M. MOTOO
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PERIODIC EXTENSIONS OF TWO-DIMENSIONAL BROWNIAN MOTION ON THE HALF PLANE, I

BY MINORU MOTOO

DEPARTMENT OF INFORMATION SCIENCES, TOKYO DENKI UNIVERSITY(†)

Introduction

In the paper and the following one [6], we shall study periodic extensions of the Brownian motion on the half plane $\bar{D}=\{(x, y): y\geqq 0\}$. By an "extension", we mean a Markov process on $\bar{D}$ whose laws before paths reach the boundary $\partial_0=\{(x, y): y=0\}$ coincide with those of the two-dimensional Brownian motion, and by a "periodic extension" we mean an extension whose laws are invariant under translation of length 2π parallel to the x-axis.

First, let us quote an extension as an example. Assume the semigroup of the extension satisfies the boundary condition

$$\alpha(x)\frac{\partial^2}{\partial x^2}+\beta(x)\frac{\partial}{\partial x}+\frac{\partial}{\partial y}=0$$

on ∂_0, where α and β are smooth periodic functions on the real line and α is positive. Then, the extension is periodic, has continuous paths and has no sojourn on the boundary ∂_0. Let functions u and m be harmonic in $D=\{(x, y): y>0\}$ (in classical sense) and smooth in $\bar{D}$, and satisfy

$$\alpha(x)u_{xx}(x, 0)+\beta(x)u_x(x, 0)+u_y(x, 0)=0$$

$$u(x+2\pi, y)-u(x, y)=2\pi$$

$$(\alpha(x)m(x, 0))_{xx}-(\beta(x)m(x, 0))_x+m_y(x, 0)=0$$

and

$$\int_0^{2\pi} m(x, y)dx=2\pi .$$

Such smooth functions u and m with $u_x>0$ and $m>0$ uniquely exist. Define

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$$\sigma(dx)=u_x(x, 0)dx\,, \qquad \mu(dx)=m(x, 0)dx\,,$$

$$k=\frac{1}{2\pi}\int_0^{2\pi}\beta(x, 0)\mu(dx)\,,$$

$$p=\frac{1}{2\pi}\int_0^{2\pi}u_x(x, 0)^2\mu(dx)\,.$$

The function u is considered as the "standard" harmonic function and m is the density of an invariant measure of the extension. The numbers k and p are considered to represent "mean drift" and "mean fluctuation", respectively, of the extension on ∂_0. Moreover, α and β are determined by σ, μ, k and p.

In general, starting from two periodic measures σ, μ and two constants k, p, which satisfy certain analytic conditions, we shall define an extension which we call $\{\sigma, \mu, k, p\}$-process. It is, in a sense, a limit of such processes as given in the above example. Let $\mathcal{P}_f$ be the class of extensions which are Feller processes with continuous paths and have no sojourn on the boundary ∂_0. Although a $\{\sigma, \mu, k, p\}$-process is not necessarily a process with continuous paths, we shall show each element of $\mathcal{P}_f$ is contained in the class of all $\{\sigma, \mu, k, p\}$-processes. Thus we can characterize it by σ, μ, k and p. To establish these facts is the main purpose of our papers.

We can not yet handle more general extensions. Without periodicity, difficulty arises from noncompactness of the boundary. By conformal mapping, investigation of periodic extensions is essentially equivalent to that of extensions of the Brownian motion on the unit disc. As for continuity of paths, it seems beyond our methods to treat extensions with paths which have jumps at ∂_0. We can not yet see whether the condition that extensions are of Feller type is essential or not.

In Chapter I, we shall first give general definitions of extensions which we shall treat. In our study, we shall exclude in advance extensions with sojourns at ∂_0 or jumps from ∂_0 into D. Then, we shall show that the problems to determine our extensions is equivalent to the problem to find systems of hitting measures of the lines $\{(x, y): y=a\}$ from points in $\{(x, y): 0<y<a\}$ for any $a>0$. In the following, we shall concentrate our study mainly on systems of hitting measures. In the last section, a continuity conditions on semigroups of extensions will be translated into certain conditions on systems of hitting measures.

In Chapter II, we shall define $\{\sigma, \mu, k, p\}$-process. Let σ and μ be periodic measures on the real line with $\sigma([0, 2\pi])=\mu([0, 2\pi])=2\pi$. We shall assume that they satisfy a certain integrability condition [P] given in [5.11]. Let k be any constant and p be any constant with $p\geqq p_0(\sigma, \mu, k)$, where $p_0(\sigma, \mu, k)$ is a constant determined by σ, μ and k in (4.14). Starting from σ, μ, k and p, we shall formulate a boundary condition. A $\{\sigma, \mu, k, p\}$-process will be defined as an extension which induces a class of harmonic functions satisfying this boundary condition. Proof of uniqueness of a $\{\sigma, \mu, k, p\}$-process for given σ,

μ, k and p is the main contents of the chapter.

In Chapter III, guided by their probabilistic meaning stated in the example, we shall construct two measures σ_P, μ_P and two constants k_P, p_P from a given extension $\boldsymbol{P}$. We shall also show that σ_P and μ_P satisfy the condition [P] and $p_P \geqq p_0(\sigma_P, \mu_P, k_P)$. In the last section, we shall give a sufficient condition that an extension $\boldsymbol{P}$ is a $\{\sigma_P, \mu_P, k_P, p_P\}$-process.

In the next paper [8], we shall prove existence of a $\{\sigma, \mu, k, p\}$-process for given σ, μ, k and p. Then we can show that, for the $\{\sigma, \mu, k, p\}$-process $\boldsymbol{P}$, σ_P, μ_P, k_P and p_P constructed in Chapter III coinside with the given σ, μ, k and p. We shall also study how properties of $\{\sigma, \mu, k, p\}$-processes can be transformed into conditions on σ, μ, k and p. Especially, we shall show that any extension $\boldsymbol{P}$ in the class $\mathscr{P}_f$ can be characterized as a $\{\sigma, \mu, k, p\}$-process with σ and μ being positive on any open interval and σ being a continuous measure.

In 1952, W. Feller ([2]) determined all possible extensions of one-dimensional diffusion processes in regular intervals and characterized them by boundary conditions with analytic forms. Probabilistic construction of these extensions was given by K. Ito and H.P. Mckean ([5]). For multi-dimensional diffusion processes, A.D. Wentzell ([15]) gave possible forms of boundary conditions which smooth extensions of processes should satisfy. Using the idea of the process on the boundary, T. Ueno and K. Sato ([13], [10]) showed existence of extensions which satisfy Wentzell's boundary conditions in general but smooth cases. Following their idea, M. Motoo ([6]) characterized extensions of fairly general class of Markov processes by processes on the boundary. In 1979, S. Watanabe ([14]) gave probabilistic construction of extensions which satisfy Wentzell's boundary conditions. For this purpose, he used Poisson point processes of Brownian excursions. Research of extensions of Markov processes, using theory of Dirichlet forms, were introduced by M. Fukushima ([3]) in 1969 and developped by H. Kunita ([4]) and M.L. Silverstein ([11]).

However, it seems that there are few papers which treat multi-dimensional singular extensions in concrete forms, except the paper of E.B. Dynkin ([1]) and that of M.L. Silverstein ([12]). The former treated extensions with singular drifts of Brownian motion on the smooth plane domain, while the latter treated symmetric extensions of certain symmetric process on the half plane. Our papers treat extensions of one of the simplest multi-dimensional diffusions, that is, the absorbing Brownian motion on the half plane and our aim is to try to find the most general extensions of it as concretely as possible. Results of our papers have already been published in ([7]).

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Contents of the paper

Contents of the next paper [6]

§0. Notations.

Throughout the paper, we shall use the following notations.

1°

R=the set of all real numbers.
D=the open upper half plane=$\{z=(x, y): y>0\}$.
$\bar{D}=\{z=(x, y): y\geqq 0\}$.

For any real interval I,

$$D^I=\{z=(x, y): y\in I\}$$

$$D=D^{(0,\infty)},\quad \bar{D}=D^{[0,\infty)}.$$

We also write

$$D^{a}=D^{(0,a)}$$
$$\partial_a=D^{(a)}=\{z=(x,\ y)\,;\ y=a\}.$$

To the above spaces, we give the ordinary metric

$$\rho(z_1,\ z_2)=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$(z_j=(x_j,\ y_j),\ j=1,\ 2).$$

2° In general, for any metric space X with metric ρ, we write

$\boldsymbol{B}(X)$=the σ-field generated by all open sets in X.
$B(X)$=the set of all $\boldsymbol{B}(X)$-measurable functions on X.
$B_b(X)=\{f\in B(X): f$ is bounded$\}$.
$\|f\|=\sup_{x\in X}|f(x)|$ for $f\in B_b(X)$.
$C(X)$=the set of all continuous functions on X.
$C_b(X)=\{f\in C(X): f$ is bounded$\}=C(X)\cap B_b(X)$.
$C_K(X)=\{f\in C(X): f$ has compact support$\}$.
$M(X)$=the set of all measures on $\boldsymbol{B}(X)$.

For $u\in X$ and $\varepsilon>0$,

$$U_\varepsilon(u)=\{v\in X:\ \rho(u,\ v)<\varepsilon\}.$$

3° If G is an open set in a Euclidean space and $G\subset X\subset\bar{G}$, we set

$C^k(X)$=the set of functions on X which are k-times differentiable in G and whose derivatives of order $\leqq k$ are continuous in X.

$C_b^k(X)=\{f\in C^k(X): f$ and its derivatives up to k are bounded$\}$.

4° We set $z+a=(x+a,\ y)$ for $a\in R$ and $z=(x,\ y)$.
For $X=R$ or D^I and a positive integer N, set

$B_{p,N}(X)=\{f\in B(X): f(u+2N\pi)=f(u)$ for any $u\in X\}$,

$B_{q,N}(X)=\{f\in B(X)$; there exists a constant C_f such that $2N\pi C_f=f(u+2N\pi)-f(u)\}$,

$$B_p(X)=B_{p,1}(X),\quad B_q(X)=B_{q,1}(X),\quad C_{p,N}(X)=C(X)\cap B_{p,N}(X),$$

$$C_{q,N}(X)=C(X)\cap B_{q,N}(X),\quad C_p(X)=C_{p,1}(X),\quad C_q(X)=C_{q,1}(X).$$

For $X=R$, set

$$M_{p,N}(R)=\{\mu\in M(R): \text{locally bounded and } \mu(du+2N\pi)=\mu(du)\},$$

$$M_p(R)=M_{p,1}(R),$$

$$[\mu]=\frac{1}{2N\pi}\mu([0, 2N\pi)) \quad \text{for } \mu\in M_{p,N}(R).$$

5° We define D^* by identifying all points in the set $\partial_0=\{y=0\}$ in $\bar{D}$, more precisely, $D^*=D\cup\{\partial\}$ with metric

$$\rho^*(z_1, z_2)=\min\{y_1+y_2, \rho(z_1, z_2)\}, \quad z_1, z_2\in D,$$

$$\rho^*(\partial, z)=\rho(z, \partial)=y, \quad z\in D,$$

$$\rho^*(\partial, \partial)=0.$$

Then, it is easily seen that D^* is a complete separable metric space.

We define a continuous mapping ι from $\bar{D}$ onto D^* by

$$\iota(z)=\begin{cases} z & \text{for } z\in D \\ \partial & \text{for } z\in\partial_0+\{y=0\}. \end{cases}$$

6° W=the set of all continuous functions $w=w(t)$ from $t\in[0, \infty)$ into D^*.

We write $w(t)=z(t, w)$.

θ_t=the (shift) mapping from W into W such that
$z(s, \theta_t w)=z(t+s, w)$ for any $s\in[0, \infty)$.

$\boldsymbol{B}_t$=the σ-field generated by all sets $\{w: z(s, w)\in A\}$, $s\leqq t$, $A\in\boldsymbol{B}(D^*)$.

$\boldsymbol{B}$=the σ-field generated by all sets $\{w: z(s, w)\in A\}$, $s<\infty$, $A\in\boldsymbol{B}(D^*)$.

A $[0, \infty)$-valued $\boldsymbol{B}$-measurable function τ on W is called Markov time if $\{\tau<t\}\in\boldsymbol{B}_t$ for any $t\in[0, \infty)$.

For any Markov time τ, we set

$$\boldsymbol{B}_\tau=\{\mathcal{A}\in\boldsymbol{B}: \mathcal{A}\cap\{\tau<t\}\in\boldsymbol{B}_t\}.$$

Replacing D^* by $\bar{D}$ in the above definitions, we also define

$$\bar{W}, \bar{W}(t)=z(t, \bar{w}), \bar{\boldsymbol{B}}_\tau, \bar{\boldsymbol{B}} \text{ and } \bar{\boldsymbol{B}}_\tau.$$

7° $P_z^{B,2}(\cdot)$ is the 2-dimensional Brownian measure on the space of continuous paths on R^2, starting from $z\in R^2$.

$P_y^{B,1}(\cdot)$ is the 1-dimensional Brownian measure on the space of continuous paths on R, starting from $y\in R$.

$P_z^{R,2}(\cdot)$ is the 2-dimensional reflecting Brownian measure on $\bar{D}$ starting from $z \in \bar{D}$.

$P_y^{R,1}(\cdot)$ is the 1-dimensional reflecting Brownian measure on $[0, \infty)$ starting from $y \in [0, \infty)$.

The measures above give Markov processes, whose transition probabilities are given by

$$P_z^{B,2}(z(t) \in A) = \frac{1}{2\pi t} \iint_A \exp\left\{-\frac{1}{2t}((\xi - x)^2 + (\eta - y)^2)\right\} d\xi d\eta \quad \text{for } A \in \boldsymbol{B}(R^2),$$

$$P_y^{B,1}(y(t) \in B) = \frac{1}{\sqrt{2\pi t}} \int_B \exp\left\{-\frac{1}{2t}(\eta - y)^2\right\} d\eta \quad \text{for } B \in \boldsymbol{B}(R),$$

$$P_z^{R,2}(z(t) \in A) = \frac{1}{2\pi t} \iint_A \left(\exp\left\{-\frac{1}{2t}((\xi - x)^2 + (\eta - y)^2)\right\} + \exp\left\{-\frac{1}{2t}((\xi - x)^2 + (\eta + y)^2)\right\}\right) d\xi d\eta \quad \text{for } A \in \boldsymbol{B}(R^2),$$

$$P_y^{R,1}(z(t) \in B) = \frac{1}{\sqrt{2\pi t}} \int_B \left(\exp\left\{-\frac{1}{2t}(\eta - y)^2\right\} + \exp\left\{-\frac{1}{2t}(\eta + y)^2\right\}\right) d\eta \quad \text{for } B \in \boldsymbol{B}([0, \infty)).$$

8° We shall use the following kernels related to the 2-dimensional Brownian motion. For $0 \leqq c < b < a$, $x \in R$ and $B \in \boldsymbol{B}(R)$,

$$ {}^a_c\Pi^a_b(x, B) = P^{B,2}_{(x,b)}(z(\sigma_a) \in B \times \{a\},\ \sigma_a < \sigma_c) = \int_B {}^{a-c}\pi^{a-b}(\xi - x) d\xi,$$

$$ {}^a_c\Pi^c_b(x, B) = P^{B,2}_{(x,b)}(z(\sigma_c) \in B \times \{c\},\ \sigma_c < \sigma_a) = \int_B {}^{a-c}\pi^{b-c}(\xi - x) d\xi,$$

where σ_a and σ_b are hitting times of $\partial_a = \{y = a\}$ and $\partial_c = \{y = c\}$ respectively, and

$$ {}^r\Pi^s(x) = \frac{\sin(\pi s/r)}{2r(\cosh(\pi x/r) - \cos(\pi s/r))} \quad (r > s > 0).$$

We write

$$Q^{a-c}(x, B) + \lim_{b \downarrow c} \frac{1}{b - c} {}^a_c\Pi^a_b(x, B) = \int_B q^{a-c}(\xi - x) d\xi,$$

and, if $x \notin \bar{B}$ ($\bar{B}$ is the closure of B),

$$P^{a-c}(x,\ B)=\lim_{b\uparrow a}\frac{1}{a-b}\,{}^{a}_{c}\Pi^{a}_{b}(x,\ B)=\int_B p^{a-c}(\xi-x)d\xi\,,$$

where

$$q^r(x)=\frac{\pi}{4r^2}\frac{1}{(\cosh(\pi x/2r))^2}\,,$$

$$p^r(x)=\frac{\pi}{4r^2}\frac{1}{(\sinh(\pi x/2r))^2}\,,$$

and further

$$\Pi^c_b(x,\ B)={}^{\infty}_{c}\Pi^c_b(x,\ B)=P^{B,2}_{(x,b)}(z(\sigma_c)\in B\times\{c\})=\int_B \pi^{b-c}(\xi-x)d\xi\,,$$

where $\pi^r(x)$ is Poisson kernel, that is

$$\pi^r(x)=\frac{1}{\pi}\frac{1}{x^2+r^2}\,.$$

We also write:

$$h_\xi(z)=\pi^y(x-\xi)=\frac{1}{\pi}\frac{y}{(x-\xi)^2+y^2}=-\frac{1}{\pi}\operatorname{Im}\left(\frac{1}{x-\xi+iy}\right),$$

$$k_\xi(z)=\frac{1}{\pi}\frac{x-\xi}{(x-\xi)^2+y^2}=\frac{1}{\pi}\operatorname{Re}\left(\frac{1}{x-\xi+iy}\right),$$

$$\tilde{h}_\xi(z)=\sum_{n=-\infty}^{\infty}h_{\xi+2n\pi}(z)=\frac{\sinh y}{2\pi(\cosh y-\cos(x-\xi))}$$

$$=-\frac{1}{2\pi}\operatorname{Im}\left(\cot\frac{(x-\xi)+iy}{2}\right),$$

$$\tilde{k}_\xi(z)=\lim_{N\to\infty}\sum_{n=-N}^{N}k_{\xi+2n\pi}(z)=\frac{\sin(x-\xi)}{2\pi(\cosh y-\cos(x-\xi))}$$

$$=\frac{1}{2\pi}\operatorname{Re}\left(\cot\frac{(x-\xi)+iy}{2}\right).$$

The functions $k_\xi(z)$ and $\tilde{k}_\xi(z)$ are harmonic conjugates of $h_\xi(z)$ and $\tilde{h}_\xi(z)$ in D, respectively.

