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SUBMANIFOLDS OF QUATERNION PROJECTIVE SPACE WITH BOUNDED SECOND FUNDAMENTAL FORM

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Abstract. Let h be the second fundamental form of a compact submanifold M of the quaternion projective space $\boldsymbol{HP}^n(1)$. For any unit vector $u\in TM$, set $\delta(u)=\|h(u,u)\|^2$. We determine all compact totally complex submanifolds of $\boldsymbol{HP}^n(1)$ (resp. all compact totally real minimal submanifolds of $\boldsymbol{HP}^n(1)$) satisfying condition $\delta(u)\leqq\frac{1}{4}$ (resp. $\delta(u)\leqq\frac{1}{12}$) for all unit vectors $u\in TM$.

1. Introduction.

Let M be a smooth m-dimensional Riemannian manifold isometrically immersed in an $(m+p)$-dimensional Riemannian manifold $\tilde{M}$. Let h denote the second fundamental form of this immersion. For each $x\in M$, h is a bilinear mapping from $TM_x\times TM_x$ into $TM_x^{\perp}$, where TM_x is the tangent space of M at x and $TM_x^{\perp}$ is the normal space. We denote by $S(x)$ the square of the length of h at $x\in M$. By Gauss' equation we have $S(x)=m(m-1)-\rho(x)$, whenever M is immersed as a minimal submanifold of $S^{m+p}(1)$ with scalar curvature $\rho(x)$ at x in M. Therefore $S(x)$ is an intrinsic invariant of M.

In 1968, J. Simons [12] discovered for the class of compact minimal m-dimensional submanifolds of the unit $(m+p)$-sphere that the totally geodesic submanifolds are isolated in the following sense: If $S(x)<n/(2-1/p)$ for all $x\in M$, then $S(x)\equiv 0$ on M, and thus M is totally geodesic. In [1], S. S. Chern, M. do Carmo, and S. Kobayashi determined all minimal submanifolds of the unit sphere satisfying $S(x)\equiv n/(2-1/p)$. Later similar results were obtained for various types of minimal submanifolds of the complex projective spaces and the quaternion projective spaces.

Let $T: UM\to M$ and UM_x denote the unit tangent bundle of M along with its fibre over $x\in M$. We set $\delta(u)=\|h(u, u)\|^2$ for $u\in UM$. Observe that $\delta(u)$ is not an intrinsic invariant of the submanifold M. However, like $S(x)$, $\delta(u)$ can be considered as a natural measure of the degree to which an immersion fails to be totally geodesic.

In [10], and [11], A. Ros proved that if M is a compact Kaehler submanifold of $\boldsymbol{CP}^n(1)$ and if $\delta(u)<1/4$, for any $u\in UM$, then M is totally geodesic in

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$\boldsymbol{CP}^n(1)$. Ros also gives a complete list of Kaehler submanifolds in $\boldsymbol{CP}^n(1)$ which satisfy the condition

$$\max_{u\in UM}\{\delta(u)\}=1/4.$$

One of the authors obtained results ([4], [5]) similar to the results of Ros for minimal submanifolds of a sphere and for totally real minimal submanifolds of $\boldsymbol{CP}^n(1)$. In the present paper we obtain analogous results for totally complex and totally real minimal submanifolds of quaternion projective space $\boldsymbol{HP}^n(1)$.

Recall the standard totally complex imbeddings [3]:

$$\tau:\ \boldsymbol{CP}^n(1)\longrightarrow \boldsymbol{HP}^n(1),$$

along with the following standard imbeddings [8]:

$$\tilde{\phi}_1:\ \boldsymbol{CP}^m(1/2)\longrightarrow \boldsymbol{CP}^k(1),\quad \text{where } k=m(m+3)/2$$

$$\tilde{\phi}_2:\ \boldsymbol{CP}^{m-s}(1)\times \boldsymbol{CP}^s(1)\longrightarrow \boldsymbol{CP}^k(1),\quad k=m+s(m-s)$$

$$\tilde{\phi}_3:\ Q^m\longrightarrow \boldsymbol{CP}^{m+1}(1),\quad m\geqq 3 \text{ and } Q \text{ is the standard complex quadric.}$$

$$\tilde{\phi}_4:\ U\left(\frac{m+4}{2}\right)/U(2)\times U(m/2)\longrightarrow \boldsymbol{CP}^k(1),\quad k=m(m+10)/8$$

$$\tilde{\phi}_5:\ SO(10)/U(5)\longrightarrow \boldsymbol{CP}^{15}(1)$$

$$\tilde{\phi}_6:\ E_6/\mathrm{Spin}(10)\times T\longrightarrow \boldsymbol{CP}^{26}(1).$$

We define the imbeddings of $\phi_i=\tau\circ\tilde{\phi}_i$, which we call the Nakagawa-Takagi imbeddings or the NT imbeddings.

