u, v), x_3(u, v))$ is a parametrization of the surface. Then, since $Ax=(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)$ is normal to the surface, we have that

$$(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)\cdot\left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)=0\,,$$

$$(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3)\cdot\left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right)=0\,,$$

or

$$\frac{\partial}{\partial u}(\lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2)=0,$$

$$\frac{\partial}{\partial v}(\lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2)=0.$$

So

$$\lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2=c, \tag{3.2}$$

where c is a constant, and we see that M^2 is part of a quadratic surface. For this quadratic surface one computes the mean curvature

$$\|H\|=\pm\frac{(\lambda_2+\lambda_3)\lambda_1^2 x_1^2+(\lambda_1+\lambda_3)\lambda_2^2 x_2^2+(\lambda_1+\lambda_2)\lambda_3^2 x_3^2}{2(\lambda_1^2 x_1^2+\lambda_2^2 x_2^2+\lambda_3^2 x_3^2)^{3/2}}.$$

From (*) and (1.1), we have that the absolute value of the mean curvature equals $(1/2)\|Ax\|$, which implies that

$$(\lambda_1^2 x_1^2+\lambda_2^2 x_2^2+\lambda_3^2 x_3^2)^2\pm((\lambda_2+\lambda_3)\lambda_1^2 x_1^2+(\lambda_1+\lambda_3)\lambda_2^2 x_2^2+(\lambda_1+\lambda_2)\lambda_3^2 x_3^2)=0. \tag{3.3}$$

From (3.2) we have that

$$x_3^2=\frac{1}{\lambda_3}(c-\lambda_1 x_1^2-\lambda_2 x_2^2).$$

If we substitute this in (3.3), we obtain a polynomial in x_1 and x_2 which has to be identically zero, so in particular the coefficients of x_1^4 and x_2^4, which are $\lambda_1^2(\lambda_1-\lambda_3)^2$ respectively $\lambda_2^2(\lambda_2-\lambda_3)^2$ have to be zero. So we find that $\lambda_1=\lambda_2=\lambda_3$. Hence M^2 is a sphere. The constant term of the polynomial, which is $c\lambda_3(c\lambda_3-\lambda_1-\lambda_2)$, also has to be zero. From this we find that $c=2$. So if we write r for the radius of the sphere we have

$$\lambda_1=\lambda_2=\lambda_3=\frac{2}{r^2}.$$

(2) *Rank of A is 2.*

By choosing a basis $\{e_1, e_2, e_3\}$ with $e_1, e_2\in\operatorname{Im} A$ and $e_3\in(\operatorname{Im} A)^\perp$, we may suppose that A and B have the form

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad\text{and}\quad B=\begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix}.$$

If $B=0$, then $\Delta x=-2H$ belongs to $\operatorname{Im} A$ which is a plane through the origin. This means that the normal on this plane is a constant tangent direction to M^2, but this isn't possible since M^2 isn't a cylinder. So we may suppose that $b_3\neq 0$.

Consider the set

$$U=\{p\in M^2 \mid (e_3)_p \notin T_pM^2\}.$$

Since

$$U=\left\{p\in M^2 \middle| \begin{vmatrix} \left.\frac{\partial x_1}{\partial u}\right|_p & \left.\frac{\partial x_2}{\partial u}\right|_p \\ \left.\frac{\partial x_1}{\partial v}\right|_p & \left.\frac{\partial x_2}{\partial v}\right|_p \end{vmatrix} \neq 0\right\},$$

this is an open set, and by the assumption that M^2 is not a cylinder, U cannot be empty. By the inverse function theorem, on U the surface is locally given as the graph of a function f in the following way

$$x=(x_1,\ x_2,\ f(x_1,\ x_2)).$$

From (1.1) we see that $\Delta x=Ax+B$ is normal to the surface, so

$$(Ax+B)\cdot\left(1,\ 0,\ \frac{\partial f}{\partial x_1}\right)=0,$$

$$(Ax+B)\cdot\left(0,\ 1,\ \frac{\partial f}{\partial x_2}\right)=0,$$

or

$$\frac{\partial f}{\partial x_1}=\frac{1}{b_3}(a_{11}x_1+a_{12}x_2+a_{13}f),$$

$$\frac{\partial f}{\partial x_2}=\frac{1}{b_3}(a_{21}x_1+a_{22}x_2+a_{23}f).$$

Since f satisfies

$$\frac{\partial^2 f}{\partial x_1\partial x_2}=\frac{\partial^2 f}{\partial x_2\partial x_1},$$

we have

$$(a_{12}-a_{21})b_3+(a_{13}a_{21}-a_{11}a_{23})x_1+(a_{13}a_{22}-a_{12}a_{23})x_2=0\,,$$

or

(3.4) $$\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}=0,$$

(3.5) $$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}=0,$$

(3.6) $$a_{12}=a_{21}\,.$$

Since A has rank 2, expressions (3.4) and (3.5) imply that

$$a_{13}=a_{23}=0\,.$$

Equation (3.6) shows that the matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is symmetric. By a coordinate transformation we may suppose that A has the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\lambda_1 \cdot \lambda_2 \neq 0$.

Suppose now that $(x_1(u, v), x_2(u, v), x_3(u, v))$ is a parametrization of the surface. Then, since $Ax+B=(\lambda_1 x_1, \lambda_2 x_2, b_3)$ is normal to the surface, we have that

$$(\lambda_1 x_1, \lambda_2 x_2, b_3)\cdot\left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right)=0,$$

$$(\lambda_1 x_1, \lambda_2 x_2, b_3)\cdot\left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right)=0,$$

or

$$\frac{\partial}{\partial u}(\lambda_1 x_1^2+\lambda_2 x_2^2+2b_3 x_3)=0,$$

$$\frac{\partial}{\partial v}(\lambda_1 x_1^2+\lambda_2 x_2^2+2b_3 x_3)=0.$$

So

$$\lambda_1 x_1^2+\lambda_2 x_2^2+2b_3 x_3=c,$$

where c is a constant, and we see that M^2 should be part of a quadratic surface. However, for this quadratic surface one computes the mean curvature

$$\|H\|=\pm\frac{\lambda_1^2\lambda_2 x_1^2+\lambda_1\lambda_2^2 x_2^2+(\lambda_1+\lambda_2)b_3^2}{2(\lambda_1^2 x_1^2+\lambda_2^2 x_2^2+b_3^2)^{3/2}}.$$

The absolute value of the mean curvature equals $(1/2)\|Ax+B\|$ by (1.1). This implies that the polynomial

$$(\lambda_1^2 x_1^2+\lambda_2^2 x_2^2+b_3^2)^2\pm(\lambda_1^2\lambda_2 x_1^2+\lambda_1\lambda_2^2 x_2^2+(\lambda_1+\lambda_2)b_3^2)$$

should be identically zero, which contradicts $\lambda_1\cdot\lambda_2\neq 0$.

(3) *Rank of A is* 1.

Since $\Delta x=-2H$, equation (*) implies that $-2H$ lies on $\operatorname{Im} A+B$ which is a line. So a vector orthogonal to a plane which contains the line $\operatorname{Im} A+B$ and the origin, is everywhere tangent to the surface M^2. This contradicts our assumption that M^2 isn't a cylinder.

(4) *Rank of A is* 0.
In this case (*) becomes

$$\Delta x = B .$$

If $B=0$, then we have by (1.1) that $H=0$, so the surface is minimal. If $B\neq 0$, equation (1.1) implies that B is a constant vector normal to M^2, so M^2 is a plane. However for a plane we have that $H=0$, which contradicts $B\neq 0$. ■

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