I. Preliminaries

§1. Definitions of processes in $\mathscr{P}$ and $\mathscr{P}_c$.

Let D^*, W, $\boldsymbol{B}$ and $\boldsymbol{B}_t$ be defined as in §0, 6°.

[1.1] Definition of the class $\mathscr{P}$.

$\boldsymbol{P}=\{P_z(A)\colon z\in D,\ A\in\boldsymbol{B}\}$ is in class $\mathscr{P}$ if and only if

(p. 1) $P_z(A)$ is a probability kernel on $D\times\boldsymbol{B}$.

(p. 2) $P_z(z(t, w)\in D)=1$ for any $t\geqq 0$ and $z\in D$.

(p. 3) $P_z(z(t+s, w)\in A\,|\,\boldsymbol{B}_t)=P_{z(t,w)}(z(s, w)\in A)$ a. s. P_z for any $A\in\boldsymbol{B}(D)$, t, $s\geqq 0$ and $z\in D$.

(p. 4) $P_z(z(t, w)\in A,\ t<\sigma_0)=P_z^{B,2}(z(t)\in A,\ t<\sigma_0)$ for any $A\in\boldsymbol{B}(D)$, $t\geqq 0$ and $z\in D$, where $P_z^{B,2}$ is the Brownian measure defined in § 0, 7°.

(p. 5) $P_{z+2\pi}(z(t, w)\in A+2\pi)=P_z(z(t, w)\in A)$ for any $z\in D$ and $A\in\boldsymbol{B}(D)$.

By (p. 4), we see:

[1.2] $P_z(z(0, w)=z)=1$ for any $z\in D$.

[1.3] *Remark.* If $z=\partial$, $P_z(\cdot)$ is not defined. However, by (p. 2) the right side of (p. 3) has a meaning.

By (p. 2) and (p. 3) together with [1.2], the system of measures $\{P_z(\cdot)\}_{z\in D}$ on $\boldsymbol{B}$ defines a Markov process on D. Set

$$P_t f(z)=E_z(f(z(t, w)) \tag{1.1}$$

where $E_z(\cdot)$ is the expectation taken by the measure P_z.

[1.4] (1) P_t maps $B_b(D)$ into $B_b(D)$ and $P_t\cdot P_s=P_{t+s}$ for any t, $s\geqq 0$.
(2) P_t maps $B_b(D)$ in $C_b(D)$ and $B_{p,N}(D)$ into $C_{p,N}(D)$ for any $t>0$.
(3) $\lim_{\substack{t\to 0\\ \zeta\to z}} P_t f(\zeta)=f(z)$ for any $f\in C_b(D)$ and $z\in D$.

Proof. (1) is obvious from (p. 1) and (p. 2). For any $a>0$ and $\zeta\in D^{[a,\infty)}$

$$P_\zeta(\sigma_0\leqq\varepsilon)=P_\zeta^{B,2}(\sigma_0\leqq\varepsilon)\leqq P_a^{B,1}(\sigma_{(0)}\leqq\varepsilon)=\phi(\varepsilon, a) \quad\text{and}$$

$$\lim_{\varepsilon\downarrow 0}\phi(\varepsilon, a)=0\,.$$

For $f\in B_b(D)$ and $t>0$, take ε such that $0<\varepsilon<t$, and set $f_\varepsilon(\zeta)=P_{t-\varepsilon}f(\zeta)$. Then,

$$P_t f(\zeta)=E_\zeta(f_\varepsilon(z(\varepsilon, w))=E_\zeta^{B,2}(f_\varepsilon(z(\varepsilon)):\ \varepsilon<\sigma_0)+E_\zeta(f_\varepsilon(z(\varepsilon)):\ \varepsilon\geqq\sigma)\,.$$

Since the semi-group of the absorbing Brownian motion maps $B_b(D)$ into $C_b(D)$, the first term in the right side is continuous. On the other hand, $|E_\zeta(f_\varepsilon(z(\varepsilon)):\ \varepsilon\geqq\sigma)|\leqq|f|\phi(\varepsilon, a)$ and ε can be chosen arbitrarily small. Hence $P_t f$ is continuous in $D^{[a,\infty)}$. Since $a>0$ is arbitrary, the first part of (2) is proved. The second part of (2) is obvious from (p. 5).

For any $z\in D$ and $f\in C_b(D)$, take a so small as $z\in D^{(a,\infty)}$. Then, for any $\zeta\in D^{[a,\infty)}$,

$$P_t f(\zeta)=E_\zeta^{B,2}(f(z(t)): t<\sigma_0)+E_\zeta(f(z(t)): t\geqq\sigma).$$

The first term tends to $f(z)$ as $\zeta\to z$ and $t\to 0$, and the second term tends to zero uniformly in $\zeta\in D^{[a,\infty)}$ as $t\to 0$. Therefore (3) is proved.

The process $\{p_z(\cdot)\}_{z\in D}$ has not strong Markov property in general. However, the following proposition holds. The definition of Markov time is given in §0, 6°.

[1.5] PROPOSITION (Strong Markov property). *For any Markov time τ and $A\in\boldsymbol{B}$,*

$$P_z(\theta_\tau w\in A \mid \boldsymbol{B}_\tau)=P_{z(\tau,w)}(A)$$

on $\{w: \tau(w)<\infty$ and $z(\tau, w)\in D\}$ a.s. P_z.

Proof. It is sufficient to prove that for any $0\leqq t_1<t_2<\cdots<t_n$, $f_k\in C_b(D)$ and $f_k\geqq 0$ $(k=1, 2, \cdots, n)$ and $A\in\boldsymbol{B}_\tau$,

$$E_z\Big[\prod_{k=1}^n f_k(z(\tau+t_k)): A,\ z(\tau)\in D,\ \tau<\infty\Big]$$

$$=E_z\Big[E_{z(\tau)}\Big[\prod_{k=1}^n f_k(z(t_k))\Big]: A,\ z(\tau)\in D,\ \tau<\infty\Big].$$

Set

$$\tau^m=\frac{l}{2^m}\quad\text{if}\ \frac{l-1}{2^m}\leqq\tau\leqq\frac{l}{2^m},\quad (l=1, 2, \cdots),$$

$$\infty\quad\text{if}\ \tau=\infty.$$

Then, by ordinary Markov property (p. 3),

$$E_z\Big[\prod_{k=1}^n f_k(z(\tau^m+t_k)): A,\ z(\tau^m)\in D^{(\varepsilon,\infty)},\ \tau<\infty\Big]$$

$$=E_z\Big[E_{z(\tau^m)}\Big\{\prod_{k=1}^n f_k(z(t_k))\Big\}: A,\ z(\tau^m)\in D^{(\varepsilon,\infty)},\ \tau<\infty\Big].$$

Since $\tau^m\downarrow\tau$ as $m\to\infty$ and

$$E_z\Big[\prod_{k=1}^n f(z(t_k)))\Big]=P_{t_1}(f_1P_{t_2-t_1}(f_2\cdots(f_{n-1}P_{t_n-t_{n-1}}f_n)\cdots))(z)$$

is continuous in $z\in D$ by [1.4] (2), we have

$$E_z\Big[\prod_{k=1}^n f_k(z(\tau+t): A,\ z(\tau)\in D^{(\varepsilon,\infty)},\ \tau<\infty\Big]$$

$$\leqq\varliminf_{n\to\infty} E_z\Big[\prod_{k=1}^n f(z(\tau^m+t_k)): A,\ z(\tau^m)\in D^{(\varepsilon,\infty)},\ \tau<\infty\Big]$$

$$\leqq \varliminf_{n\to\infty} E_z\Big[E_{z(\tau^m)}\Big\{\prod_{k=1}^n f(z(t_k))\Big\}: A,\ z(\tau^m)\in D^{(\varepsilon,\infty)},\ \tau<\infty\Big]$$

$$\leqq E_z\Big[E_{z(\tau)}\Big[\prod_{k=1}^n f(z(t_k))\Big]: A,\ z(\tau)\in D^{[\varepsilon,\infty)},\ \tau<\infty\Big].$$

Similarly, we have

$$E_z\Big[\prod_{k=1}^n f_k(z(\tau+t_k)): A,\ z(\tau)\in D^{[\varepsilon,\infty)},\ \tau<\infty\Big]$$

$$\geqq E_z\Big[E_{z(\tau)}\Big[\prod_{k=1}^n f(z(t_k))\Big]: A,\ z(\tau)\in D^{(\varepsilon,\infty)},\ \tau<\infty\Big].$$

Letting ε tend to zero, we prove the proposition.

[1.6] PROPOSITION. *Let $\boldsymbol{P}$ in $\mathfrak{P}$ be given. Then, the processes $(y(t, w), P_{(x,y)})$ and $(y(t), P_y^{R,1})$ are identical in law for $z=(x, y)\in D$, where $z(t, w)=(x(t, w), y(t, w))$ for $w\in W$ and $P_y^{R,1}$ is the one-dimensional reflecting Brownian measure given in* § 0, 7°.

Proof. By Markov property of $(z(t, w), P_z)$ and $(y(t), P_y^{R,1})$, it is sufficient to prove

$$P_t f(z)=P_t^{R,1}f(y) \tag{1.2}$$

for any nonnegative f in $C_b((0, \infty))$ and $z=(x, y)\in D$, where $P_t^{R,1}f(y)=E_y^{R,1}(f(y(t)))$ and $P_t f(z)=E_z(f(y(t)))$. Choose a positive a and set $\rho\equiv\sigma+\sigma_a(\theta_\sigma(w))$. Then ρ is a Markov time and $z(\rho(w), w)\in\partial_a$ if $\rho<\infty$. For the indicator function $I_{[a,\infty)}$ of the interval $[a, \infty)$, set

$$A(t)=A_a(t, w)=\int_0^t I_{[a,\infty)}(y(s, w))ds\,,$$

then $A(\sigma)=A(\rho)$. For $\lambda>0$, set

$$G_\lambda^a f(z)=E_z\Big(\int_0^\infty e^{-\lambda A(t)}f(y(t, w))dA(t)\Big).$$

Then by strong Markov property given in [1.5] and (p. 4),

$$G_\lambda^a f(z)=E_y^{R,1}\Big(\int_0^{\sigma_0} e^{-\lambda\tilde{A}(t)}f(y(t))d\tilde{A}(t)\Big)+E_z(e^{-\lambda A(\sigma)}G_\lambda^a f(z(\rho)): \rho<\infty) \tag{1.3}$$

$$\leqq E_y^{R,1}\Big(\int_0^{\sigma_0} e^{-\lambda\tilde{A}(t)}f(y(t))d\tilde{A}(t)\Big)+E_y^{R,1}(e^{-\lambda\tilde{A}(\sigma_0)})\sup_x G_\lambda^a f(x, a)$$

for any $z=(x, y)\in D$, where $\sigma_0=\sigma_{\{0\}}$ and $\tilde{A}(t)=\int_0^t I_{[a,\infty)}(y(s))ds$ for reflecting Brownian path $y(\cdot)$. Set

$$\tilde{G}_\lambda^a f(y)=E_y^{R,1}\left(\int_0^\infty e^{-\lambda A(t)}f(y(t))d\tilde{A}(t)\right).$$

By a similar argument, we have

$$\tilde{G}_\lambda^a f(y)=E_y^{R,1}\left(\int_0^{\sigma_0} e^{-\lambda\tilde{A}(t)}f(y(t))d\tilde{A}(t)\right)+E_y^{R,1}(e^{-\lambda\tilde{A}(\sigma_0)})\tilde{G}_\lambda^a f(a). \tag{1.4}$$

Letting $z=(x, a)$ in (1.3) and $y=a$ in (1.4) and subtracting (1.4) from (1.3), we have

$$\sup_x G_\lambda^a f(x, a)-\tilde{G}_\lambda^a f(a)\leqq E_a^{R,1}(e^{-\lambda\tilde{A}(\sigma_0)})(\sup_x G_\lambda^a f(x, a)-\tilde{G}_\lambda^a f(a)).$$

Since $E_a^{R,1}(e^{-\lambda\tilde{A}(\sigma_0)})<1$, it follows that

$$\sup_x G_\lambda^a f(x, a)\leqq\tilde{G}_\lambda^a f(a).$$

Using (1.3) and (1.4) again, we have for any $z=(x, y)$

$$G_\lambda^a f(z)\leqq\tilde{G}_\lambda^a f(y).$$

Since

$$\lim_{a\to 0} G_\lambda^a f(z)=G_\lambda f(z)=\int_0^\infty e^{-\lambda t}P_t f(z)dt$$

and

$$\lim_{a\to 0}\tilde{G}_\lambda^a f(y)=\tilde{G}_\lambda f(y)=\int_0^\infty e^{-\lambda t}P_t^{R,1}f(y)dt,$$

$G_\lambda f(z)\leqq\tilde{G}_\lambda f(y)$ holds for any $z=(x, y)\in D$, nonnegative f in $C_b((0, \infty))$, and $\lambda>0$. On the other hand $G_\lambda 1(z)=1/\lambda=\tilde{G}_\lambda 1(y)$ by (p. 2). Hence we finally have $G_\lambda f(z)=\tilde{G}_\lambda f(y)$ for any $z\in D$, nonnegative f in $C_b((0, \infty))$, and $\lambda>0$. Since $P_t f(z)$ is continuous in t by (3) in [1.4], (1.2) is proved.(†)

[1.7] COROLLARY.

$$P_z(\sigma_a<\infty)=1 \quad \textit{for any } a>0 \textit{ and } z\in D.$$

Proof. By [1.6], $P_z(\sigma_a<\infty)=P_y^{R,1}(\sigma_{(a)}<\infty)=1$.

Now we shall define the class of continuous processes on $\bar{D}$ in a way similar to definition [1.1].

[1.8] DEFINITION OF THE CLASS $\bar{\mathcal{P}}$. $\boldsymbol{P}=\{\bar{P}_z(\bar{A}): z\in D, \bar{A}\in\bar{\boldsymbol{B}}\}$ is in class $\bar{\mathcal{P}}$ if and only if

($\bar{\text{p}}$. 1) $\bar{P}_z(\bar{A})$ is a probability kernel on $D\times\bar{\boldsymbol{B}}$.

(†) The proof of [1.6] is given by T. Shiga.

(p̄. 2) $\bar{P}_z(z(t, w)\in D)=1$ for any $t\geqq 0$ and $z\in D$.

(p̄. 3) $\bar{P}_z(z(t+s, w)\in A\,|\,\bar{\boldsymbol{B}}_t)=\bar{P}_{z(t,\bar{w})}(z(s, \bar{w})\in A)$ a.s. $\bar{P}_z$
for any $A\in\boldsymbol{B}(\bar{D})$, t, $s\geqq 0$ and $z\in D$.

(p̄. 4) $\bar{P}_z(z(t, \bar{w})\in A,\ t<\sigma_0)=P_z^{B,2}(z(t)\in A,\ t<\sigma_0)$
for any $t\geqq 0$, $z\in D$ and $A\in\boldsymbol{B}(D)$.

(p̄. 5) $\bar{P}_{z+2\pi}(z(t, \bar{w})\in A+2\pi)=\bar{P}_z(z(t, \bar{w})\in A)$ for any $z\in D$ and $A\in\boldsymbol{B}(D)$.

In the sequel we shall also write $\bar{z}(t)=z(t, \bar{w})=\bar{w}(t)$. Note that we do not define $\bar{P}_z(\cdot)$ for $z\in\partial_0$.

The continuous mapping ι from $\bar{D}$ into D^*,

$$\iota(z)=\begin{cases} z & z\in D\,, \\ \partial & z\in\partial_0\,, \end{cases} \tag{1.5}$$

induces a measurable mapping from $(\bar{W}, \bar{\boldsymbol{B}})$ into $(W, \boldsymbol{B})$ by

$$z(t, \iota\bar{w})=\iota(z(t, \bar{w})) \tag{1.6}$$

for $\bar{w}\in\bar{W}$. For any $\bar{\boldsymbol{P}}$ in $\bar{\mathcal{P}}$, let

$$\iota\bar{P}_z(A)=\bar{P}_z(\iota^{-1}A) \tag{1.7}$$

for $A\in\boldsymbol{B}$. Noting (p̄. 2), we can easily see:

[1.9] For any $\bar{\boldsymbol{P}}$ in $\bar{\mathcal{P}}$, $\iota\bar{\boldsymbol{P}}=\{\iota P_z(A)\colon z\in D,\ A\in\boldsymbol{B}\}$ is in $\mathcal{P}$. And the mapping ι from $\bar{\mathcal{P}}$ into $\mathcal{P}$ is injection.

[1.10] DEFINITION OF THE CLASS $\mathcal{P}_c$. We set $\mathcal{P}_c=\iota\bar{\mathcal{P}}$ and identify elements in $\mathcal{P}_c$ with those in $\bar{\mathcal{P}}$.

[1.11] PROPOSITION. *Let $\boldsymbol{P}$ be in $\mathcal{P}$. If there exists a set W_0 in $\boldsymbol{B}$ and a mapping ϕ from W_0 into $\bar{W}$ such that $P_z(W_0)=1$ for any $z\in D$, ϕ is measurable and $\bar{z}(t, \phi(w))=z(t, w)$ whenever $z(t, w)\in D$, then $\boldsymbol{P}$ is in $\mathcal{P}_c$.*

Proof. Set $W_r=\{w\colon z(\gamma, w)\in D$ for any rational $\gamma\}$ and $W_0^*=W_0\cap W_r$. Then by (p. 2) $P_z(W_0^*)=1$ for any $z\in D$. If $w\in W_0^*$ and $z(t, w)=\partial$, there exists a rational sequence $\{\gamma_n\}$ such that $z(\gamma_n, w)\in D$, $\gamma_n\to t$ and $z(\gamma_n, w)\to\partial$. Since $z(\gamma_n, w)=\bar{z}(\gamma_n, \phi(w))$, we see $\bar{z}(t, \phi(w))\in\partial_0$. Therefore, for any $w\in W_0^*$, $z(t, w)\in D$ if and only if $z(t, \phi(w))\in D$. Hence $\iota\cdot\phi(w)=w$ on W_0^*. Set $\phi P_z(\bar{A})=P_z(\phi^{-1}(\bar{A}))$ for any $z\in D$ and $\bar{A}\in\bar{\boldsymbol{B}}$. Then, we can easily check $\bar{\boldsymbol{P}}=\{\phi P_z(\cdot)\}_{z\in D}$ satisfies condition (p̄. 1)$\sim$(p̄. 5) in [1.8]. Thus we have $\boldsymbol{P}=\iota\phi\boldsymbol{P}=\iota\bar{\boldsymbol{P}}$ for $\bar{\boldsymbol{P}}=\phi\boldsymbol{P}\in\bar{\mathcal{P}}$.