THEOREM 1. *Let M be a compact totally complex submanifold of real dimension $2m$, immersed in the quaternion projective space $\boldsymbol{HP}^n(1)$. If $\delta(u)\leqq 1/4$ for all $u\in UM$, then either*

(i) $\delta(u)\equiv 0$ *and M is totally geodesic in $\boldsymbol{HP}^n(1)$,*

or

(ii) $\mathrm{Max}\{\delta(u)\}=1/4$ *and M is an imbedded submanifold congruent to one of the NT-imbeddings.*

Note that the real dimensions of M for the imbeddings $\phi_1, \phi_2, \cdots, \phi_6$ are $2m$, $2m$, $2m$, $2m$, 20 and 32 respectively.

THEOREM 2. *Let $\phi: M\to\boldsymbol{HP}^n(1)$ be a totally complex immersion of a compact Kaehler manifold M into $\boldsymbol{HP}^n(1)$. Let H denote the holomorphic sectional curvature of M. If $H>1/2$, then M is totally geodesic. If $H\geqq 1/2$ and M is not totally geodesic, then ϕ is congruent to one of the six NT-imbeddings.*

Recall the totally real imbeddings [2]:

$$\nu: \boldsymbol{RP}^n(1/4) \longrightarrow \boldsymbol{HP}^n(1),$$

and the first standard imbeddings of projective spaces:

$$\tilde{\phi}_1: \boldsymbol{RP}^2(1/12) \longrightarrow \boldsymbol{RP}^4(1/4)$$

$$\tilde{\phi}_2: \boldsymbol{CP}^2(1/3) \longrightarrow \boldsymbol{RP}^7(1/4)$$

$$\tilde{\phi}_3: \boldsymbol{HP}^2(1/3) \longrightarrow \boldsymbol{RP}^{13}(1/4)$$

$$\tilde{\phi}_4: \mathrm{Cay}\boldsymbol{P}^2(1/3) \longrightarrow \boldsymbol{RP}^{25}(1/4).$$

THEOREM 3. *Let M be a compact totally real minimal submanifold of dimension m, immersed in the quaternion projective space $\boldsymbol{HP}^n(1)$. If $\delta(u) \leqq 1/12$ for all $u \in UM$, then either*

(i) $\delta(u) \equiv 0$ *and M is totally geodesic in $\boldsymbol{HP}^n(1)$*

or

(ii) $\mathrm{Max}\{\delta(u)\} = 1/12$ *and M is either congruent to one of the imbeddings $\phi_i = \nu \circ \tilde{\phi}_i$ or to the immersion $\phi_5 = \phi_1 \circ \pi$, where $\pi: S^2(1/12) \to \boldsymbol{RP}^w(1/12)$ is the covering map.*

Note that the dimension of M for the mappings $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ are 2, 4, 8, 16, and 2 respectively.

2. Quaternion Kaehler Manifolds.

Let N be a differentiable manifold of dimension $4n$, and assume that there is a 3-dimensional vector bundle V, [6], consisting of tensors of type (1, 1) over N satisfying the following condition: in any coordinate neighborhood U of N there is a local base $\{I, J, K\}$ of V called a *canonical local base* of V, such that

$$\begin{aligned} &I^2 = J^2 = K^2 = -Id \\ &IJ = -JI = K; \quad JK = -KJ = I; \quad KI = -KI = J, \end{aligned} \tag{2.1}$$

where Id denotes the identity tensor field of type (1, 1). If N is a manifold and V is a bundle over N satisfying the above condition then (N, V) is called an *almost quaternion* manifold. If g is a Riemannian metric for (N, V) such that $g(\phi X, Y) + g(X, \phi Y) = 0$, holds for any cross section ϕ of V, with $X, Y \in TN$, then (N, V, g) is called an *almost quaternion metric* manifold.

Assume that the Riemannian connection ∇ of (N, V, g) satisfies the following condition: if ϕ is a local cross section of the bundle V, then $\nabla_X \phi$ is also a local cross section of V, where X is an arbitrary vector field. In this case $N = (N, V, g)$ is called a *Kaehler quaternion* manifold.