§ 2. System of harmonic measures.

[2.1] DEFINITION. $\boldsymbol{H}=\{\boldsymbol{H}^a(z, A): a>0,\ z\in D^a,\ A\in\boldsymbol{B}(R)\}$ is in class $\mathcal{H}$ if and only if

(h. 1) $H^a(z, A)$ is a probability kernel on $D^a\times\boldsymbol{B}(R)$ for any fixed $a>0$.

For $\phi\in C_b(R)$, set

$$H^a\phi(z)=\int H^a(z, d\xi)\phi(\xi). \tag{2.1}$$

(h. 2) For $0<b<a$ and $z\in D^b$

$$H^a\phi(z)=\int H^b(z, d\xi)\int H^a((\xi, b), d\eta)\phi(\eta).$$

(h.3) For $a>0$ and $z\in D^a$, $H^a\phi$ is harmonic in D^a and $\lim_{z\to(x,a)} H^a\phi(z)=\phi(x)$.

(h. 4) $$H^a\phi(z+2\pi)=\int H^a(z, d\xi)\phi(\xi+2\pi) \quad \text{for } a>0 \text{ and } z\in D^a.$$

Here, the meaning of harmonic function is the ordinary one, that is, $f(z)$ is harmonic in an open set U if and only if

$$\Delta f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0 \qquad \text{in } U.$$

For $0<b<a$, we shall also write

$$H_b^a(x, A)=H^a((x, b), A), \tag{2.2}$$

$$H_b^a\phi(x)=H^a\phi(x, b),$$

where $A\in\boldsymbol{B}(R)$ and $\phi\in C_b(R)$.

[2.2] Definition [2.1] is equivalent to any one of the following two conditions (1) and (2).

(1) (h.1) holds and (h.2), (h.3) and (h.4) holds for any $\phi\in C_K(R)$.

(2)

($\bar{\text{h}}$.1) $H_b^a(x, A)$ defined in (2.2) is a probability kernel on $R\times\boldsymbol{B}(R)$ for $0<b<a$.

($\bar{\text{h}}$.2) For $0<c<b<a$,

$$H_c^a=H_c^bH_b^{a\,(\dagger)}.$$

(†) $H_c^bH_b^a(x, A)=\int H_c^b(x, d\xi)H_b^a(\xi, A)$.

In the sequel, similar notations are used for products of kernels

($\bar{\text{h}}$.3) For $0<c<b<a$,

$$H_b^a={}_c^a\Pi_b^a+{}_c^a\Pi_b^c H_c^a .$$

($\bar{\text{h}}$.4) For $0<b<a$, $x\in R$ and $A\in \boldsymbol{B}(R)$,

$$H_b^a(x, A)=H_b^a(x+2\pi, A+2\pi).$$

The kernels ${}_c^a\Pi_b^a$ and ${}_c^a\Pi_b^c$ are given in § 0, 8°.

Combining ($\bar{\text{h}}$.2) and ($\bar{\text{h}}$.3), we have

$$H_b^a={}_c^a\Pi_b^a+{}_c^a\Pi_b^c H_c^b H_b^a . \tag{2.3}$$

[2.3] DEFINITION. Let $\boldsymbol{P}$ in $\mathscr{P}$ be given. We define

$$H_{\boldsymbol{P}}^a(z, A)=P_z(z(\sigma_a(w), w)\in A\times\{a\}) \tag{2.4}$$

for any $a>0$, $z\in D^a$ and $A\in \boldsymbol{B}(R)$, where σ_a is the hitting time of ∂_a. And set

$$\boldsymbol{H}(\boldsymbol{P})=\boldsymbol{H}_{\boldsymbol{P}}=\{H_{\boldsymbol{P}}^a(z, A): a>0, z\in D^a, A\in \boldsymbol{B}(R)\}. \tag{2.5}$$

Then, we can easily see $\boldsymbol{H}(\boldsymbol{P})$ is in class $\mathscr{H}$. For, $H_{\boldsymbol{P}}^a(z, R)=1$ follows from corollary [1.7], (h.2) is a consequence of [1.7] and strong Markov property given in [1.5]. The conditions (h.3) and (h.4) follow from (p.4) and (p.5) in [1.1], respectively.

The following theorem is fundamental. The proof will be given in [9].

[2.4] THEOREM. *The mapping* $\boldsymbol{P}\to\boldsymbol{H}(\boldsymbol{P})$ *given by* (2.4) *and* (2.5) *is bijection between* $\mathscr{P}$ *and* $\mathscr{H}$.

[2.5] DEFINITION. Let $\{{}^n\boldsymbol{H}\}_{n=1}^{\infty}$ be a sequence in $\mathscr{H}$ and let $\boldsymbol{H}\in\mathscr{H}$. Then we shall say $\{{}^n\boldsymbol{H}\}_{n=1}^{\infty}$ converges to $\boldsymbol{H}$ in $\mathscr{H}$ and write ${}^n\boldsymbol{H}\to\boldsymbol{H}$ if and only if ${}^nH^a\phi(z)\to H^a\phi(z)$ $(n\to\infty)$ for any $a>0$, $z\in D^a$ and $\phi\in C_K(R)$.

Let $\{{}^n\boldsymbol{P}\}_{n=1}^{\infty}$ be a sequence in $\mathscr{P}$ and let $\boldsymbol{P}\in\mathscr{P}$. We define ${}^n\boldsymbol{P}\to\boldsymbol{P}$ in $\mathscr{P}$ if and only if ${}^n\boldsymbol{H}_{\boldsymbol{P}}\to\boldsymbol{H}_{\boldsymbol{P}}$ in $\mathscr{H}$.

[2.6] *Remark.* If ${}^n\boldsymbol{H}\to\boldsymbol{H}$ in $\mathscr{H}$, then ${}^nH^a\phi(z)\to H^a\phi(z)$ for any $a>0$, $z\in D^a$ and $\phi\in C_b(R)$. For, ${}^nH^a1=H^a1=1$ holds.

For a fixed $\phi\in C_b(R)$, $\{H^a\phi\}_{\boldsymbol{H}\in\mathscr{H}}$ is a family of functions uniformly bounded and harmonic in D^a with boundary function $H^a\phi(x, a)=\phi(x)$ on ∂_a, therefore we have:

[2.7] Let $\{{}^n\boldsymbol{H}\}$ be in $\mathscr{H}$, ϕ be a function in $C_b(R)$ and a be any positive number. Then, we can select a subsequence $\{n_k\}$ such that $\{{}^{m_k}\boldsymbol{H}^a\phi\}$ converges uniformly on any compact subset of $D^{(0,a]}$.

[2.8] PROPOSITION. *Let* $\{{}^n\boldsymbol{H}\}_{n=1}^{\infty}$ *be a sequence in* $\mathscr{H}$ *such that, for any fixed* $a>0$ *and* $z\in D^a$, *the set of measures* $\{{}^nH^a(z,\cdot)\}$ *is tight, that is, for any* $\varepsilon>0$ *there exists a number* $N=N(\varepsilon, a, z)$ *such that*

$$\sup_n {}^nH^a(z, \{y: |y-x|>N\})<\varepsilon . \tag{2.6}$$

Then $\{{}^n\boldsymbol{H}\}$ *has a convergent subsequence.*

Proof. Notice that, for any $\phi\in C_K(R)$ and $a>0$, $\{{}^nH^a\phi\}$ are uniformly bounded and harmonic in D^a and $C_K(R)$ is separable. Then, for any $a>0$, we can select a subsequence $\{n_k\}$ such that, for any ϕ, $\{{}^{n_k}H^a\phi\}$ converges to some harmonic function $H^a\phi$ uniformly in any compact subset of D^a. Moreover $H^a\phi(z)$ can be represented by kernel $H^a(z, d\xi)$, that is,

$$H^a\phi(z)=\int H^a(z, d\xi)\phi(\xi) .$$

The assumption (2.6) assures $H^a(z, R)=1$. Let $\{a_m\}$ be a set of positive number which decreases to zero. By taking a subsequence again, we may suppose $\{{}^{n_k}H^{a_m}\phi(z)\}$ converges to $H^{a_m}\phi(z)$ for any a_m, $z\in D^{a_m}$ and $\phi\in C_K(R)$. Let the subsequence $\{n_k\}$ be fixed. Now for any fixed $a>0$, choose a_m and b so small that

$$0<b<a_m<a .$$

Let g be one of limit functions of subsequences of $\{{}^{n_k}H^a\phi\}$ in D^a for a fixed ϕ in $C_K(R)$. Then, by (2.3), $g_m(x)=g(x, a_m)$ is a solution of

$$g_m(x)={}^a_b\Pi^a_{a_m}\phi(x)+{}^a_b\Pi^b_{a_m}H^{a_m}_b g_m(x) .$$

Since

$${}^a_b\Pi^b_{a_m}H^{a_m}_b(x, R)=\frac{a-a_m}{a-b}<1 ,$$

$g(z)$ is determined on ∂_{a_m}, independently of the subsequence. Noting that $\lim_{z\to(x,a)} g(z)=\phi(x)$ and that g is harmonic in D^a, we can see $g(z)$ is determined in $D^{(a_m, a]}$. Since a_m can be choosen arbitrarily small, g is determined in D^a. Thus, we have proved that $\{{}^{n_k}H^a\phi(z)\}$ converges for any $a>0$, $z\in D^a$ and $\phi\in C_K(R)$. Set $\lim_{k\to\infty} {}^{n_k}H^a\phi(z)=H^a\phi(z)$. Then $H^a\phi(z)$ is represented as

$$H^a\phi(z)=\int H^a(z, d\xi)\phi(\xi) \quad \text{with } H^a(z, R)=1.$$

Set $\boldsymbol{H}=\{H^a(z, A): a>0, z\in D^a, A\in\boldsymbol{B}(R)\}$. The above arguments show that $\boldsymbol{H}$ satisfies (h.1) and (h.2) in [2.1]. Noting [2.7], we can easily prove (h.3). The property (h.4) is obvious. Therefore $\boldsymbol{H}\in\mathscr{H}$ and ${}^{n_k}\boldsymbol{H}\to\boldsymbol{H}$.

§ 3. Continuity of $P_t f$ in $\bar{D}$.

In this section, we shall fix a process $\boldsymbol{P}$ in $\mathfrak{P}$ and set $\boldsymbol{H}=\boldsymbol{H}_{\boldsymbol{P}}$. We shall discuss the relation between continuity of functions $P_t f(z)=E_z(f(z(t)))$ in $\bar{D}$ and continuity of functions $H^a\phi(z)$ in $\bar{D}^a=D^{[0,a]}\subset\bar{D}$.

[3.1] DEFINITION. For f in $C_b(D^a)$, we write $f\in C_b(\bar{D}^a)$ if f can be extended to a continuous function on $\bar{D}^a$. In this case, we denote the extension by $\tilde{f}$.

It is obvious that $\tilde{f}$ is unique if it exists.

[3.2] CONDITION [C] (Continuity Condition). (1) $P_t f$ is in $C_b(\bar{D})$ for any $t>0$ and $f\in C_K(D)$.

(2) If $\{f_n\}$ is a nonnegative sequence in $C_K(D)$ such that $f_n\uparrow 1$ in D, then $P_t f_n\uparrow 1$ in $\bar{D}$ for any $t>0$.

[3.3] CONDITION [H. C]. (1) $H^a\phi$ is in $C_b(\bar{D}^a)$ for any $a>0$ and ϕ in $C_K(R)$.

(2) If $\{\phi_n\}$ is a nonnegative sequence in $C_K(R)$ such that $\phi_n\uparrow 1$ in R, then $H^a\phi_n\uparrow 1$ in $\bar{D}^a$.

[3.4] Under [C] it holds that

(1) $P_t f\in C_b(\bar{D})$ $(t>0)$ for any $f\in C_b(D)$

(2) $P_t f_n\uparrow 1$ in $\bar{D}$ $(t>0)$ for any nonnegative sequence $\{f_n\}$ in $C_b(D)$ such that $f_n\uparrow 1$ in D.

Proof. Assume [C] holds. Let $\{g_n\}$ be a nonnegative sequence in $C_K(D)$ such that $g_n\uparrow 1$ in D. Then for any $t>0$, $P_t g_n$ converges to 1 uniformly on any compact subset of $\bar{D}$ by Dini's theorem. Let $f\in C_b(D)$. Since

$$|P_t f g_n(z)-P_t f g_m(z)|\leqq\|f\|\,|P_t g_n(z)-P_t g_m(z)|,$$

$P_t f g_n$ converges uniformly on any compact subset of $\bar{D}$. Therefore, $\tilde{g}(z)=\lim_{n\to\infty} P_t f g_n(z)$ exists and is continuous in $\bar{D}$. Noting $\tilde{g}(z)=\lim P_t f g_n(z)=P_t f(z)$ for z in D, we see that (1) is proved. (2) is obvious, for we can choose a nonnegative sequence $\{g_n\}$ in $C_K(D)$ such that $g_n\uparrow 1$ and $g_n\leqq f_n$.

In a similar way, we can show:

[3.5] Under [H.C], it holds that

(1) $H^a\phi\in C_b(\bar{D}^a)$ $(a>0)$ for $\phi\in C_b(R)$.

(2) $H^a\phi_n\uparrow 1$ in $\bar{D}^a$ for any nonnegative sequence $\{\phi_n\}$ in $C_b(R)$ such that $\phi_n\uparrow 1$ in R.

[3.6] For $a>0$ and $\phi\in C_K(R)$, let g be a continuous extention of $H^a\phi$ to D such that

$$\|g\|_D \leqq \|H^a\phi\|_{D^a} \leqq \|\phi\|_R .$$

Here $\|f\|_A$ denotes $\sup_{x\in A}|f(x)|$. Then, for any $z\in D^{a/2}$ and $t>0$

$$|H^a\phi(z)-P_t g(z)| \leqq \|\phi\|\varepsilon_a(t),$$

where $\varepsilon_a(t)$ is independent of z, ϕ and g, and $\lim_{t\downarrow 0}\varepsilon_a(t)=0$.

Proof. Let $\sigma=\sigma_a$ be the hitting time of ∂_a. For $z\in D^a$

$$P_t g(z)=E_z(H^a\phi(z(t))\,;\,t<\sigma)+E_z(g(z(t))\,;\,t\geqq\sigma)$$
$$=E_z(\phi(z(\sigma))\,;\,t<\sigma)+E_z(g(z(t))\,;\,t\geqq\sigma)$$

and $H^a\phi(z)=E_z(\phi(z(\sigma)))$. Therefore

$$|H^a\phi(z)-P_t g(z)| \leqq (\|\phi\|+\|g\|)P_z(t\geqq\sigma) \leqq 2\|\phi\|P_z(t\geqq\sigma).$$

Especially, for $z\in D^{a/2}$,

$$P_z(t\geqq\sigma)=P_z(y(s)\geqq a \text{ for some } s\leqq t)$$
$$=P_y^{R,1}(y(s)\geqq a \text{ for some } s\leqq t) \quad (\text{by [1.6]})$$
$$\leqq P_{a/2}^{R,1}(y(s)\geqq a \text{ for some } s\leqq t).$$

Hence $\varepsilon_a(t)=2P_{a/2}^{R,1}(y(s)\geqq a$ for some $s\leqq t)$ has the desired properties.

[3.7] [C] implies [H.C].

Proof. Let ϕ be in $C_K(R)$. For any $a>0$, let g be selected as in [3.6]. By [C] and [3.4], $P_t g$ is in $C_b(\bar{D})$. Therefore, by [3.6], $H^a\phi(z)$ is approximated uniformly in $z\in D^{a/2}$ by continuous functions $P_t g$ $(t>0)$ on $\bar{D}^{a/2}$. Since $H^a\phi$ is continuous in $D^{[a/2,a]}$ in general, we have proved (1) in [H.C]. Let ϕ_n be in $C_K(R)$ such that $\phi_n\uparrow 1$. Then $H^a\phi_n(z)\uparrow 1$ for z in $D^{(0,a]}$. We can select a continuous extension g_n of each $H^a\phi_n$ to D such that $g_n\uparrow 1$ and $\|g_n\|\leqq \|H^a\phi_n\|\leqq 1$. Then, by [3.4] $P_t g_n$ is in $C_b(\bar{D})$ and $P_t g_n\uparrow 1$ in $\bar{D}$. Moreover, by [3.6]

$$\|H^a\phi_n-P_t g_n\|_{\bar{D}^{a/2}} \leqq \varepsilon_a(t).$$

Since, $\varepsilon_a(t)$ can be taken arbitrarily small, $H^a\phi_n(z)\uparrow 1$ holds for any z in $\bar{D}^{a/2}$. (2) in [H.C] is proved.

[3.8] Let f be in $C_K(D)$. Then

$$\lim_{t\to 0}\sup_{|r-s|\leqq t}\|P_r f-P_s f\|_D=0$$

Proof. Since $\|P_r f - P_s f\| = \|P_s(P_{r-s} f - f)\| \leqq \|P_{r-s} f - f\|$ if $r > s$, it is sufficient to prove

$$\varepsilon_f(t) = \sup_{0 \leqq s \leqq t} \|P_s f - f\| \longrightarrow 0 \quad \text{as } t \to 0 .$$

Choose $a > 0$ such that $f \equiv 0$ in D^a. For $z \in D^{a/2}$,

$$\sup_{s \leqq t} |P_s f(z) - f(z)| = \sup_{s \leqq t} |P_s f(z)| \leqq \|f\| P_z(\sigma_a \leqq t)$$

where σ_a is the hitting time of ∂_a. And, in the proof of [3.6], we have seen

$$P_z(\sigma_a \leqq t) \leqq \frac{1}{2} \varepsilon_a(t) \equiv P^{R, \frac{1}{2}}_{a/2}(y(s) \geqq a \text{ for some } s \leqq t) .$$

On the other hand, for $z \in D^{[a/2, \infty)}$ and for any δ satisfying $0 < \delta < a/2$,

$$\sup_{s \leqq t} |P_s f(z) - f(z)| \leqq \varepsilon(f, \delta) + 2\|f\| \sup_{s \leqq t} P_z(|z(s) - z| \geqq \delta)$$

where $\varepsilon(f, \delta) = \sup_{|\zeta - z| < \delta, \zeta, z \in D} |f(\zeta) - f(z)|$. Let $\tau_\delta = \inf\{t : |z(t) - z(0)| \geqq \delta\}$, and σ be the hitting time of ∂. Then $\tau_\delta < \sigma$ if $z(0) \in D^{[a/2, \infty)}$. Therefore we have for $z \in D^{[a/2, \infty)}$

$$\begin{aligned} P_z(|z(t) - z| \geqq \delta) &\leqq P_z(\tau_\delta \leqq t \text{ and } \tau_\delta < \sigma) \\ &= P_z^{B, 2}(\tau_\delta \leqq t) \quad \text{by (p. 4) in [1.1]} \\ &= P_0^{B, 2}(\tau_\delta \leqq t) \equiv \bar{\varepsilon}(\delta, t) \end{aligned}$$

and $\lim_{t \to 0} \bar{\varepsilon}(\delta, t) = 0$ for fixed $\delta > 0$. We have thus proved

$$\sup_{0 \leqq s \leqq t} \|P_s f - f\|_D \leqq \text{Max} \left(\frac{1}{2} \|f\| \varepsilon_a(t),\ \varepsilon(f, \delta) + 2\|f\| \bar{\varepsilon}(\delta, t) \right)$$

and $\overline{\lim}_{t \to 0} \|P_t f - f\| \leqq \varepsilon(f, \delta)$. Since δ can be taken arbitrarily small, the proof is completed.

Combining [3.8] and (2) in [1.4], we can also show:

[3.9] $P_t f(z)$ is continuous in $(t, z) \in [0, \infty) \times D$ for any $f \in C_K(D)$.

[3.10] [H.C] implies $P_t f$ is in $C_b(\bar{D})$ for any $f \in C_K(D)$ and $t > 0$.

Proof. Let f in $C_K(D)$ and $t > 0$ be given. For $a > 0$, set

$$u_a(z) = E_z(f(z(\sigma_a + t))) = E_z(P_t f(z(\sigma_a))) ,$$

where σ_a is the hitting time of $D^{[a, \infty)}$. Then,

$$u_a(z)=\begin{cases} P_t f(z) & \text{if } y \geqq a\,, \\ H^a(P_t f)(z) & \text{if } y<a\,. \end{cases}$$

Since $P_t f(\cdot, a)$ belongs to $C_b(R)$, u_a has a continuous extension $\tilde{u}_a$ in $\bar{D}$ by [3.5] and (2) in [1.4]. Let $0<s<t$. Then

$$|P_t f(z)-u_a(z)| \leqq |E_z(P_{t-\sigma_a} f(z(\sigma_a))-P_t f(z(\sigma_a))\,;\,\sigma_a<s)|+2\|f\| P_z(\sigma_a \geqq s)$$
$$\leqq \varepsilon(f, s)+2\|f\| P_z(\sigma_a \geqq s)\,,$$

where $\varepsilon(f, s)=\sup_{|r-u|\leqq s} \|P_r f-P_u f\|$. By [1.6]

$$P_z(\sigma_a \geqq s)=P_y^{R,1}(\bar{\sigma}_a \geqq s) \leqq P_0^{R,1}(\bar{\sigma}_a \geqq s) \equiv \bar{\varepsilon}(a, s)\,,$$

where $\bar{\sigma}_a$ is the hitting time of $[a, \infty)$ for one-dimensional reflecting Brownian motion. Therefore, $\|P_t f-u_a\|_D \leqq \varepsilon(f, s)+2\|f\|\bar{\varepsilon}(a, s)$. Noting $\lim_{a\to 0} \bar{\varepsilon}(a, s)=0$ for fixed $s>0$ and $\lim_{s\to 0} \varepsilon(f, s)=0$ by [3.8], we see that $P_t f$ can be approximated uniformly in D by functions u_a $(a>0)$, which are in $C_b(\bar{D})$. Therefore $P_t f$ is in $C_b(\bar{D})$.

[3.11] Assume [H.C]. Let $\{f_n\}$ be a sequence of functions in $C_K(D)$ such that $f_n \uparrow 1$. Then $P_t f_n \uparrow 1$ in $\bar{D}$ for $t>0$.

Proof. By [3.10], $P_t f_n$ is in $C_b(\bar{D})$. We have $P_t f_n(z) \uparrow 1$ for $(t, z)\in[0, \infty) \times D$. For any compact set $K\subset D$, $P_s f_n(z)\uparrow 1$ uniformly in $(s, z)\in[0, t]\times K$. Let $\{\phi_m\}$ be a sequence in $C_K(R)$ such that $\phi_m \uparrow 1$. Let K_m be the support of ϕ_m. Then for $z\in D^a$

$$1-P_t f_n(z) \leqq E_z(1-P_{t-\sigma_a} f_n(z(\sigma_a))\,;\,\sigma_a<t,\ z(\sigma_a)\in K_m)$$
$$+P_z(\sigma_a \geqq t)+P_z(z(\sigma_a)\notin K_m)\,,$$

where σ_a is the hitting time of $D^{[a,\infty)}$. Set

$$\varepsilon(n\,;\,t, m)=\sup_{(s,z)\in[0,t]\times K_m} (1-P_s f_n(z)))\,.$$

Then $\lim_{n\to\infty} \varepsilon(n\,;\,t, m)=0$, for fixed t and m. In the proof of [3.10] we have shown

$$P_z(\sigma_a \geqq t) \leqq \bar{\varepsilon}(a, t)=P_0^{R,1}(\bar{\sigma}_a \geqq t)\,,$$
$$P_z(z(\sigma_a)\notin K_m) \leqq E_z(1-\phi_m(z(\sigma_a)))=1-H^a(\phi_m(\cdot, a))(z)\,.$$

Therefore, we have

$$1-P_t f_n(z) \leqq \varepsilon(n\,;\,t, m)+\bar{\varepsilon}(a, t)+1-H^a(\phi_m(\cdot, a))(z)$$

for any z in $\bar{D}^a$. Making n and then m tend to infinity, we have

$$\overline{\lim_{n\to\infty}}(1-P_t f_n(z)) \leqq \tilde{\varepsilon}(a, t) \quad \text{for any } a>0, \text{ and } z\in D^{[0,a]}.$$

Since $\lim_{n\to\infty}(1-P_t f_n(z))=0$ holds in general for $z\in D$, [3.11] is proved.

By [3.7], [3.10] and [3.12], we have proved:

[3.12] THEOREM. *The condition [C] is equivalent to the condition [H. C].*

II. Formulation and Uniqueness of *B*-processes.

§4. Formulations of *B*-solutions and *B*-processes.

We shall prepare several lemmas on periodic harmonic functions on the upper half plane.

[4.1] Let u and v be in $C_{p,N}(D^{[b,a]})$ and harmonic in $D^{(b,a)}$ $(b<a)$, and v be a harmonic conjugate of u. Then, the values of integrals

$$\frac{1}{2N\pi}\int_0^{2N\pi} u(x, y)dx, \quad \frac{1}{2N\pi}\int_0^{2N\pi} v(x, y)dx$$

are independent of y $(b<y<a)$.

Proof. By Green's formula, we can easily show independence of the values of the integrals.

[4.2] Let u be in $C_{p,N}(D^{[a,\infty)})$ and bounded harmonic in $D^{(a,\infty)}$, and v be a harmonic conjugate of u in $D^{(a,\infty)}$. Then, u and v have the following representations:

$$u(z)=\int_R h_\xi(x, y-a)u(\xi, a)d\xi \tag{4.1}$$

$$=\frac{1}{N}\int_0^{2N\pi} \tilde{h}_{\xi/N}(x/N, (y-a)/N)u(\xi, a)d\xi,$$

$$v(z)=\lim_{T\to\infty}\int_{-T}^{T} k_\xi(x, y-a)u(\xi, a)d\xi+c \tag{4.2}$$

$$=\frac{1}{N}\int_0^{2N\pi} \tilde{k}_\xi(x/N, (y-a)/N)u(\xi, a)d\xi+c$$

for $z\in D^{(a,\infty)}$, where h_ξ, k_ξ, $\tilde{h}_\xi$ and $\tilde{k}_\xi$ are the functions defined in §0, 8°, and c is a constant.

[4.3] Let u in $C_{p,N}(D)$ be harmonic in D and bounded in $D^{[a,\infty)}$ for each $a>0$. Let v be a harmonic conjugate of u in D. Then v is also in $C_{p,N}(D)$ and bounded in $D^{[a,\infty)}$ for every $a>0$. Moreover,

$$(4.3)\qquad \begin{cases} \lim_{y\to\infty} u(z)=\dfrac{1}{2N\pi}\displaystyle\int_0^{2N\pi} u(x,\, y)dx\,, \\ \lim_{y\to\infty} v(z)=\dfrac{1}{2N\pi}\displaystyle\int_0^{2N\pi} v(x,\, y)dx \quad (y>0)\,. \end{cases}$$

Note that the right sides in (4.3) are independent of y by [4.1].

Proof. The first half is an immediate consequence of the representation (4.2). Noting (4.1) and (4.2), and $\lim_{y\to\infty} \tilde{h}_\xi(z)=1/2\pi$ and $\lim_{y\to\infty} \tilde{k}_\xi(z)=0$ (uniformly in x), we can easily get the latter half.

[4.4] Let u and f in $C_{p,N}(D)$ be harmonic in D and bounded in $D^{[a,\infty)}$ for each $a>0$. Let v and g be harmonic conjugates of u and f, respectively. Set

$$\alpha=\frac{1}{2N\pi}\int_0^{2N\pi} u(x,\, y)dx\,, \quad \beta=\frac{1}{2N\pi}\int_0^{2N\pi} v(x,\, y)dx\,,$$

$$\gamma=\frac{1}{2N\pi}\int_0^{2N\pi} f(x,\, y)dx\,, \quad \delta=\frac{1}{2N\pi}\int_0^{2N\pi} g(x,\, y)dx\,,$$

which are independent of y by [4.1] and [4.3]. Then, we have

$$(4.4)\qquad \frac{1}{2N\pi}\int_0^{2N\pi}(ug+vf)(x,\, y)dx=\alpha\delta+\beta\gamma\,.$$

Proof. Set $w=ug+vf=\mathrm{Im}\,(u+iv)(f+ig)$. Then w is harmonic in D, bounded in $D^{[a,\infty)}$ for each $a>0$ and in $C_{p,N}(D)$. Therefore by (4.3) in [4.3]

$$\lim_{y\to\infty}(ug+vf)=\frac{1}{2N\pi}\int_0^{2N\pi}(ug+vf)(x,\, y)dx$$

and

$$\lim_{y\to\infty} u=\alpha, \quad \lim_{y\to\infty} v=\beta, \quad \lim_{y\to\infty} f=\gamma, \quad \lim_{y\to\infty} g=\delta,$$

which proves (4.4).

[4.5] For $a>0$, let u and f in $C_{p,N}(D^a)$ be harmonic in D^a, and v and g be harmonic conjugates of u and f, respectively. Assume v and g be also in $C_{p,N}(D^a)$ and

$$(4.5)\qquad \int_0^{2N\pi}(ug+vf)(x,\, y)dx=0 \quad (0<y<a)\,.$$

Then we can show:

(1) There is a solution $U(z)$ $(z\in D^a)$ of

$$(4.6)\qquad \begin{cases} U_x=ug+vf\,, \\ U_y=uf-vg \end{cases}$$

such that U is in $C_{p,N}(D^a)$ and harmonic in D^a. Such a U is unique up to an additive constant.

(2) For $0<c<b<a$,

$$\int_0^{2N\pi} f(x, b)U(x, b)\,dx-\int_0^{2N\pi} f(x, c)U(x, c)\,dx \tag{4.7}$$
$$=\iint_{\substack{c\leq y\leq b\\ 0<x<2N\pi}} u(f^2+g^2)\,dx\,dy\,.$$

Proof. Since

$$ug+vf-i(uf-vg)=-i(u+iv)(f+ig)$$

is analytic in D^a, we can easily show (1), using (4.5). Let F be a harmonic function in D^a such that $F_x=g$ and $F_y=f$. Applying Green's formula, we have

$$\int_0^{2N\pi} fU(x, b)dx-\int_0^{2N\pi} fU(x, c)dx=\int_0^{2N\pi} F_yU(x, b)dx$$
$$+\int_b^c F_xU(2N\pi, y)dy+\int_{2N\pi}^0 F_yU(x, c)dx+\int_c^b F_xU(0, y)dy$$
$$=\iint_{\substack{0<x<2N\pi\\ c<y<b}} (F_xU_x+F_yU_y)dx\,dy$$
$$=\iint_{\substack{0<x<2N\pi\\ c<y<b}} u(f^2+g^2)dx\,dy\,.$$

The proof is thus completed.

Now, we shall give the definitions of $\boldsymbol{B}$-solutions and $\boldsymbol{B}$-processes. Let σ and μ be in $M_p(R)$ and

$$[\sigma]=\frac{1}{2\pi}\sigma([0, 2\pi))=1\,,$$

$$[\mu]=\frac{1}{2\pi}\mu([0, 2\pi))=1\,,$$

and k be any real constant. Set, for $z\in D$,

$$\begin{cases} s(z)=\int_R h_\xi(z)\sigma(d\xi)=\int_{[0,2\pi)} \tilde{h}_\xi(z)\sigma(d\xi)\,, \\ m(z)=\int_R h_\xi(z)\mu(d\xi)=\int_{[0,2\pi)} \tilde{h}_\xi(z)\mu(d\xi)\,, \end{cases} \tag{4.8}$$

$$(4.9)\qquad \begin{cases} t(z)=\int_{[0,2\pi)} \tilde{k}_\xi(z)\sigma(d\xi)+k\,, \\ l(z)=\int_{[0,2\pi)} \tilde{k}_\xi(z)\mu(d\xi)-k\,. \end{cases}$$

Then, we can easily see that

$$(4.10)\qquad \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} s(x,\,y)dx=1\,, & \frac{1}{2\pi}\int_0^{2\pi} t(x,\,y)dx=k\,, \\ \frac{1}{2\pi}\int_0^{2\pi} m(x,\,y)dx=1\,, & \frac{1}{2\pi}\int_0^{2\pi} l(x,\,y)dx=-k\,, \end{cases}$$

for any $y>0$ and that t and l are harmonic conjugates of s and m, respectively. Therefore by [4.4]

$$(4.11)\qquad \frac{1}{2\pi}\int_0^{2\pi}(mt+sl)(x,\,y)dx=0$$

for any $y>0$. By (1) in [4.5] the equation,

$$(4.12)\qquad \begin{cases} \tilde{U}_x=mt+ls \\ \tilde{U}_y=ms-lt \end{cases}$$

has a solution $\tilde{U}$, which is in $C_p(D)$ and harmonic in D. Henceface, when we say $\tilde{U}$ is a solution of (4.12), we mean that $\tilde{U}$ is in $C_p(D)$ and harmonic in D and satisfies (4.12).

[4.6] CONDITION [P*] (Condition on Existence of Positive Solution). Any solution $\tilde{U}$ of (4.12) satisfies

$$(4.13)\qquad \inf_{z\in D}\tilde{U}(z)>-\infty\,.$$

Since the solution of (4.12) is unique up to an additive constant, we have:

[4.7] If some solution of (4.12) satisfies (4.13), then every solution of (4.12) satisfies (4.13). If the condition [P*] is satisfied, then there exists a unique minimum nonnegative solution U^0 of (4.12).

Condition [P*] is a restriction on σ, μ and k. In theorem [5.11], we shall show that [P*] is equivalent to a condition [P], which is a certain integrability condition on μ and σ, and, in fact, which depends only on σ and μ.

Assume [P*] and let U^0 be the minimum nonnegative solution of (4.12). Set

$$(4.14)\qquad p_0\equiv p_0(\sigma,\,\mu,\,k)=\inf_{y>0}\frac{1}{2\pi}\int_0^{2\pi}U^0(x,\,y)s(x,\,y)dx\,.$$

By (2) in [4.5], for $0<b<a$

$$\int_0^{2\pi} U^0 s(x, a)dx - \int_0^{2\pi} U^0 s(x, b)dx = \iint_{\substack{0<x<2\pi \\ b<y<a}} m(s^2+t^2)dxdy \geqq 0 .$$

Therefore, we have:

[4.8] $$p_0 = \lim_{y\to 0} \frac{1}{2\pi}\int_0^{2\pi} U^0(x, y)s(x, y)dx \geqq 0 .$$

[4.9] DEFINITION OF CLASS $\mathcal{B}$. $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ is in class $\mathcal{B}$ if and only if

(b. 1) σ and μ is in $M_p(R)$ and $[\sigma]=[\mu]=1$.

(b. 2) k is a real constant.

(b. 3)* σ, μ and k satisfy the condition [P*].

(b. 4) p is a nonnegative constant such that $p \geqq p_0(\sigma, \mu, k)$.

[4.10] *Remark.* We shall see in § 5(†) that (b.3)* can be replaced by

(b. 3) σ and μ satisfy the condition [P].

Since any solution $\tilde{U}$ of (4.12) is given by $\tilde{U}=U^0+c$, we have

$$\lim_{y\to 0}\frac{1}{2\pi}\int_0^{2\pi}\tilde{U}(x, y)s(x, y)dx = p_0 + c .$$

[4.11] For any $p \geqq p_0$, there exists a unique nonnegative solution U of (4.12) such that

(4.15) $$\lim_{y\to 0}\frac{1}{2\pi}\int_0^{2\pi} U(x, y)s(x, y)dx = p .$$

[4.12] DEFINITION. Let $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathcal{B}$ be given. For z in D, set

$$s(z)=s(\boldsymbol{B})(z)=\int_{[0,2\pi)} \tilde{h}_\xi(z)\sigma(d\xi) ,$$

$$t(z)=t(\boldsymbol{B})(z)=\int_{[0,2\pi)} \tilde{k}_\xi(z)\sigma(d\xi)+k ,$$

$$m(z)\equiv m(\boldsymbol{B})(z)=\int_{[0,2\pi)} \tilde{h}_\xi(z)\mu(d\xi) ,$$

$$l(z)\equiv l(\boldsymbol{B})(z)=\int_{[0,2\pi)} \tilde{k}_\xi(z)\mu(d\xi)-k .$$

The function $U(z)=U(\boldsymbol{B})(z)$ on D is defined as the unique solution of (4.12) such that

(†) See theorem [5.11].

$$\lim_{y\to 0}\frac{1}{2\pi}\int_0^{2\pi}U(x, y)s(x, y)dx=p\,.$$

[4.13] DEFINITION OF $D^a_{q,N}$. For $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ and $a>0$, a function ϕ belongs to $D^a_{q,N}\equiv D^a_{q,N}(\boldsymbol{B})$ if and only if the following are satisfied:

(1) ϕ is harmonic in D^a and in $C_{q,N}(D^a)\cap C(D^{(0,a]})$.