Let $x \in N$ and $X \in TN_x$. Consider the 4-dimensional subspace $Q(x)$ in TN_x defined by

$$Q(X)=\mathrm{Span}_R\{X, IX, JX, KX\}.$$

We call this the Q-section generated by X. If for all $x\in N$, and $X\in TN_x$, and $Y, Z\in Q(X)$, the sectional curvature $\sigma(Y, Z)=c$ (a constant), then we say that N is a Kaehler quaternion manifold of constant Q-sectional curvature c. In addition, such a manifold is called a *quaternion space-form.*

The curvature operator R of a quaternionic space-form $N=(N, V, g)$ has the form:

$$R(X, Y)Z=\frac{c}{4}[\Lambda(Y, Z)X-\Lambda(X, Z)Y-2\Gamma(X, Y)Z] \tag{2.2}$$

where c is the Q-sectional curvature,

$$\Lambda(Y, Z)X=g(Y, Z)X+g(IY, Z)IX+g(JY, Z)JX+g(KY, Z)KX$$

and

$$\Gamma(X, Y)Z=g(IX, Y)IZ+g(JX, Y)JZ+g(KX, Y)KZ.$$

It is well known that the quaterion projective space $\boldsymbol{HP}^n(c)$ is a compact $4n$-dimensional quaternion space-form.

3. Totally Complex Submanifolds.

Let $(\tilde{M}, V, \tilde{g})$ be a Kaehler quaternion manifold and let M be a Riemannian manifold immersed in $\tilde{M}$ isometrically by $F: M\to\tilde{M}$. A submanifold M is called a *totally complex* submanifold of $\tilde{M}$ [3], if the following two conditions are satisfied:

(i) There exists a global section I of $F^*(V)$ satisfying

$$\tilde{\nabla}_X I=0$$

for any $X\in TM$.

(ii) For each $x\in M$, there exists a neighborhood $U(x)\subset M$ and a canonical local base $\{I, J, K\}$ of $F^*(V)$ over $U(x)$ adapted to I such that

$$I(TM_y)=TM_y\,;\quad J(TM_y)\perp TM_y\,;\quad K(TM_y)\perp TM_y$$

for each $y\in U(x)$.

It follows from this definition, that any totally complex submanifold of a Kaehler quaternion manifold is even dimensional. In fact, it is easy to see that it has a natural Kaehler structure. Let h be the second fundamental form of M. We define

$$T_1(X, Y, Z)=\tilde{g}(h(X, Y), JZ),$$

and

$$T_2(X, Y, Z)=\tilde{g}(h(X, Y), KZ)$$

for $X, Y, Z\in TM_x$, $x\in M$. To simplify notation, we henceforth write $\tilde{g}(\,,\,)=\langle\,,\,\rangle$.

LEMMA 3.1, [13]. *Assume that M is a totally complex submanifold of a Kaehler quaternion manifold then*

(i) $h(IX, Y)=h(X, IY)=Ih(X, Y)$

for $X, Y\in TM_x$, $x\in M$.

(ii) T_1 *and* T_2 *are symmetric with respect to all three arguments.*

(iii) $T_i(IX, Y, Z)=T_i(X, IY, Z)=T_i(X, Y, IZ)$ *for* $i=1, 2$, *and for* $X, Y, Z\in TM_x$, $x\in M$.

By Lemma 3.1, $h(IX, IY)=-h(X, Y)$. It follows that any totally complex submanifold of Kaehler quaternion manifold is minimal. We shall need the following to prove Theorem 1.

LEMMA 3.2, [11]. *Let S be a k-covariant tensor field on a compact Riemannian manifold N. Then*

$$\int_{UN}(\nabla S)(u, \cdots, u\,; u)du=0,$$

where ∇ is the Riemann connection on N, UN is the unit tangent bundle of N, and du is the canonical volume element on UN.

For the remainder of this section we shall assume that M is a totally complex compact submanifold of real dimension $2m$ in the quaternionic projective space $\boldsymbol{HP}^n(1)$. We shall denote by $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ the Riemannian connections on $\boldsymbol{HP}^n$, on M, and the normal connection on M, respectively. We recall that $\delta(u)=\|h(u, u)\|^2$, where $u\in UM$.