(2) $\int_0^{2N\pi}(m\phi_y-l\phi_x)(x, y)dx=0$ for $0<y<a$.

(3) For any b $(0<b<a)$, there exists a constant $K=K(b, \phi)$ such that

$$|\phi_x(z)|\leqq Ks(z) \quad \text{in } D^b\,.$$

We also write

$$D_{q,N}=\bigcap_{a>0}D^a_{q,N},\quad D^a_q=D^a_{q,1}\quad\text{and}\quad D_q=D_{q,1}\,.$$

Let $u=u(z)$ be the harmonic function in D that satisfies

$$\begin{cases} u_x=s, \quad u_y=-t \quad \text{in } D \\ u(0, 1)=0\,. \end{cases} \tag{4.16}$$

Noting (4.11) and

$$u(z+2\pi)-u(z)=\int_0^{2\pi}s(x+\xi, y)d\xi=2\pi\,,$$

we can easily show:

[4.14] u is in D_q. We shall write $u(z)\equiv u(\boldsymbol{B})(z)$.

From (1) and (3) in [4.13], we can easily have:

[4.15] Let ϕ be in $D^a_{q,N}$. Then, there exists a periodic signed measure σ_ϕ of period $2N\pi$ on R, which is of bounded variation in each finite interval and

$$\lim_{y\to 0}\int_0^{2N\pi}g(x)\phi_x(x, y)dx=\int_0^{2N\pi}g(x)\sigma_\phi(dx) \tag{4.17}$$

for any g in $C_{p,N}(R)$. Moreover, σ_ϕ is absolutely continuous with respect to σ and $d\sigma_\phi/d\sigma$ has a bounded periodic version. Especially $\sigma_u=\sigma$.

[4.16] DEFINITION OF $\boldsymbol{B}_N$-SOLUTION. Let N be a positive integer and let $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ be given. For any f in $C_{q,N}(R)$ and any $a>0$, we say a function ϕ on $D^{(0,a]}$ is a $\boldsymbol{B}_N$-solution for f in D^a if and only if the following are satisfied:

(1) ϕ is in $D^a_{q,N}$.

(2) $\phi=f$ on $\partial_a=\{y=a\}$.

(3)

(4.18) $$\lim_{y\to 0}\int_0^{2N\pi} g(x)(U\phi_x-U(\phi)s)(x, y)dx=0$$

for any g in $C_{p,N}(R)$, where $U(\phi)$ satisfies

(4.19) $$U(\phi)_x=-m\phi_y+l\phi_x\,,\quad U(\phi)_y=m\phi_x+l\phi_y \quad \text{in } D^a\,.$$

[4.17] *Remark.* By (2) of [4.13] and [4.5], the function $U(\phi)$ in [4.16] is in $C_{p,N}(D^a)$. Since $U(\phi)$ is determined up to an additive constant by (4.19) and since

$$\lim_{y\downarrow 0}\int_0^{2N\pi}(U\phi_x-U(\phi)s)(x, y)dx=0$$

by (4.18), we see that $U(\phi)$ is uniquely determined by a $\boldsymbol{B}_N$-solution ϕ.

In definition [4.16], suppose that U, $U(\phi)$, ϕ, l, m and s can be extended to smooth functions on $D^{[0,a)}$ and that $s>0$ on $\partial_0=\{y=0\}$. Then by (4.18)

$$U\phi_x-U(\phi)s=0 \quad \text{on } \partial_0\,,$$

or

(4.20) $$U(\phi)=\frac{U}{s}\phi_x \quad \text{on } \partial_0\,.$$

Combining (4.20) with (4.19), we have

(4.21) $$\left(\frac{U}{s}\phi_x\right)_x+m\phi_y-l\phi_x=0 \quad \text{on } \partial_0\,,$$

which gives a boundary condition for ϕ. We can show (4.20) actually holds if s and m can be extended to smooth functions on $\bar{D}$ and if they are positive on ∂_0.

[4.18] Let u be the function defined by (4.16). Then $u(z)$ restricted on D^a is a $\boldsymbol{B}_N$-solution for $f=u(\cdot, a)$ in D^a. Moreover $U(u)=U(\boldsymbol{B})$ in D^a.

Proof. u is in D_q^a by [4.14]. Note that $U=U(\boldsymbol{B})$ is a solution of (4.19) for $\phi=u$ and (4.18) holds trivially for $U(u)=U$.

[4.19] DEFINITION OF $\boldsymbol{B}$-PROCESS. Let $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ be given. Then we say a process $\boldsymbol{P}$ in $\mathscr{P}$ is a $\boldsymbol{B}$-process, if and only if, for any N, $H_p^a f(z)$ is a $\boldsymbol{B}_N$-solution for f in D^a for any $a>0$ and any f in $C_{p,N}(R)$. In this case, we shall also say $\boldsymbol{P}$ is in class $\mathscr{P}_{\mathscr{B}}$.

§ 5. Explicit representation of the function U.

Let ρ and ν be periodic signed measures with period 2π and of bounded variation in each finite interval. Define

$$\begin{cases} \tilde{h}_\rho(z)=\int_{[0,2\pi)} \tilde{h}_\xi(z)\rho(d\xi)\,, \\ \tilde{k}_\rho(z)=\int_{[0,2\pi)} \tilde{k}_\xi(z)\rho(d\xi)\,, \\ [\rho]=\frac{1}{2\pi}\int_{[0,2\pi)} d\rho\,, \end{cases} \tag{5.1}$$

and $\tilde{h}_\nu$, $\tilde{k}_\nu$, $[\nu]$ similarly. Let a and b be any constants. In this section we shall obtain concrete representations of the solution $V=V(z\,;\rho,\nu,a,b)$ of the equation

$$\begin{cases} V_x=\tilde{h}_\nu(\tilde{k}_\rho+a)+(\tilde{k}_\nu+b)\tilde{h}_\rho\,, \\ V_y=\tilde{h}_\nu\tilde{h}_\rho-(\tilde{k}_\nu+b)(\tilde{k}_\rho+a) \end{cases} \tag{5.2}$$

in D. Especially, we shall obtain a representation of the function U defined in [4.12]. As a corollary we can reformulate the condition [P*] in [4.6] in a more concrete form.

1°. For x and ξ in R, set

$$F(x,\xi)=\begin{cases} n+\frac{1}{2} & \text{if } x=\xi+2n\pi\,, \\ n & \text{if } x\in(\xi+2(n-1)\pi,\ \xi+2n\pi)\,, \end{cases} \tag{5.3}$$

and for $z=(x,y)$ in D and ξ in R, set

$$G(z,\xi)=\int_0^{2\pi}\Big(F(\zeta,\xi)-\frac{\zeta}{2\pi}\Big)\tilde{h}_\zeta(z)d\zeta+\frac{x}{2\pi}\,. \tag{5.4}$$

[5.1] The function $G(z,\xi)$ is harmonic in D and belongs to $C_q(D)$ with respect to z, and belongs to $C_p(R)$ with respect to ξ. Moreover

$$\begin{cases} G_x(z,\xi)=\tilde{h}_\xi(z)\,, \\ G_y(z,\xi)=-\tilde{k}_\xi(z) \quad \text{for } z\in D\,, \end{cases} \tag{5.5}$$

and

$$\lim_{y\downarrow 0} G(z,\xi)=F(x,\xi)\,. \tag{5.6}$$

Proof. The first part is obvious. Since

$$G(z, \xi)=\int_{\xi}^{2\pi}\tilde{h}_{\zeta}(z)d\zeta-\frac{1}{2\pi}\int_{0}^{2\pi}\zeta\tilde{h}_{\zeta}(z)d\zeta+\frac{x}{2\pi},$$

we have $G_x=\tilde{h}_{\xi}$ by easy calculation. Therefore $G_y=-\tilde{k}_{\xi}(z)+c$. Since G is bounded in y for fixed x, we have $c=0$. Noting $\tilde{h}_{\zeta}(x, y)d\zeta\to\delta_x(d\zeta)$ weakly as $y\to 0$ and $\tilde{h}_{\zeta}(x, y)=\tilde{h}_{-\zeta}(-x, y)$. We see (5.6) from the definition of F.

For any periodic signed measure ρ with period 2π and of bounded variation in each finite interval, set

$$\left\{\begin{aligned} F(x, \rho)&=\int_{[0,2\pi)}F(x, \xi)\rho(d\xi) \quad x\in R, \\ G(z, \rho)&=\int_{[0,2\pi)}G(z, \xi)\rho(d\xi) \quad z\in D. \end{aligned}\right. \tag{5.7}$$

Then by [5.1], we can easily see:

[5.2] $G(z, \rho)$ is harmonic in D and belongs to $C_q(D)$. It belongs to $C_p(D)$ if and only if $[\rho]=0$. Moreover,

$$\left\{\begin{aligned} G_x(z, \rho)&=\tilde{h}_{\rho}(z), \\ G_y(z, \rho)&=-\tilde{k}_{\rho}(z) \end{aligned}\right. \tag{5.8}$$

and

$$\lim_{y\to 0} G(z, \rho)=F(x, \rho). \tag{5.9}$$

2°. We shall give a solution $W=W(z, \xi, \eta)$ of

$$\left\{\begin{aligned} W_x&=\tilde{h}_{\eta}\tilde{k}_{\xi}+\tilde{k}_{\eta}\tilde{h}_{\xi}, \\ W_y&=\tilde{h}_{\eta}\tilde{h}_{\xi}-\tilde{k}_{\eta}\tilde{k}_{\xi}. \end{aligned}\right. \tag{5.10}$$

Since

$$\tilde{h}_{\xi}+i\tilde{k}_{\xi}=\frac{i}{2\pi}\cot\frac{z-\xi}{2},$$

$$\tilde{h}_{\eta}+i\tilde{k}_{\eta}=\frac{i}{2\pi}\cot\frac{z-\eta}{2}$$

and

$$W_x-iW_y=\frac{i}{4\pi^2}\cot\frac{z-\xi}{2}\cot\frac{z-\eta}{2},$$

we have

$$W=-\frac{1}{4\pi^2}\operatorname{Im}\int\cot\left(\frac{z-\xi}{2}\right)\cot\left(\frac{z-\eta}{2}\right)dz. \tag{5.11}$$

a) The case $\xi\equiv\eta \pmod{2\pi}$. Noting

$$\cot^2\left(\frac{z-\xi}{2}\right)=\frac{d}{dz}\left(-2\cot\left(\frac{z-\xi}{2}\right)-z\right),$$

we set

(5.12) $$W(z,\xi,\xi)=-\frac{1}{\pi}\tilde{h}_{\xi}(z)+\frac{y}{4\pi^2},$$

which is a solution of (5.10) for $\xi\equiv\eta(2\pi)$.

b) The case $\xi\not\equiv\eta \pmod{2\pi}$. Since

$$\cot\left(\frac{z-\xi}{2}\right)\cot\left(\frac{z-\eta}{2}\right)=\frac{d}{dz}\left\{2\cot\left(\frac{\xi-\eta}{2}\right)\left(\log\sin\frac{z-\xi}{2}-\log\sin\frac{z-\eta}{2}-z\right\}\right.,$$

(5.13) $$W=\frac{-1}{2\pi^2}\cot\frac{\xi-\eta}{2}\left(\operatorname{Arg}\sin\frac{z-\xi}{2}-\operatorname{Arg}\sin\frac{z-\eta}{2}\right)+\frac{y}{4\pi^2}+C$$

is a solution of (5.10). However, for further analysis, it is convenient to select a suitable constant $C=C(\xi,\eta)$.

We shall define a function $S=S(z,\xi,\eta)$ for $\xi\not\equiv\eta(2\pi)$ in the following way.

(i) If $0<\cot((\xi-\eta)/2)<\infty$, that is, if there exists a ξ_0 such that $\xi_0\equiv\xi(2\pi)$ and $\eta<\xi_0<\eta+\pi$, then set

(5.14) $$S(z,\xi,\eta)=\frac{1}{\pi}\left(\operatorname{Arg}\sin\frac{z-\xi_0}{2}-\operatorname{Arg}\sin\frac{z-\eta}{2}\right).$$

where the branch of the right side in (5.14) is given as Figure(†).

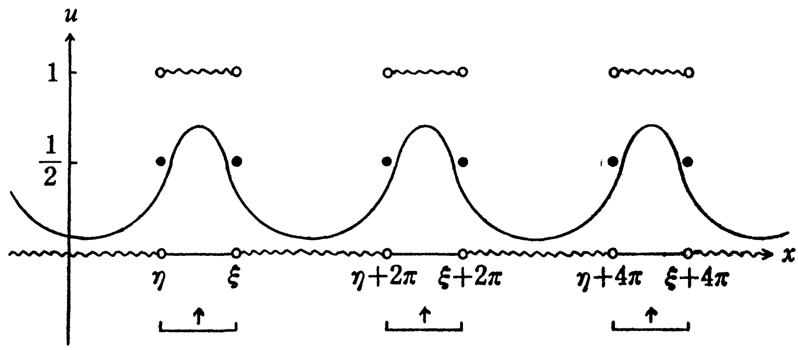

Figure.

A branch $\left(\operatorname{Arg}\sin i=\frac{\pi}{2}\right)$ of $u=S(z,\xi,\eta)=\frac{1}{\pi}\left(\operatorname{Arg}\sin\frac{z-\xi}{2}-\operatorname{Arg}\sin\frac{z-\eta}{2}\right)$ when $\eta<\xi<\eta+\pi$, where $z=x+iy$ and $y>0$ is fixed. The broken line is the limit curve when $y\downarrow 0$. Arrows represent domains of increase of S as y decreases.

(†) In the left side, the branch with $\operatorname{Arg}\sin i=\frac{\pi}{2}$ have to be adopted.

(ii) If $0>\cot((\xi-\eta)/2)>-\infty$, set

$$S(z, \xi, \eta)=S(z, \eta, \xi). \tag{5.15}$$

(iii) If $\cot((\xi-\eta)/2)=0$ or $\eta\equiv\xi+\pi(2\pi)$, then set

$$S(z, \xi, \eta)=0. \tag{5.16}$$

We can easily see:

[5.3] Let $\xi\not\equiv\eta \pmod{2\pi}$.

(1) $S(z, \xi, \eta)$ is harmonic in $z\in D$, and $0\leqq S\leqq 1$.
(2) $S(z, \xi, \eta)$ is periodic with period 2π in x, ξ and η, and $S(z, \xi, \eta)=S(z, \eta, \xi)$.
(3) There exists

$$S_0(x, \xi, \eta)=\lim_{y\to 0} S(z, \xi, \eta) \tag{5.17}$$

for all x, ξ and η ($z=x+iy$, $\xi\not\equiv\eta \pmod{2\pi}$).
(4) $S(z, \xi, \eta)$ increases to $S_0(x, \xi, \eta)$ as y decreases to zero on the set $S_+=\{(z, \xi, \eta): S_0(x, \xi, \eta)>0\}$.
More precisely, S_0 is given by

$$S_0(x, \xi, \eta)=\begin{cases}\text{undefined} & \text{if } \xi\equiv\eta(2\pi),\\ 1 & \text{if } (x, \xi, \eta)\in\bigcup_{n,m} A_{n,m},\\ \frac{1}{2} & \text{if } (x, \xi, \eta)\in\bigcup_{n,m} B_{n,m},\\ 0 & \text{otherwise,}\end{cases}$$

where

$$A_{n,m}=\{(x, \xi, \eta): |\xi-\eta-2(m-n)\pi|<\pi,\ (\xi-x-2m\pi)(\eta-x-2n\pi)>0\}$$

$$B_{n,m}=\{(x, \xi, \eta): \xi=x+2n\pi,\ \eta=x+2m\pi\}$$

and n and m run over all integers.

For $\xi\neq\eta(2\pi)$, define

$$T(z, \xi, \eta)=\frac{1}{2\pi}\left|\cot\frac{\xi-\eta}{2}\right|S(z, \xi, \eta), \tag{5.18}$$

$$W(z, \xi, \eta)=-T(z, \xi, \eta)+\frac{y}{4\pi^2} \tag{5.19}$$

for $z\in D$ and $x, y\in R$. Then, by (5.13) and definition of S, we have:

[5.4] For $\xi\not\equiv\eta \pmod{2\pi}$, $W(z, \xi, \eta)$ is a solution of (5.10).

We also note:

[5.5] Let $\xi\not\equiv\eta\,(\mathrm{mod}\,2\pi)$.

(1) $T(z, \xi, \eta)$ is positive harmonic in $z\in D$.

(2) $T(z, \xi, \eta)$ is periodic with period 2π in x, ξ and η, and $T(z, \xi, \eta)= T(z, \eta, \xi)$.

(3) $0\leqq T(z, \xi, \eta)\leqq\dfrac{1}{2\pi}\left|\cot\dfrac{\xi-\eta}{2}\right|$.

(4) $0\leqq T(z, \xi, \eta)\leqq\dfrac{1}{4}\dfrac{\sin y}{\cosh y-1}$ for $y>0$.

(5) $T_0(x, \xi, \eta)=\lim\limits_{y\to 0} T(z, \xi, \eta)$ exists for all x, ξ and η, and

$$T_0(x, \xi, \eta)=\frac{1}{2\pi}\left|\cot\frac{\xi-\eta}{2}\right|S_0(x, \xi, \eta).$$

Proof. (1), (2), (3) and (5) are easily shown by using [5.3]. For $y>0$ and $|\xi-\eta|<\pi$ we have (4), since

$$\left|\cot\frac{\xi-\eta}{2}\right|\leqq\frac{1}{|\sin(\xi-\eta/2)|}\leqq\frac{\pi}{|\xi-\eta|}\quad\text{and}$$

$$\frac{1}{|\xi-\eta|}\left|\operatorname{Arg}\sin\frac{z-\xi}{2}-\operatorname{Arg}\sin\frac{z-\eta}{2}\right|\leqq\sup_x\left|\frac{d}{dx}\operatorname{Arg}\sin\frac{x+iy}{2}\right|$$

$$\leqq\frac{1}{2}\sup_x\left|\operatorname{Im}\cot\frac{x+iy}{2}\right|\leqq\frac{1}{2}\frac{\sinh y}{\cosh y-1}.$$

Noting T is periodic in ξ and η, we can show (4) in general.

[5.6]

(1) $S(z, \xi, \eta)=\displaystyle\int_0^{2\pi}\tilde{h}_\zeta(z)S_0(\zeta, \xi, \eta)d\zeta$.

(2) $T(z, \xi, \eta)=\displaystyle\int_0^{2\pi}\tilde{h}_\zeta(z)T_0(\zeta, \xi, \eta)d\zeta$.

Proof. Since S_0 is a boundary function of the bounded harmonic function S by (5.17), we have (1). By definition of T and T_0, (2) is a consequence of (1).

In conclusion, we have defined $W=W(z, \xi, \eta)$ by

$$W(z, \xi, \eta)=\begin{cases}-\dfrac{1}{\pi}\tilde{h}_\xi(z)+\dfrac{y}{4\pi^2} & \text{if } \xi\equiv\eta(2\pi),\\[2ex] -T(z, \xi, \eta)+\dfrac{y}{4\pi^2} & \text{if } \xi\not\equiv\eta(2\pi),\end{cases}\tag{5.20}$$

which is a solution of (5.10).

3° Let ρ and ν be periodic signed measures with period 2π and of bounded variation in each finite interval. Set $\tilde{h}_\rho$, $\tilde{k}_\rho$, $\tilde{h}_\nu$, $\tilde{k}_\nu$, $[\rho]$ and $[\nu]$ as in (5.1). Integrating both sides of (5.18) by ρ and ν, we have:

[5.7] Set

$$W(z, \rho, \nu)\equiv\iint_{[0,2\pi)\times[0,2\pi)} W(z, \xi, \eta)\rho(d\xi)\nu(d\eta) \tag{5.21}$$

$$=-\iint_{\xi\neq\eta;\ \xi,\eta\in[0,2\pi)} T(z, \xi, \eta)\rho(d\xi)\nu(d\eta)$$

$$-\int_{[0,2\pi)} \tilde{h}_\xi(z)d(\rho\cdot\nu)(\xi)+[\rho][\nu]y\,,$$

where

$$(\rho\cdot\nu)(A)=\iint_{A\times A} I_{\{\xi=\eta\}}\rho(d\xi)\nu(d\eta) \tag{5.22}$$

$$=\sum_{\xi_i\in A}\rho(\{\xi_i\})\nu(\{\xi_i\}) \quad \text{for } A\in \boldsymbol{B}(R)\,.$$

Then $W(z, \rho, \nu)$ is a solution of the equation

$$\begin{cases} W_x=\tilde{h}_\nu\tilde{k}_\rho+\tilde{k}_\nu\tilde{h}_\rho\,, \\ W_y=\tilde{h}_\nu\tilde{h}_\rho-\tilde{k}_\nu\tilde{k}_\rho \quad \text{in } D\,. \end{cases} \tag{5.23}$$

Let a and b be any constants. Since

$$\tilde{h}_\nu(\tilde{k}_\rho+a)+(\tilde{k}_\nu+b)\tilde{h}_\rho=\tilde{h}_\nu\tilde{k}_\rho+\tilde{k}_\nu\tilde{h}_\rho+a\tilde{h}_\nu+b\tilde{h}_\rho\,,$$

$$\tilde{h}_\nu\tilde{h}_k-(\tilde{k}_\nu+b)(\tilde{k}_\rho+a)=\tilde{h}_\nu\tilde{h}_\rho-\tilde{k}_\nu\tilde{k}_\rho-(a\tilde{k}_\nu+b\tilde{k}_\rho)-ab\,.$$

We have, combining [5.2] and [5.7]:

[5.8]

$$V(z, \rho, \nu, a, b)\equiv W(z, \rho, \nu)+G(z, a\nu+b\rho)-aby \tag{5.24}$$

is a solution of the equation (5.2).

4° Now, let σ and μ be in $M_p(R)$ and $[\sigma]=[\mu]=1$, and k be any constant. Set

$$s=\tilde{h}_\sigma,\quad m=\tilde{h}_\mu,\quad t=\tilde{k}_\sigma+k,\quad l=\tilde{k}_\mu-k\,,$$

as in [4.12]. Then, by [5.7] and [5.8], we immediately have:

[5.9] PROPOSITION.

$$(5.25)\qquad \bar{U}\equiv\bar{U}(z,\sigma,\mu,k)=-\iint_{\xi\neq\eta;\ \xi,\eta\in[0,2\pi)}T(z,\xi,\eta)\sigma(d\xi)\mu(d\eta)$$

$$-\int_{[0,2\pi)}\tilde{h}_{\xi}(z)(\sigma\cdot\mu)(d\xi)+kG(z,\mu-\sigma)+(1+k^2)y$$

is a solution of the equation,

$$(5.26)\qquad \begin{cases}\tilde{U}_x=mt+ls,\\ \tilde{U}_y=ms-lt \quad \text{in } D.\end{cases}$$

[5.10] *Remark.* Any solution of (5.26) is given by $\bar{U}$+constant.

[5.11] THEOREM. *Let σ and μ be in $M_p(R)$ such that $[\sigma]=[\mu]=1$, and k be any constant. Then $\{\sigma,\mu,k\}$ satisfies the condition [P^*] if and only if the following condition is fulfilled:*

Condition [P]
the measures σ and μ have no common having positive mass and

$$\tilde{T}_0(x,\sigma,\mu)=\iint_{(\xi,\eta)\in A(x)}\left|\cot\frac{\xi-\eta}{2}\right|\sigma(d\xi)\mu(d\eta)$$

is bounded in x, where

$$A(x)=\{(\xi,\eta):\xi\neq\eta,\ |\xi-\eta|<\pi,\ (\xi-x)(\eta-x)<0\}.$$

Proof. By definition of S_0, we can easily see that condition [P] is equivalent to boundedness of the function

$$(5.27)\qquad T_0(x;\sigma,\mu)=\frac{1}{2\pi}\iint_{\xi\neq\eta;\ \xi,\eta\in[0,2\pi)}\left|\cot\frac{\xi-\eta}{2}\right|S_0(x,\xi,\eta)\sigma(d\xi)\mu(d\eta)$$

in x. On the other hand, by (5.25), a solution $\tilde{U}$ of (5.26) is bounded below in D if and only if

$$T(z,\sigma,\mu)=\iint T(z,\xi,\eta)\sigma(d\xi)\mu(d\eta)$$

is bounded and

$$\tilde{h}_{(\sigma\cdot\mu)}=\int\tilde{h}_{\xi}(z)(\sigma\cdot\mu)(d\xi)$$

is bounded.

Since $(\sigma\cdot\mu)$ is a discrete measure, $\tilde{h}_{(\sigma\cdot\mu)}(z)$ is bounded if and only if $(\sigma\cdot\mu)=0$ or σ and μ have no common point having positive mass. By (2) in [5.6] and Fubini's theorem

$$T(z, \sigma, \mu)=\int \tilde{h}_\zeta(z)T_0(\zeta, \sigma, \mu)d\zeta$$

and $T(z, \sigma, \mu)$ is bounded in z if $T_0(x, \sigma, \mu)$ is bounded in x. On the other hand, by (5) in [5.5]

$$T_0(x, \sigma, \mu)=\iint \lim_{y\to 0} T(z, \xi, \eta)\sigma(d\xi)\mu(d\eta)$$

$$\leqq \lim_{y\to 0} T(z, \sigma, \mu),$$

and $T_0(x, \sigma, \mu)$ is bounded in x if $T(z, \sigma, \mu)$ is bounded in z. [5.11] is proved.

[5.12] COROLLARY. *Let* $B=\{\sigma, \mu, k, p\}$ *in* $\mathcal{B}$. *Then* $U=U(B)$ *defined in* [4.11] *can be given by*

$$U(z)=-\iint_{\xi\neq\eta;\ \xi,\eta\in[0,\pi)} T(z, \xi, \eta)\sigma(d\xi)\mu(d\eta)+kG(z, \mu-\sigma)+(l+k^2)y+C \tag{5.28}$$

where C *is a constant determined by* (4.15). *Especially,* $U(z)$ *is bounded in* D^a *for any* $a>0$.

[5.13] DEFINITION. For $B=\{\sigma, \mu, k, p\}$, we set

$$U_0(x)\equiv-\iint_{\xi\neq\eta;\ \xi,\eta\in[0,2\pi)} T_0(x, \xi, \eta)\sigma(d\xi)\mu(d\eta)+kF(x, \mu-\sigma)+C \tag{5.29}$$

where C is the constant given in (5.28) and F is defined in (5.7). This function U_0 is a version of boundary function of U in the following sense.

[5.14] U_0 is bounded and in $B_p(R)$. Moreover

$$U_B(z)=\int_{[0,2\pi)} U_0(\xi)\tilde{h}_\xi(z)d\xi, \tag{5.30}$$

where $U_B=U(B)$ is defined in [4.12].

5°. Let $B=\{\sigma, \mu, k, p\}$ be in $\mathcal{B}$. For ϕ in $D_q^a(B)$, we shall investigate a representation of a solution $\tilde{U}(\phi)$ of

$$\begin{cases} U(\phi)_x=-m\phi_y+l\phi_x, \\ U(\phi)_y=m\phi_x+l\phi_y. \end{cases} \tag{5.31}$$

First, we shall prepare two simple lemmas.

[5.15] Let ψ in $C_q(D^a)$ be harmonic in D^a. If $\psi_x(x, y)dx\to 0$ as $y\to 0$ in weak sense as measures on torus $[0, 2\pi)$, then, ψ is in $C_p(D^a)$ and there exists a constant c and a periodic function ψ^0 in $C^\infty(R)$ such that

$$\psi(z)-c=\mathcal{O}(y),$$

$$\phi_x(z)=\mathcal{O}(y),$$

$$\phi_y(z)-\phi^0(x)=\mathcal{O}(y) \quad (y\to 0).$$

Proof. Note that ϕ_x and ϕ_y are in $C_p(D^a)$, for ϕ is in $C_q(D^a)$. Since $\int_0^{2\pi}\phi_x(x, y)dx$ is independent of y by [4.1], we can immediately see ϕ is in $C_p(D^a)$ by assumption. For any $b<a$, $z\in D^b$ and $\varepsilon<y$,

$$\phi_x(z)=\int_0^{2\pi}\phi_x(\xi, b)^{b-\varepsilon}\tilde{\pi}^{b-y}(\xi-x)d\xi+\int_0^{2\pi}\phi_x(\xi, \varepsilon)^{b-\varepsilon}\tilde{\pi}^{y-\varepsilon}(\xi-x)d\xi$$

where ${}^r\tilde{\pi}^s(x)=\sum_{n=-\infty}^{\infty}{}^r\pi^s(x+2n\pi)$ and ${}^r\pi^s$ is given in § 0, 8°. Letting ε tend to 0, we have by assumption

$$\phi_x(z)=\int_0^{2\pi}\phi_x(\xi, b)^b\tilde{\pi}^{b-y}(\xi-x)d\xi,$$

which can be extended to a periodic harmonic function $\tilde{\phi}_x$ in $R\times(-b, b)$ and $\tilde{\phi}_x(x, 0)\equiv 0$. Therefore, ϕ can be also extended to a periodic harmonic function $\tilde{\phi}$ in $R\times(-b, b)$ with $\tilde{\phi}(x, 0)=c$. Setting $\phi^0(x)\equiv\tilde{\phi}_y(x, 0)$ we can get [5.15].

[5.16] Set $m=\tilde{h}_\mu(z)=\int_{[0,2\pi)}\tilde{h}_\xi(z)\mu(d\xi)$ and $l=\int_{[0,2\pi)}\tilde{k}_\xi(z)\mu(d\xi)-k$ for μ in $M_p(R)$ with $[\mu]=1$, and $\tilde{h}_{\phi\cdot\mu}(z)=\int_{[0,2\pi)}\tilde{h}_\xi(z)\phi(\xi)\mu(d\xi)$ for a function ϕ in $C^\infty(R)\cap C_p(R)$. Then

(1) $\int_0^{2\pi}ym(x, y)dx=o(1)$,

(2) $\int_0^{2\pi}y\,|l(x, y)|\,dx=o(1)$,

(3) $\int_0^{2\pi}|\phi(x)m(x, y)-\tilde{h}_{\phi\cdot\mu}(x, y)|\,dx=o(1) \quad (y\to 0)$.

Proof. (1) is obvious. Since

$$\int y\,|l(x, y)|\,dx\leqq\frac{1}{2\pi}\iint_{\xi,\, x\in[0,2\pi)}\frac{y\,|\sin(\xi-x)|}{\cosh y-\cos(\xi-x)}\mu(d\xi)dx+2\pi\,|k|\,y$$

and $\dfrac{y\sin(\xi-x)}{\cosh y-\cos(\xi-x)}$ tends to 0 boundedly as $y\to 0$, (2) is proved. Since

$$\int_0^{2\pi}|\phi(x)m(x, y)-\tilde{h}_{\phi\cdot\mu}(x, y)|\,dx$$

$$\leqq\int_0^{2\pi}|\phi(x)-\phi(\xi)|\frac{|\sinh y|}{\cosh y-\cos(\xi-x)}\mu(d\xi)dx \longrightarrow 0 \quad (y\to 0),$$

(3) is obtained.

Let $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ be given, and ϕ be in $D_q^a(\boldsymbol{B})$. Let σ_ϕ be the periodic signed measure defined in [4.15]. Set $\phi^*(z)=G(z, \sigma_\phi)$, where $G(z, \sigma_\phi)$ is the function defined by (5.7), and set $\tilde{\phi}=\phi-\phi^*$. Then, by [5.2],

$$\tilde{\phi}_x=\phi_x-\tilde{h}_{\sigma_\phi}(z),$$

$$\tilde{\phi}_y=\phi_y+\tilde{k}_{\sigma_\phi}(z).$$

Noting

$$\int_0^{2\pi} g(x)\tilde{h}_{\sigma_\phi}(x, y)dx \longrightarrow \int_{[0,2\pi)} g(\xi)\sigma_\phi(d\xi)$$

for any g in $C_p(R)$, we see $\tilde{\phi}_x dx \to 0$ $(y\to 0)$ in weak sense. Set

$$W(z)=W(z, \sigma_\phi, \mu)=\iint W(z, \xi, \eta)\sigma_\phi(d\xi)\mu(d\eta),$$

where $W(z, \xi, \eta)$ is given by (5.20). Then W is a solution of

$$W_x=-m\phi_y^*+l_0\phi_x^*,$$

$$W_y=m\phi_x^*+l_0\phi_y^*$$

and

$$l_0=\tilde{k}_\mu(z)=l+k.$$

Let $\tilde{U}(\phi)$ be any solution of (5.31). Then

$$(\tilde{U}(\phi)-W)_x=-m\phi_y+l\tilde{\phi}_x-k\phi_x^* \quad \text{in } D^a.$$

Therefore, since $\tilde{\phi}_x dx\to 0$ $(y\to 0)$ in weak sense, it follows from [5.15] and [5.16] that there exists a function $\tilde{\phi}^0$ in $C_p(R)$, and we have

$$\{\tilde{U}(\phi)-W+G(z, k\sigma_\phi+\tilde{\phi}^0\cdot\mu)\}_x dx$$

$$=-(\tilde{\phi}_y-\tilde{\phi}^0)m dx-(\tilde{\phi}^0 m-\tilde{h}_{\tilde{\phi}^0\cdot\mu})dx+l\tilde{\phi}_x dx \longrightarrow 0 \quad (y\to 0)$$

in weak sense, where $\tilde{\phi}^0\cdot\mu$ is the measure defined by $\tilde{\phi}^0\cdot\mu(d\xi)=\tilde{\phi}^0(\xi)\mu(d\xi)$. Applying [5.15] again, we have

$$\tilde{U}(\phi)-W+G(z, k\sigma_\phi+\tilde{\phi}^0\cdot\mu)=\text{const}+\mathcal{O}(y) \quad (y\to 0),$$

and $\tilde{U}(\phi)-W+G(z, k\sigma_\phi+\tilde{\phi}_0\cdot\mu)$ is in $C_p(D^a)$. By (2) in [4.13] $\tilde{U}(\phi)$ is in $C_p(D^a)$ and W is also in $C_p(D)$ by definition. Therefore $G(z, k\sigma_\phi+\tilde{\phi}^0\cdot\mu)$ is in $C_p(D^a)$ or

$$\int_{[0,2\pi)}(k d\sigma_\phi+\tilde{\phi}^0 d\mu)=0.$$

By [4.15] $d(\sigma_\phi\cdot\mu)=0$ follows from $d(\sigma\cdot\mu)=0$; the latter is a consequence of the condition [P]. Using the representation of W given in [5.11], we finally have:

[5.17] THEOREM. *Let $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ be given. Then, for any ϕ in $D_q^a=D_q^a(\boldsymbol{B})$, a solution $\tilde{U}(\phi)$ of* (5.31) *can be expressed by*

$$\tilde{U}(\phi)(z)=-\iint_{\xi\neq\eta;\ \xi,\ \eta\in[0,2\pi)} T(z\,;\xi,\eta)\sigma_\phi(d\xi)\mu(d\eta) \tag{5.32}$$

$$-G(z,\ k\sigma_\phi+\tilde{\phi}^0\cdot\mu)+C+\mathcal{O}(y)$$

for small y. Here T and G are given by (5.18) *and* (5.7), *respectively, $\tilde{\phi}^0(x)$ is the boundary function of $\phi_y+\tilde{k}_{\sigma_\phi}$ on ∂_0 which is in $C^\infty(R)\cap C_p(R)$, $\int_{[0,2\pi)}(k\cdot d\sigma_\phi+\tilde{\phi}^0\cdot d\mu)=0$, and C is a constant.*

Since $\int_{[0,2\pi)}(kd\sigma_\phi+\tilde{\phi}^0\cdot d\mu)=0$, the function $G(z,\ k\sigma_\phi+\tilde{\phi}^0\cdot\mu)$ is bounded in D. And by [4.15]

$$\left|\iint T(z,\xi,\eta)\sigma_\phi(d\xi)\mu(d\eta)\right|\leqq K\iint T(z,\xi,\eta)\sigma(d\xi)\mu(d\eta)\,.$$

Therefore, as a consequence of condition [P], we have:

[5.18] The solution $\tilde{U}(\phi)$ of (5.31) is bounded in D^b for each $b<a$.