LEMMA 3.3. *Assume that $\delta(u)\leqq 1/4$ for all $u\in UM$. Then*

(i) $\tilde{\nabla}h\equiv 0$, (*i.e. the second fundamental form is parallel*).

(ii) $\tilde{g}(h(X, Y), JZ)=\tilde{g}(h(X, Y), KZ)=0$ *for all* $X, Y, Z\in TM_x$, $x\in M$.

Proof. We shall use the method of Ros [11]. The first and second covariant derivatives of h are given by

$$(\tilde{\nabla}h)(X, Y\,; Z)=\nabla_Z^{\perp}(h(X, Y))-h(\nabla_Z X, Y)-h(X, \nabla_Z Y),$$

and

$$(\tilde{\nabla}^2 h)(X, Y\,; Z\,; W)=\nabla_W^{\perp}((\tilde{\nabla}h)(X, Y\,; Z))-(\tilde{\nabla}h)(\nabla_W X, Y\,; Z)$$
$$-(\tilde{\nabla}h)(X, \nabla_W Y\,; Z)-(\tilde{\nabla}h)(X, Y\,; \nabla_W Z).$$

Using equation (2.2), we can write the Codazzi equation as:

$$(\tilde{\nabla}h)(X_1, X_2, X_3)=(\tilde{\nabla}h)(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) \tag{3.2}$$

for any permutation σ, and for any $X_1, X_2, X_3\in TM_x$, $x\in M$, (i.e. $(\tilde{\nabla}h)$ is symmetric in all three arguments). We obtain the following Ricci identity:

(3.3) $$(\tilde{\nabla}^2 h)(X, Y; Z; W)-(\tilde{\nabla}^2 h)(X, Y; W; Z) = -R^{\perp}(Z, W)h(X, Y)+h(R(Z, W)X, Y)+h(X, R(Z, W)Y),$$

where R and $R^{\perp}$ denote the curvature tensors associated with ∇ and $\nabla^{\perp}$, respectively. Since M has a Kaehler structure, we have

(3.4) $$IR(X, IX)X=R(X, IX)IX.$$

Let t be the 4-covariant tensor field on M defined by

$$t(X, Y, Z, W)=\langle h(X, Y), h(Z, W)\rangle.$$

Now, for any $u\in UM$, we have

$$(\nabla t)(u, u, u, u; u)=2\langle(\tilde{\nabla} h)(u, u; u), h(u, u)\rangle$$

and

(3.5) $$(\nabla^2 t)=(u, u, u, u; u; u) =2\langle(\tilde{\nabla}^2 h)(u, u; u; u), h(u, u))+2\|(\tilde{\nabla} h)(u, u; u)\|.$$

Using equations (3.1) through (3.5) and applying Lemma 3.1, we obtain:

(3.6) $$\begin{aligned}&(\nabla^2 t)(Iu, Iu, Iu, Iu; Iu; Iu)\\ &=2\langle(\tilde{\nabla}^2 h)(Iu, u; u; Iu), h(u, u)\rangle+2\|(\tilde{\nabla} h)(u, u; u)\|^2\\ &=2\langle(\tilde{\nabla}^2 h)(Iu, u; Iu, u), h(u, u)\rangle+2\langle R^{\perp}(Iu, u)Ih(u, u), h(u, u)\rangle\\ &\quad-4\langle R(Iu, u)Iu, A_{h(u,u)}u\rangle+2\|(\tilde{\nabla} h)(u, u; u)\|^2.\end{aligned}$$

By Lemma 3.1,

(3.7) $$A_{I\xi}=IA_{\xi}=-A_{\xi}I.$$

Using the Ricci equation, (2.2), and (3.7), we obtain

(3.8) $$\begin{aligned}&\langle R^{\perp}(Iu, u), Ih(u, u), h(u, u)\rangle\\ &=-\frac{1}{2}\|h(u, u)\|^2-2\|A_{h(u,u)}(u)\|^2+\frac{1}{2}\langle h(u, u), Ju\rangle^2+\frac{1}{2}\langle h(u, u), Ku\rangle^2.\end{aligned}$$

Now, by Gauss' equation and using (2.2) and (3.7) we have

(3.9) $$\langle R(Iu, u)Iu, A_{h(u,u)}(u)\rangle=-\|h(u, u)\|^2+2\|A_{h(u,u)}(u)\|^2.$$