[5.19] DEFINITION. Let $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ be given. For ϕ in $D_q^a(\boldsymbol{B})$ and a solution $\tilde{U}(\phi)$ of (5.31) given by (5.32), we define

$$\tilde{U}_0(\phi)(x)=-\iint_{\xi\neq\eta;\ \xi,\ \eta\in[0,2\pi)} T_0(x,\xi,\eta)\sigma_\phi(d\xi)\mu(d\eta)-F(x,\ k\sigma_\phi+\tilde{\phi}^0\cdot\mu)+, \tag{5.33}$$

where T_0 is defined by (5) of [5.5] and F by (5.7), and $\tilde{\phi}^0$ and C are the same as given in (5.32).

By Fubini's theorem, we can easily have:

[5.20] The function $\tilde{U}_0(\phi)$ defined in (5.33) is a version of boundary function on ∂_0 of the function $\tilde{U}(\phi)$ given by (5.32), that is,

$$\tilde{U}(\phi)(z)=\int_0^{2\pi}\tilde{h}_\xi(z)\tilde{U}_0(\phi)(\xi)d\xi+\mathcal{O}(y)\,.$$

[5.21] *Remark.* $U=U(\boldsymbol{B})$ defined in [4.12] is a solution of (5.31) for $\phi=u$ by [4.14]. We can easily see U_0 defined in [5.13] is a version of boundary function of $U=U(u)$ in the sense stated in [5.20]. Note that $u_y=-t=-\tilde{k}_\sigma-k$ and $\phi_0=-k$ in this case.

§6. Proof of the fundamental lemma.

The purpose of this section is to prove the following lemma, which is essential in the proof of uniqueness of $\boldsymbol{B}_N$-solutions.

[6.1] LEMMA. *Let* $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ *in* $\mathscr{B}$ *and* ϕ *in* $D_q^\alpha(\boldsymbol{B})$ *be given, and* $\tilde{U}(\phi)$ *be any solution of* (5.31) *in* §5, 5°. *Then,*

$$\lim_{y\to 0}\int_0^{2\pi} g(x)\tilde{U}(\phi)(x, y)\psi(x, y)dx=\int_{[0,2\pi)} g(x)\tilde{U}_0(\phi)(x)\sigma_\psi(dx) \tag{6.1}$$

for any g *in* $C_p(R)$ *and any* ψ *in* $D_q^\alpha(\boldsymbol{B})$. *Here* $\tilde{U}_0(\phi)$ *is defined in* [5.19] *and* σ_ψ *in* [4.15].

Let ρ be in $M_p(R)$ η in $[0, 2\pi)$. Set

$$\begin{cases} T_a(\rho, \rho, \eta)=\iint_{[0,2\pi)\times[0,2\pi)} T((\zeta, a), \xi, \eta)\rho(d\zeta)\rho(d\xi), \\ T_0(\rho, \rho, \eta)=\iint_{[0,2\pi)\times[0,2\pi)} T_0(\zeta, \xi, \eta)\rho(d\zeta)\rho(d\xi), \\ Q_a(\rho, \rho, \eta)=\iint_{[0,2\pi)\times[0,2\pi)} \tilde{h}_p(x, a)W((x, a), \xi, \eta)dx\rho(d\xi), \end{cases} \tag{6.2}$$

where T is defined in (5.18), T_0 in (5) [5.5] and W in (5.20).

[6.2]
(1) $Q_a(\rho, \rho, \eta)$ decreases as a decreases.
(2) $Q_a(\rho, \rho, \eta)=-T_{2a}(\rho, \rho, \eta)+a[\rho]^2$, if $\rho(\{\eta\})=0$.
(3) $\lim_{a\to 0} Q_a(\rho, \rho, \eta)=-T_0(\rho, \rho, \eta)$, if $\rho(\{\eta\})=0$.

Proof. $W(z, \rho, \eta)=\int_{[0,2\pi)} W(z, \xi, \eta)\rho(d\xi)$ is a solution of

$$W_x=\tilde{h}_\eta\tilde{k}_\rho+\tilde{k}_\eta\tilde{h}_\rho,$$
$$W_y=\tilde{h}_\eta\tilde{h}_\rho-\tilde{k}_\eta\tilde{k}_\rho.$$

Therefore by (2) [4.5],

$$Q_a(\rho, \rho, \eta)-Q_b(\rho, \rho, \eta)=\int_b^a dy\int_{[0,2\pi)} \tilde{h}_\eta(\tilde{h}_\rho^2+\tilde{k}_\rho^2)dx\geqq 0,$$

for $b<a$ and (1) is proved. Assume $\rho(\{\eta\})=0$. By (2) [5.6]

$$\iint_{[0,2\pi)\times[0,2\pi)} \tilde{h}_p(x, a)T((x, a), \xi, \eta)dx\rho(d\xi)$$
$$=\iint_{[0,2\pi)\times[0,2\pi)\times[0,2\pi)} \rho(d\zeta)\rho(d\xi)d\zeta' T_0(\zeta', \xi, \eta)\int_0^{2\pi}\tilde{h}_\zeta(x, a)\tilde{h}_{\zeta'}(x, a)dx$$
$$=\iint_{[0,2\pi)\times[0,2\pi)\times[0,2\pi)} \rho(d\zeta)\rho(d\xi)d\zeta'\tilde{h}_{\zeta'}(\zeta, 2a)T_0(\zeta', \xi, \eta)$$
$$=T_{2a}(\rho, \rho, \eta).$$

Since, by (5.20)

$$W((x, a), \xi, \eta)=-T((x, a), \xi, \eta)+\frac{a}{4\pi^2}$$

for $\xi\neq\eta$, we have proved (2).

Since $T_a\geqq 0$ and $\lim_{y\to 0} T(z, \xi, \eta)=T_0(x, \xi, \eta)$ by (5) [5.5], it follows from Fatou's lemma that

$$\varliminf_{a\to 0} T_a(\rho, \rho, \eta)\geqq T_0(\rho, \rho, \eta),$$

and by (2) we have

(6.3) $$\varlimsup_{a\to 0} Q_a(\rho, \rho, \eta)\leqq -T_0(\rho, \rho, \eta).$$

Set $A_n=\{\xi: |\cot(\xi-\eta/2)|\leqq n\}\subset R$ and define $d\rho_n=I_{A_n}\cdot d\rho$. Then ρ_n is in $M_p(R)$ and $0\leqq T(z, \xi, \eta)\leqq n/2\pi$ for $\xi\in A_n$ by (3) [5.5]. Therefore $\lim_{a\to 0} T_a(\rho_n, \rho_n, \eta)= T_0(\rho_n, \rho_n, \eta)$ by the bounded convergence theorem. Therefore, by (1) and (2),

$$Q_a(\rho_n, \rho_n, \eta)\geqq \lim_{b\to 0} Q_b(\rho_n, \rho_n, \eta)=-T_0(\rho_n, \rho_n, \eta).$$

Since $\rho(\{\eta\})=0$ and $I_{A_n}\uparrow I_{\{\xi\neq\eta(2\pi)\}}$, the monotone convergence theorem establishes

$$\lim_{n\to\infty} T_0(\rho_n, \rho_n, \eta)=T_0(\rho, \rho, \eta),$$

and

$$\lim_{n\to\infty} Q_a(\rho_n, \rho_n, \eta)=Q_a(\rho, \rho, \eta)$$

by (2). Therefore

(6.4) $$Q_a(\rho, \rho, \eta)\geqq -T_0(\rho, \rho, \eta).$$

Combining (6.4) with (6.3), we finish the proof of (3).

Let $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ be given, and set

(6.5) $$\begin{cases} T_a(\sigma, \sigma, \mu)\equiv\iiint_{[0,2\pi)\times[0,2\pi)\times[0,2\pi)}\sigma(d\zeta)\sigma(d\xi)\mu(d\eta)T((\zeta, a), \xi, \eta). \\ T_0(\sigma, \sigma, \mu)=\iiint_{[0,2\pi)\times[0,2\pi)\times[0,2\pi)}\sigma(d\zeta)\sigma(d\xi)\mu(d\eta)T_0(\zeta, \xi, \eta). \end{cases}$$

[6.3] $\lim_{a\to 0} T_a(\sigma, \sigma, \mu)=T_0(\sigma, \sigma, \mu)$.

Proof. By condition [P] in [5.11],

(6.6) $$\mu\{\eta: \sigma(\{\eta\})>0\}=0,$$

and by (2) [6.2]

$$T_{2a}(\sigma, \sigma, \mu)=-\int_{[0,2\pi)} Q_a(\sigma, \sigma, \eta)\mu(d\eta)+2\pi a.$$

By (2) [6.2], $Q_a(\sigma, \sigma, \eta)$ is bounded above in a for $0<a\leqq a_0<\infty$, and by (6.6) and (1) and (3) in [6.2], $Q_a(\sigma, \sigma, \eta)$ converges to $-T_0(\sigma, \sigma, \eta)$ decreasingly as $a\to 0$. Therefore [6.3] is proved by the monotone convergence theorem.

[6.4] Let σ_1 and σ_2 be periodic signed measures with period 2π. Assume that σ_j $(j=1, 2)$ are absolutely continuous with respect to σ and $|d\sigma_j/d\sigma|\leqq K$. Then,

$$\lim_{a\to 0} T_a(\sigma_1, \sigma_2, \mu)=T_0(\sigma_1, \sigma_2, \mu),$$

where $T_a(\sigma_1, \sigma_2, \mu)$ and $T_0(\sigma_1, \sigma_2, \mu)$ are defined in the way similary to (6.5).

Proof. By (4) [5.3], $S((\zeta, a), \xi, \eta)$ increases to $S_0(\zeta, \xi, \eta)$ on the set $\tilde{S}_+=\{(\zeta, \xi, \eta): S_0(\zeta, \xi, \eta)>0,\ 0\leqq\zeta, \xi, \eta<2\pi\}$ as $a\to 0$. Noting (5.18), we have by the monotone convergence theorem

$$\text{(6.7)}\qquad \begin{aligned}&\lim_{a\to 0}\iiint_{\tilde{S}_+} T((\zeta, a), \xi, \eta)\sigma(d\zeta)\sigma(d\xi)\mu(d\eta)\\ &=\iiint_{\tilde{S}_+} T_0(\zeta, \xi, \eta)\sigma(d\zeta)\sigma(d\xi)\mu(d\eta)\\ &=T_0(\sigma, \sigma, \mu).\end{aligned}$$

Combining (6.7) with [6.3], we have

$$\text{(6.8)}\qquad \lim_{a\to 0}\iiint_{\tilde{S}_+^c} T((\zeta, a), \xi, \eta)\sigma(d\zeta)\sigma(d\xi)\mu(d\eta)=0,$$

where $\tilde{S}_+^c=\{0\leqq\xi, \eta, \zeta<2\pi\}-\tilde{S}_+$. Therefore,

$$\begin{aligned}&|T_a(\sigma_1, \sigma_2, \mu)-T_0(\sigma_1, \sigma_2, \mu)|\\ &\leqq\iiint_{\tilde{S}_+}\{T_0(\zeta, \xi, \eta)-T((\zeta, a), \xi, \eta)\}K^2\sigma(d\zeta)\sigma(d\xi)\mu(d\eta)\\ &\quad+\iiint_{\tilde{S}_+^c} T((\zeta, a), \xi, \eta)K^2\sigma(d\zeta)\sigma(d\xi)\mu(d\eta).\end{aligned}$$

By (6.7) and (6.8), the right hand side tends to zero as a tends to zero. This proves [6.4].

Proof of lemma [6.1]. Set

$$\text{(6.9)}\qquad I(y)=I(y\,; g, \phi, \psi)=\int_0^{2\pi} g(x)\tilde{U}(\phi)(x, y)\psi_x(x, y)dx.$$

Since $\tilde{U}(\phi)(z)$ is bounded near ∂_0 by [5.18] and $\psi_x(x, y)dx-\tilde{h}_{\sigma_\psi}(x, y)dx\to 0$ in weak sense, we have

$$I(y)=\iint_{[0,2\pi)\times[0,2\pi)}\sigma_\psi(d\zeta)dx\,g(x)\tilde{U}(\phi)(x, y)\tilde{h}_\zeta(x, y)+o(1)\quad(y\to 0).$$

Since $\int_0^{2\pi} |g(x)-g(\zeta)| \tilde{h}_\zeta(x, y)d\zeta = o(1)$ $(y\to 0)$ uniformly in x,

$$I(y) = \iint_{[0,2\pi)\times[0,2\pi)} \sigma_\psi(d\zeta)dx\, g(\zeta)\tilde{U}(\phi)(x, y)\tilde{h}_\zeta(x, y) + o(1) \quad (y\to 0).$$

And by [5.20]

$$I(y) = \iiint_{[0,2\pi)\times[0,2\pi)\times[0,2\pi)} \sigma_\psi(d\zeta)dx\,d\xi\, g(\zeta)\tilde{h}_\xi(x, y)\tilde{h}_\zeta(x, y)\tilde{U}_0(\phi)(\xi) + o(1)$$

$$= \iint_{[0,2\pi)\times[0,2\pi)} \sigma_\psi(d\zeta)d\xi\, g(\zeta)\tilde{h}_\xi(\zeta, 2y)\tilde{U}_0(\phi)(\xi) + o(1) \quad (y\to 0).$$

Therefore, by definition of $\tilde{U}_0(\phi)$ in [5.19],

$$I(y) = -T_{2y}(g\cdot\sigma_\psi, \sigma_\phi, \mu) - \int_{[0,2\pi)} \sigma_\psi(d\zeta)g(\zeta)G((\zeta, 2y), k\sigma_\phi + \tilde{\phi}^0\cdot\mu)$$

$$+ C\int_{[0,2\pi)} g(\zeta)\sigma_\psi(d\zeta) + o(1) \quad (y\to 0).$$

Since by [4.15] $\left|\frac{dg\cdot\sigma_\psi}{d\sigma}\right| \leqq \|g\| \left|\frac{d\sigma_\psi}{d\sigma}\right|$ and $\left|\frac{d\sigma_\psi}{d\sigma}\right|$ is bounded, it follows from [6.4] that

$$\lim_{y\to 0} T_{2y}(g\cdot\sigma_\psi, \sigma_\phi, \mu) = T_0(g\cdot\sigma_\psi, \sigma_\phi, \mu).$$

On the other hand, since $\int_0^{2\pi}(k\,d\sigma_\phi + \psi^0 d\mu) = 0$, $G((\zeta, 2y), k\sigma_\phi + \tilde{\phi}^0\cdot\mu)$ converges to $F(\zeta, k\sigma_\phi + \tilde{\phi}^0\cdot\mu)$ boundedly by virtue of [5.2]. Therefore,

$$\lim_{y\to 0} I(y) = -T_0(g\cdot\sigma_\psi, \sigma_\phi, \mu) - \int_{[0,2\pi)} g(\zeta)\sigma_\psi(d\zeta)F(\zeta, k\sigma_\phi + \tilde{\phi}^0\cdot\mu)$$

$$+ C\int_{[0,2\pi)} g(\zeta)\sigma_\psi(d\zeta).$$

Using [5.19] again, we finally have

$$\lim_{y\to 0} I(y) = \int_{[0,2\pi)} g(\zeta)\sigma_\psi(d\zeta)\tilde{U}_0(\phi)(\zeta),$$

which proves lemma [6.1].

[6.5] Let $\boldsymbol{B} = \{\sigma, \mu, k, p\}$ in $\mathscr{B}$ be given. Then

$$(1) \qquad \lim_{y\to 0} \int_0^{2\pi} g(x)U(z)\psi_x(z)dx = \int_{[0,2\pi)} g(x)U_0(x)\sigma_\psi(dx)$$

for any g in $C_p(R)$ and any ψ in D_q^a. Here $U \equiv U_P$ is defined in [4.12] and U_0

in [5.13].

(2) $U_0 \geqq 0$.

(3) $\int_{[0,2\pi)} U_0(x)\sigma(dx)=p$.

Proof. By remark [5.21], (1) is an immediate consequence of lemma [6.1]. By Fatou's lemma $\varliminf_{y\to 0} T(z, \sigma, \mu) \geqq T_0(x, \sigma, \mu)$. Therefore, by [5.12],

$$0 \leqq \varlimsup_{y\to 0} U(z) = -\varliminf_{y\to 0} T(z, \sigma, \mu) + k \lim_{y\to 0} G(z, \sigma-\mu) + C$$

$$\leqq -T_0(x, \sigma, \mu) + kF(x, \sigma-\mu) + C = U_0(x),$$

which proves (2). Let u be the function defined in (4.15) and let $g=1$ and $\phi=u$ in (1). Then, noting [4.12], we have

$$\int_{[0,2\pi)} U_0(x)\sigma(dx) = \lim_{y\to 0} \int_0^{2\pi} U(z)s(z)dx = p,$$

which proves (3).

§ 7. Uniqueness of B_N-solutions and B-processes.

Thoughout this section, $\boldsymbol{B}=\{\sigma, \mu, k, p\}$ in $\mathscr{B}$ is fixed. The functions s, t, m, l and U are defined in [4.12], u in (4.16) and the class $D^a_{q,N}$ in [4.13].

Let ϕ be a $\boldsymbol{B}_N$-solution for f in D^a. Then, by lemma [6.1] and (1) [6.5], we see that (4.18) in definition [4.16] is equivalent to

$$U_0(x)d\sigma_\phi = U_0(\phi)(x)d\sigma, \tag{7.1}$$

where $U_0(\phi)$ is the boundary version of a solution $U(\phi)$ of (4.19) which is defined as in [5.20], and U_0 is defined in [5.13].