It follows from (3.2), (3.6), (3.8) and (3.9) that

(3.10) $$\begin{aligned}&(\nabla^2 t)(Iu, Iu, Iu, Iu; Iu; Iu)\\ &=-2\langle(\tilde{\nabla}^2 h)(u, u; u; u), h(u, u)\rangle+3\|h(u, u)\|^2-12\|A_{h(u,v)}(u)\|^2\\ &\quad+\langle h(u, u), Ju\rangle^2+\langle h(u, u), Ku\rangle^2+2\|(\tilde{\nabla} h)(u, u; u)\|^2.\end{aligned}$$

Taking the sum of (3.5) and (3.10), we obtain

$$(3.11)\qquad \begin{aligned}&(\nabla^2 t)(u, u, u, u; u, u)+(\nabla^2 t)(Iu, Iu, Iu, Iu; Iu; Iu)\\ &=3(\|h(u, u)\|^2-4\|A_{h(u,u)}(u)\|^2)+\langle h(u, u), Ju\rangle^2\\ &\quad+\langle h(u, u), Ku\rangle^2+4\|\tilde{\nabla}h(u, u; u)\|^2.\end{aligned}$$

Integrating (3.11) over UM and applying Lemma 3.2, we have

$$(3.12)\qquad \begin{aligned}&3\int_{UM}(\|h(u, u)-4\|A_{h(u,u)}(u)\|^2)du\\ &+\int_{UM}(\langle h(u, u), Ju\rangle^2+\langle h(u, u), Ku\rangle^2 du+4\int_{UM}\|\tilde{\nabla}h(u, u; u)\|^2 du=0.\end{aligned}$$

Now observe that by the hypothesis of this lemma $\|h(u, u)\|\leqq 1/4$, hence by Schwartz' inequality:

$$\|A_\xi(u)\|^2\leqq(\text{maximal eigenvalue of } A_\xi)^2\leqq 1/4 \quad (\|\xi\|=1).$$

Therefore,

$$\|h(u, u)\|^2-4\|A_{h(u,u)}(u)\|^2=\|h(u, u)\|(1-4\|A_\xi u\|^2)\geqq 0$$

where $h(u, u)=\|h(u, u)\|\xi$. It now follows from (3.12) that

$$\langle h(u, u), Ju\rangle=\langle h(u, u), Ku\rangle=0$$

and

$$(\tilde{\nabla}h)(u, u; u)=0$$

for each $u\in UM$. Now, using Lemma 3.1 and equation (3.2), we obtain by polarization

$$\langle h(X, Y), JZ\rangle=\langle h(X, Y), KZ\rangle=0,$$

and

$$(\tilde{\nabla}h)(X, Y; Z)=0,$$

for each $X, Y, Z\in TM_x$, $x\in M$. This completes the proof of the lemma.

Proof of Theorem 1. By Lemma 3.3(i) M has a parallel second fundamental form. All submanifolds of $\boldsymbol{HP}^n(1)$ which have parallel second fundamental form have been classified by K. Tsukada in [13]. Lemma 3.3(ii) shows that if the submanifold M in Theorem 1 is not totally geodesic, then it is of the type (C-C) in Tsukada's classification ([13], Proposition 3.2). It follows from the classification in [13], that the complete list of all submanifolds of the type (C-C) with parallel second fundamental form is given by the NT imbeddings ϕ_i, $i=1, \cdots, 6$. It is known that for each NT imbedding

$$\max_{u\in UM}\{\delta(u)\}=1/4.$$

Moreover, this maximum is achieved at every point of M. This completes the proof of Theorem 1.

Proof of Theorem 2. By (2.2) and Gauss' equation we have

$$H(u)=\langle R(u, Iu)Iu, u\rangle=1-2\delta(u),$$

for any $u\in UM$. Hence the conditions $H(u)\geqq 1/2$ is equivalent to the condition $\delta(u)\leqq 1/4$. This proves the theorem.

4. Maximal directions.

Let M be a compact m-dimensional Riemannian manifold isometrically immersed in an $(m+p)$-dimensional Riemannian manifold. As in the previous section we let h denote the second fundamental form, and we define $\delta(u)$ by $\delta(u)=\|h(u, u)\|^2$ for $u\in UM$. Assume that for some $u\in UM_x$, we have

$$\delta(u)=\max_{v\in UM}\{\delta(u)\},$$

then we say that u is a *maximal direction* at $x\in M$. We say that an orthonormal frame $\{e_1, \cdots, e_{m+p}\}$ is adapted, if $\{e_1, \cdots, e_m\}$ is a frame for TM, and $\{e_{m+1}, \cdots, e_{m+p}\}$ is a frame for $TM^{\perp}$. Whenever $\{e_1, \cdots, e_{m+p}\}$ is an adapted frame we use the notation:

$$h_{ij}=h(e_i, e_j) \qquad i, j=1, \cdots, m.$$

LEMMA 4.1, [5]. *If* $\{e_1, \cdots, e_{m+p}\}$ *is an adapted frame at* $x\in M$ *such that* e_1 *is a maximal direction at* x, *then*

$$\langle h_{11}, h_{1i}\rangle=0 \qquad i=2, 3, \cdots, m \tag{4.1}$$

where $\langle\ ,\ \rangle$ *denotes* $\tilde{g}(\ ,\ )$ *in* $\tilde{M}$.

COROLLARY. *Diagonalizing the symmetric bilinear form* $b(X, Y)=\langle h_{11}, h(X, Y)\rangle$, *we can always find an adapted frame* $\{e_1, \cdots, e_{m+p}\}$ *such that*

(4.2) e_1 *is a given maximal direction at* x,

$$\langle h_{11}, h_{ij}\rangle=0, \qquad i\neq j,\ i, j=1, 2, \cdots, m. \tag{4.3}$$

LEMMA 4.2 [5] (Variational Inequality). *For any adapted frame satisfying conditions* (4.2) *and* (4.3),

$$\|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle-2\|h_{1i}\|^2\geqq 0, \qquad i=2, 3, \cdots, m. \tag{4.4}$$

Let us define a 4-covariant tensor field t on M by the formula

$$t(X, Y, Z, W)=\langle h(X, Y), h(Z, W\rangle), \tag{4.5}$$

where $X, Y, Z, W \in TM_x$, $x \in M$. The following result is a cosequence of *J.* Simon's formula for Δh, ([12], [1]).

LEMMA 4.3 [5]. *For any adapted frame satisfying conditions* (4.2) *and* (4.3) *we have*

$$
\begin{aligned}
(4.6)\qquad & \frac{1}{2}(\Delta t)(e_1, e_1, e_1, e_1) \\
&= \sum_{i=1}^{m} [4\langle \tilde{R}(e_1, e_i)h_{11}, h_{1i}\rangle + \langle \tilde{R}(e_i, h_{11})e_i, h_{11}\rangle - \langle h_{11}, h_{ii}\rangle^2 \\
&\quad + 2(\|h_{11}\|^2 - \langle h_{11}, h_{ii}\rangle)(\langle \tilde{R}(e_1, e_i)e_i, e_1\rangle - \|h_{1i}\|^2) \\
&\quad + \|(\tilde{\nabla} h)(e_1, e_1; e_i)\|^2] + m\langle \tilde{R}(e_1, h_{11})e_1, H\rangle + m\|h_{11}\|^2\langle h_{11}, H\rangle,
\end{aligned}
$$

where Δ is the Laplace operator, $\tilde{R}$ is the curvature tensor of $\tilde{M}$, H is the mean curvature vector.

Let s be a k-covariant tensor field on M. Suppose that $u \in UM_x$ satisfies

$$s(u, \cdots, u) = \max_{v \in UM_x} \{s(v, \cdots, v)\}.$$

In such a case we say that u is a maximal direction for s at x. For any $x \in M$, we define

$$f_s(x) = s(u, \cdots, u)$$

where u is a maximal direction for s at x. The following result is an obvious generalization of [7], (Proposition 3.1).

LEMMA 4.4 [5] (Generalized Bochner's Lemma). *Let M be a compact Riemannian manifold and s a k-covariant tensor field on M. If*

$$(\Delta s)(u, \cdots, u) \geqq 0$$

for any maximal direction for s, then f_s is constant on M, and $(\Delta s)(u, \cdots, u) = 0$ for any maximal direction u for the tensor s.

5. Totally Real Minimal Submanifolds.

Let $\tilde{M} = (\tilde{M}, V, \tilde{g})$ denote a quaternion Kaehler manifold and M be a Riemannian submanifold isometrically immersed in $\tilde{M}$. We say that M is a totally real submanifold of $\tilde{M}$, [2], if

$$\theta(TM_x) \perp TM_x$$

for any $x \in M$, and any $\theta \in V_x$, where V_x is the fibre of V over x. Recall that h is the second fundamental form, and set

$$T_1(X, Y, Z) = \langle h(X, Y), IZ\rangle$$

$$T_2(X, Y, Z)=\langle h(X, Y), JZ\rangle$$

$$T_3(X, Y, Z)=\langle h(X, Y), KZ\rangle$$

where $\langle\ ,\ \rangle$ denotes the metric $\tilde{g}(\ ,\ )$.

LEMMA 5.1 [13]. *$T_i(X, Y, Z)$ is symmetric in all three arguments for each $i=1, 2, 3$.*

Proof of Theorem 3. Let $x\in M$ and let $\{I, J, K\}$ denote a cannonical local base of V defined in some neighborhood $U(x)\subset \boldsymbol{HP}^n(1)$. Let u denote a maximal direction for t at x, and let $\{e_1, \cdots, e_{4n}\}$ denote an adapted frame at x satisfying conditions (4.2) and (4.3). In addition assume that if w is an element of the frame $\{e_1, \cdots, e_{4n}\}$, then Iw, Jw, Kw are also elements of this frame. Using equation (2.2), Lemma 5.1 and the minimality condition $H=0$, we can rewrite (4.6) in the following form:

$$\begin{aligned}(5.1)\quad &\frac{1}{2}(\Delta t)(e_1, e_1, e_1, e_1)\\ &=3m\|h_{11}\|^2\Big(\frac{1}{12}-\|h_{11}\|^2\Big)+\sum_{i=1}^{m}(\|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle)(\|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle-2\|h_{1i}\|^2)\\ &\quad+2\sum_{i=1}^{m}(\|h_{11}\|^4-\langle h_{11}, h_{ii}\rangle^2)+\frac{1}{4}\sum_{i=1}^{m}(\langle h_{11}, Ie_i\rangle^2+\langle h_{11}, Je_i\rangle^2+\langle h_{11}, Ke_i\rangle^2)\\ &\quad+\sum_{i=1}^{m}\|(\tilde{\nabla}h)(e_1, e_1; e_i)\|^2.\end{aligned}$$

Now, since $\delta(u)\leqq 1/12$ for any $u\in UM$, we have that $\|h_{11}\|^2\leqq 1/12$. Therefore, using the Cauchy-Schwartz inequality along with the variational inequality (4.4) we have that each term on the right hand side in (5.1) is non-negative. By Lemma 4.4, $(\Delta t)(e_1, e_1, e_1, e_1)=0$. Hence

$$(5.2)\qquad \|h_{11}\|^2\Big(\frac{1}{12}-\|h_{11}\|^2\Big)=0;$$

$$(5.3)\qquad \|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle)(\|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle-2\|h_{1i}\|^2)=0,\quad i=2, \cdots, m;$$

$$(5.4)\qquad \|h_{11}\|^4-\langle h_{11}, h_{ii}\rangle^2=0,\quad i=2, \cdots, m;$$

$$(5.5)\qquad \langle h_{11}, Ie_i\rangle=\langle h_{11}, Je_i\rangle=\langle h_{11}, Ke_i\rangle=0,\quad i=1, \cdots, m;$$

$$(5.6)\qquad (\tilde{\nabla}h)(e_1, e_1; e_i)=0,\quad i=1, \cdots, m.$$

Now, if $\delta(u)<1/12$ for all $u\in UM$, then $h_{11}=0$ by (5.2), and we conclude that M is totally geodesic. Assume, therefore, that

$$\max_{u\in UM}\delta(u)=1/12,$$

then $\|h_{11}\|=1/\sqrt{12}$. By (5.4), we have

$$\|h_{11}\|^4=\langle h_{11}, h_{ii}\rangle^2\leqq\|h_{11}\|^2\|h_{ii}\|^2\leqq\|h_{11}\|^4.$$

Hence, $h_{ii}=\pm h_{11}$ for each $i=1, \cdots, m$. By assumption M is minimal and therefore m is even, $m=2r$. After a suitable renaming of indices we can write

$$h_{11}=h_{22}=\cdots=h_{rr}=-h_{r+1,r+1}=\cdots=-h_{2r,2r}.$$

Assume that $1\leqq\lambda, \mu, \nu, \xi\leqq r$, and let $\bar{\lambda}=\lambda+r$, then

$$h_{\lambda\lambda}=h_{11},\ h_{\bar{\lambda}\bar{\lambda}}=-h_{11}. \tag{5.7}$$

Applying equations (4.4) and (5.7) we obtain that $h_{1\lambda}=0$, $\lambda\neq1$. In addition equation (5.7) implies that each element of the frame, e_i, is a maximal direction for δ. Consequently,

$$h_{\lambda\mu}=h_{\bar{\lambda}\bar{\mu}}=0,\quad \lambda\neq\mu. \tag{5.8}$$

Using equations (5.7) and (5.3) we have $\|h_{1\bar{\lambda}}\|^2=\|h_{11}\|^2$, therefore

$$\|h_{\lambda\bar{\mu}}\|^2=\|h_{11}\|^2=1/12. \tag{5.9}$$

Now since e_i is a maximal direction for each i, we have

$$\left\|h\left(e_1+\tau\sum_{i=2}^m x^ie_i,\ e_1+\tau\sum_{i=2}^m x^ie_i\right)\right\|^2\leqq\left(1+\sum_{i=2}^m(x^i)^2\tau^2\right)^2\|h_{11}\|^2 \tag{5.10}$$

for $\tau, x^2, \cdots, x^m\in\boldsymbol{R}$. Expanding in terms of τ and using equations (4.3), (5.8), and (5.9), we obtain that

$$-4\tau^2\sum_{\bar{\lambda}\neq\bar{\mu}}\langle h_{1\bar{\lambda}}, h_{1\bar{\mu}}\rangle x^{\bar{\lambda}}x^{\bar{\mu}}+0(\tau^3)\leqq0$$

for all real $\tau, x^2, \cdots, x^m$. Hence $\langle h_{1\bar{\lambda}}, h_{1\bar{\mu}}\rangle=0$, $\bar{\lambda}\neq\bar{\mu}$. Since each direction e_i is maximal, we have

$$\langle h_{\lambda\bar{\mu}}, h_{\lambda\bar{\nu}}\rangle=0,\quad \bar{\mu}\neq\bar{\nu};\qquad \langle h_{\lambda\bar{\nu}}, h_{\mu\bar{\nu}}\rangle=0,\quad \lambda\pm\mu. \tag{5.11}$$

Once more expanding (5.10) in terms of τ we find that

$$\tau^3\sum_{i j, k\neq1}\langle h_{1i}, h_{jk}\rangle x^ix^jx^k+0(\tau^4)\leqq0.$$

Hence, $\langle h_{1i}, h_{jk}\rangle+\langle h_{1j}, h_{ki}\rangle+\langle h_{1k}, h_{ij}\rangle=0$, $i, j, k\neq1$. By (5.7), (5.8), (5.11), and since each e_i is a maximal direction, we obtain

$$\langle h_{\lambda\bar{\nu}}, h_{\mu\bar{\xi}}\rangle+\langle h_{\lambda\bar{\xi}}, h_{\mu\bar{\nu}}\rangle=0, \tag{5.12}$$

where either $\lambda\neq\mu$ or $\bar{\nu}\neq\bar{\xi}$. Using (4.3), (5.7)-(5.9), (5.11), and (5.12), we obtain by direct computation that $\delta(u)=1/12$ for any $u\in UM$. B. O'Neill [9], calls an immersion λ-isotropic if $\|h(u, u)\|=\lambda$ for any $u\in UM$. Therefore, the immersion under consideration is $\sqrt{1/12}$-isotropic.

By (5.6), $(\tilde{\nabla}h)(X, X; Y)=0$. Using polarization we obtain

(5.13) $$(\tilde{\nabla}h)(X, Y, Z)=0,$$

for $X, Y, Z\in TM_x$, $x\in M$. Using equation (5.5), and applying polarization, we obtain

(5.14) $$\langle h(X, Y), IZ\rangle=\langle h(X, Y), JZ\rangle=\langle h(X, Y), KZ\rangle=0,$$

for $X, Y, Z\in TM_x$, $x\in M$.

The second fundamental form of the immersion is parallel by equation (5.13). All totally real minimal isometric immersions into $\boldsymbol{HP}^n(1)$ with parallel second fundamental form were classified by K. Tsukada [13]. There are two possible types of such immersions, which are denoted as (R-R)-type and (R-C)-type (Proposition 3.2, [13]). It follows from (5.14) that our immersion is not of type (R-C). Among all totally real minimal isometric immersions of type (R-R) with parallel second fundamental form only ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5 are $\frac{1}{\sqrt{12}}$ isotropic. This completes the proof of Theorem 3.

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