[7.1] PROPOSITION. *For f in $C_q(R)$ and $a>0$, a $\boldsymbol{B}_1$-solution for f in D^a is unique, if it exists.*

Proof. Let ϕ_1 and ϕ_2 be $\boldsymbol{B}_1$-solutions for f in D^a. Then, by definition it is easy to see $\phi=\phi_1-\phi_2$ is a $\boldsymbol{B}_1$-solution for 0 in D^a. Since ϕ_j are in $C_q(D^{(0,a]})$,

$$\phi_j(z+2\pi)-\phi_j(z)=f(x+2\pi)-f(x).$$

Therefore ϕ is in $C_p(D^{(0,a]})$. Moreover ϕ is in $C^\infty(D^{(0,a]})$, for ϕ is harmonic in D^a and $\phi=0$ on $\partial_a=\{y=a\}$. Therefore, ϕ is smooth on ∂_a and

$$\lim_{y\to a}\int_0^{2\pi} U(\phi)\phi_x dx = -\lim_{y\to a}\int_0^{2\pi} U(\phi)_x \phi dx$$

$$=-\lim_{y\to a}\int_0^{2\pi}(-m\phi_y+l\phi_x)\phi dx=0\,.$$

On the other hand, by (2) [4.5]

$$0=\lim_{y\to a}\int_0^{2\pi}U(\phi)\phi_x dx$$

$$=\int_0^{2\pi}U(\phi)(x,\ b)\phi_x(x,\ b)dx+\int_{\substack{0\leqq x<2\pi\\ b<y<a}}m(\phi_x^2+\phi_y^2)dxdy\ ,$$

for any $b<a$. By (7.1), lemma [6.1] and (2) of [6.5], we have

$$\lim_{b\to 0}\int_0^{2\pi}U(\phi)(x,\ b)\phi_x(x,\ b)dx=\int_{[0,2\pi)}U_0(\phi)d\sigma_\phi$$

$$=\int_{[0,2\pi)}U_0(\phi)\frac{d\sigma_\phi}{d\sigma}d\sigma=\int_{[0,2\pi)}U_0\frac{d\sigma_\phi}{d\sigma}d\sigma_\phi$$

$$=\int_{[0,2\pi)}U_0\Big(\frac{d\sigma_\phi}{d\sigma}\Big)^2 d\sigma\geqq 0\,.$$

Therefore, $\int_{\substack{0\leqq x<2\pi\\ 0<y<a}}m(\phi_x^2+\phi_y^2)dxdy\leqq 0$. Since $m>0$ in D^a and $\phi=0$ on ∂_a, we have $\phi=\phi_1-\phi_2=0$ in D^a.

Now, we shall prove uniqueness of $\boldsymbol{B}_N$-solution for f in $C_{q,N}(R)$. First, we note:

[7.2] Let ρ be in $M_{p,N}(R)$. Define ρ_N in $M(R)$ by

$$\int f(x)d\rho_N=\frac{1}{N}\int f\Big(\frac{x}{N}\Big)d\rho \tag{7.2}$$

for any f in $C_K(R)$. Then ρ_N is in $M_p(R)$ and

$$\frac{1}{2\pi}\int_{[0,2\pi)}d\rho_N=\frac{1}{2N\pi}\int_{[0,2N\pi)}d\rho\,. \tag{7.3}$$

[7.3] Let $\boldsymbol{B}=\{\sigma,\ \mu,\ k,\ p\}$ be in $\mathscr{B}$. Then $\boldsymbol{B}^*=\{\sigma_N,\ \mu_N,\ k,\ p/N\}$ is also in $\mathscr{B}$, and

$$s_N(z)=s(\boldsymbol{B}^*)(z)=s(Nz)\,,$$

$$t_N(z)=t(\boldsymbol{B}^*)(z)=t(Nz)\,,$$

$$m_N(z)=m(\boldsymbol{B}^*)(z)=m(Nz)\,,$$

$$l_N(z)=l(\boldsymbol{B}^*)(z)=l(Nz)\,,$$

$$U_N(z)=U(\boldsymbol{B}^*)(z)=\frac{1}{N}U(Nz)\,.$$

Proof. We shall check the conditions in [4.9] for $\boldsymbol{B}^*$. The condition (b. 1) is obvious by (7.2), while (b. 2) is trivial. We have

$$s_N(z)=\int h_\xi(z)\sigma_N(d\xi)=\frac{1}{N}\int h_{\xi/N}(z)\sigma(d\xi)$$

$$=\frac{1}{N}\int h_{\xi/N}\Big(\frac{1}{N}Nz\Big)\sigma(d\xi)=s(Nz),$$

because $(1/N)h_{\xi/N}(z/N)=h_\xi(z)$. In the same way, we have $t_N(z)=t(Nz)$, $m_N(z)=m(Nz)$ and $l_N(z)=l(Nz)$. Set $\tilde{U}(z)=(1/N)U(Nz)$. Then $\tilde{U}_x=U_x(Nz)$ and $\tilde{U}_y=U_y(Nz)$. Therefore, it is easy to see that $\tilde{U}$ is a solution of

$$\tilde{U}_x=m_Nt_N+l_Ns_N,$$

$$\tilde{U}_y=m_Ns_N-l_Nt_N.$$

Since $\tilde{U}$ is nonnegative, (b. 3)* in [4.9] holds. Finally,

$$\frac{1}{2\pi}\int_0^{2\pi}\tilde{U}(x, y)s_N(x, y)dx=\frac{1}{2N\pi}\int_0^{2\pi}U(Nx, Ny)s(Nx, Ny)dx$$

$$=\frac{1}{2N^2\pi}\int_0^{2N\pi}U(x, Ny)s(x, Ny)dx=\frac{1}{2N\pi}\int_0^{2\pi}U(x, Ny)s(x, Ny)dx$$

and

$$\lim_{y\to 0}\frac{1}{2\pi}\int_0^{2\pi}\tilde{U}(x, y)s_N(x, y)dx=\frac{1}{N}p,$$

which proves (b. 4) and $\tilde{U}=U_N$.

[7.4] Let $\boldsymbol{B}$ and $\boldsymbol{B}^*$ be as in [7.3]. Then ϕ is in $D^a_{q,N}(\boldsymbol{B})$ if and only if $\phi_N(z)=(1/N)\phi(Nz)$ is in $D^{a/N}_q(\boldsymbol{B}^*)$.

Proof. Noting $(\phi_N)_x(z)=\phi(Nz)$ and $(\phi_N)_y=\phi(Nz)$, we can easily check the conditions in [4.13].

[7.5] Let $\boldsymbol{B}$ and $\boldsymbol{B}^*$ be as in [7.3] and let f be in $C_{q,N}(R)$. Then, ϕ is a $\boldsymbol{B}_N$-solution for f in D^a if and only if $\phi_N(z)=(1/N)\phi(Nz)$ is a $\boldsymbol{B}^*_1$-solution for $f_N(x)=(1/N)f(Nx)$ in $D^{a/N}$.

Proof. First, we note f_N is in $C_q(R)$. Let ϕ be a $\boldsymbol{B}_N$-solution for f in D^a. Then, by [7.4], ϕ_N is in $D^{a/N}_q(\boldsymbol{B}^*)$. It is obvious that $\phi_N(x, a/N)=f_N(x)$. Set $V_N(\phi_N)(z)=(1/N)U(\phi)(Nz)$ $(z\in D^{a/N})$. Then $V_N(\phi)$ is a solution of

$$V_N(\phi_N)_x=-m_N(\phi_N)_y+l_N(\phi_N)_x,$$

$$V_N(\phi_N)_y=m_N(\phi_N)_x+l_N(\phi_N)_y,$$

for $U(\phi)$ is a solution of (4.19) in [4.16]. Let g be in $C_p(R)$. Since ϕ is a $\boldsymbol{B}_N$-

solution and $g(x/N)$ is in $C_{p,N}(R)$, we have

$$\lim_{y\to 0}\int_0^{2\pi} g(x)(U_N\cdot(\phi_N)_x - V_N(\phi_N)\cdot s)(x, y)dx$$

$$=\lim_{y\to 0}\frac{1}{N}\int_0^{2\pi} g(x)(U\phi_x - U(\phi)s)(Nx, Ny)dx$$

$$=\lim_{y\to 0}\frac{1}{N^2}\int_0^{2N\pi} g\left(\frac{x}{N}\right)(U\phi_x - U(\phi)s)(x, Ny)dx$$

$$=0.$$

Therefore ϕ_N is a $\boldsymbol{B}_1^*$-solution for f_N in $D^{a/N}$. In a similar manner the converse is also proved.

Combining [7.5] with proposition [7.1], we have:

[7.6] THEOREM. *Let $\boldsymbol{B}$ in $\mathcal{B}$ be given. Then, for any N, any f in $C_{q,N}(R)$ and any $a>0$, a $\boldsymbol{B}_N$-solution for f in D^a is unique, if it exists.*

[7.7] THEOREM. *Let $\boldsymbol{B}$ in $\mathcal{B}$ be given. Then a $\boldsymbol{B}$-process is unique, if it exists.*

Proof. Let $\boldsymbol{P}$ be a $\boldsymbol{B}$-process. By theorem [7.6] and the definition of $\boldsymbol{B}$-process, $H_{\boldsymbol{P}}^a f$ is uniquely determined for any f in $C_{p,N}(R)$ and $a>0$. Let f be any function in $C_K(R)$, and set

$$f_N(x)=\sum_{k=-\infty}^{\infty} f(x+2kN\pi) \quad (N=1, 2, \cdots).$$

Then f_N is in $C_{p,N}(R)$ and $\lim_{N\to\infty} f_N(x)=f(x)$ boundedly for any $x\in R$. Therefore, $H_{\boldsymbol{P}}^a f(x)=\lim H^a f_N(x)$ is uniquely determined. By theorem [2.4], we can see uniquess of a $\boldsymbol{B}$-process.

III. Construction of $\boldsymbol{B}_P$.

§ 8. Construction of μ_P and k_P and condition [V].

In this section, we shall construct, for a given process $\boldsymbol{P}$ in $\mathcal{P}$, a nonnegative periodic harmonic function m_P in D and its supporting measure μ_P on ∂_0. In the last part of the section, we shall also show that $m_P(z)$ is the density function of an invariant measure of the process $\boldsymbol{P}$. Given a process P in P, we write the harmonic measures $H_{\boldsymbol{P}}^a(z, A)$ induced by P as $H^a(z, A)$.

[8.1] Let ${}^r\pi^s(x)$ $(0<s<r,\ x\in R)$ be defined as in § 0.8°. Then, ${}^r\pi^s(x)$ and ${}^r\tilde{\pi}^s(x)=\sum_{n=-\infty}^{\infty} {}^r\pi^s(x+2n\pi)$ are C^∞ functions with bounded derivatives in x, and

$$\int_{-\infty}^{\infty}\left|\frac{d^n}{dx^n}{}^r\pi^s(x)\right|dx<\infty.$$

Proof. By definition, we can see

$$\frac{d^n}{dx^n}{}^r\pi^s(x)=\frac{P_n(\cosh(\pi x/r), \sinh(\pi x/r))}{(\cosh(\pi x/r)-\cos(\pi s/r))^{n+1}}$$

where $P_n(x, y)$ is a polynomial of degree n. Hence [8.1] is proved easily.

[8.2] The kernel $H_b^a(x, A)=H_P^a((x, b)$ $(b<a)$ defined in [2.3] has a density function $h_b^a(x, \xi)$ in $C_b^\infty(R\times R)$. Define

$$\tilde{h}_b^a(x, \xi)=\sum_{n=-\infty}^{\infty} h_b^a(x, \xi+2n\pi).$$

It is also in $C^\infty(R\times R)$ and periodic in x and ξ.

Proof. Let $c<b<a$. Set $R=\sum_{n=0}^{\infty}(H_c^b\ {}_c^a\Pi_b^c)^n H_c^b$. Since $\|H_c^b\ {}_c^a\Pi_b^c\|=\dfrac{a-b}{a-c}<1$, R is a bounded positive kernel and $R(x, A)=R(x+2\pi, A+2\pi)$. By ($\bar{\text{h}}$.2) and ($\bar{\text{h}}$.3) in [2.2]

$$H_c^a=H_c^b\,H_b^a=H_c^b({}_c^a\Pi_b^a+{}_c^a\Pi_b^c\,H_c^a).$$

Therefore $H_c^a=R\ {}_c^a\Pi_b^a$, and

$$H_b^a={}_c^a\Pi_b^a+{}_c^a\Pi_b^c\,H_c^a={}_c^a\Pi_b^a+{}_c^a\Pi_b^c\,R\ {}_c^a\Pi_b^a.$$

The kernel $H_b^a(x, A)$ has a density

$$h_b^a(x, \xi)={}^{a-c}\pi^{a-b}(\xi-x)+\int_{R\times R}{}^{a-c}\pi^{b-c}(\eta-x)d\eta\,R(\eta, d\zeta){}^{a-c}\pi^{a-b}(\xi-\zeta).$$

Noting [8.1], we can get [8.2].

[8.3] Definitions. Let $K(x, A)$ be a kernel defined for $x\in R$ and $x\notin\bar{A}\in\mathcal{B}(R)$. Here $\bar{A}$ is the closure of A. We use notation

$$\int_{[x]}^{*}K(x, d\xi)F(x, \xi)=\lim_{\varepsilon\to 0}\int_{|\xi-x|>\varepsilon}K(x, d\xi)F(x, \xi)$$

for F in $C(R\times R)$, if the limit exists. Set

$$P^r f(x)=\int_{[x]}^{*}p^r(\xi-x)(f(\xi)-f(x)) \tag{8.1}$$

for $r>0$ and $f\in C_b^2(R)$, and

$$Q^r f(x)=\int q^r(\xi-x)f(\xi)d\xi \tag{8.2}$$

for $r>0$ and $f\in B_b(R)$, where $p^r(x)$ and $q^r(x)$ are defined in § 0, 8°.

Using explicit forms of each kernel, we can easily see:

[8.4]

(8.3) $$\lim_{b\uparrow a}\frac{1}{a-b}\int {}^a_c\Pi^a_b(x, d\xi)(f(\xi)-f(x))=P^{a-c}f(x)$$

for $0<c<a$ and f in $C^2_b(R)$.

(8.4) $$\lim_{b\uparrow a}\frac{1}{a-b}\int {}^a_c\Pi^c_b(x, d\xi)(f(\xi)-f(x))=Q^{a-c}f(x)$$

for $0<c<a$ and $f\in B_b(R)$.

[8.5] For f in $C^2_b(R)$

(8.5) $$B^a f(x)=\lim_{b\uparrow a}\frac{1}{a-b}\int H^a_b(x, d\xi)(f(\xi)-f(x))$$

exists, and $B^a f$ is expressed by a kernels $B^a(x, d\xi)$ as

(8.6) $$B^a f(x)=\int^*_{[x]} B^a(x, d\xi)(f(\xi)-f(x)).$$

The kernel $B^a(x, A)$ ($x\in R$, $A\in \boldsymbol{B}(R)$ and $x\notin \bar{A}$) satisfies for any $c<a$

(8.7) $$B^a(x, A)=P^{a-c}(x, A)+Q^{a-c}H^a_c(x, A).$$

Proof. For $c<b<a$, we have

$$H^a_b={}^a_c\Pi^a_b+{}^a_c\Pi^c_b\, H^a_c.$$

by $(\bar{\text{h}}.2)$ and [2.2]. By [8.4], we can immediately show

$$\lim_{b\uparrow a}\frac{1}{a-b}\int H^a_b(x, d\xi)(f(\xi)-f(x))$$

exists and is equal to

$$P^{a-c}f+\int (Q^{a-c}H^a_c)(x, d\xi)(f(\xi)-f(x))$$

for any f in $C^2_b(R)$.

[8.6]
(1) $B^a(x, A)=B^a(x+2\pi, A+2\pi)$ if $x\notin \bar{A}$.
(2) $B^a(x, A)$ has a density $b^a(x, \xi)$ in $C^\infty(R\times R)$ off diagonal, which satisfies

(8.8) $$b^a(x, \xi)=p^{a-c}(\xi-x)+\int_R q^{a-c}(\eta-x)h^a_c(\eta, \xi)d\eta.$$

Proof. (1) is obvious by (8.7). (8.8) follows from (8.7), [8.2] and explicit forms of $p^r(x)$ and $q^r(x)$ in § 0, 8°.

[8.7] For f in $C_b(R)$, set $\phi = H^a f$. Then

$$\phi_y(x, b) + B^b\phi(x, b) = 0 \tag{8.9}$$

holds for any $b \leqq a$, where

$$B^b\phi(x, b) = \int_{[x]}^{*} B^b(x, d\xi)(\phi(\xi, b) - \phi(x, b)).$$

If f belongs to $C_b^2(R)$, (8.9) holds for any $b \leqq a$.

Proof. If f belongs to $C_b^2(R)$, then by [8.5]

$$\phi_y(x, a) = \lim \frac{-1}{a-b}(H_b^a f(x) - f(x)) = -B^a f(x).$$

Noting $\phi(x, a) = f(x)$, we have proved (8.8) for $b = a$. For $b < a$, $\phi = H^b(H_b^a f)$ in D^b and $H_b^a f$ is in $C_b^2(R)$ for any f in $C_b(R)$. Therefore [8.7] is proved.

[8.8] For f in $C_b^2(R)$, set

$$Pf(x) = \frac{1}{\pi}\int_{[x]}^{*} \frac{d\xi}{(\xi - x)^2}(f(\xi) - f(x)). \tag{8.10}$$

Then for $a > 0$, there exists a function $m_a(x)$ in $C_p(R)$ such that

$$\int_0^{2\pi} m_a(x)(Pf(x) + B^a f(x))dx = 0 \tag{8.11}$$

for any f in $C_p^2(R)$, and

$$\int_0^{2\pi} m_a(x)dx = 2\pi. \tag{8.12}$$

Moreover, $m_a(x)$ is uniquely determined by (8.11) and (8.12).

Proof. By (8.7), for any $c < a$

$$(P + B^a)f(x) = Pf(x) + P^{a-c}f(x) + \int Q^{a-c}H_c^a(x, d\xi)(f(\xi) - f(x))$$

for f in $C_b^2(R)$. As an operator in $C_p(R)$, $P + P^{a-c}$ with domain $C_p^2(R)$ is a core of the generator of an additive process (process with stationary independent increment) on the torus $T = R/(2\pi)$, whose transition probability has a positive smooth density. Since $Q^{a-c}H_c^a$ is bounded on $C_p(R)$ and maps $C_p^\infty(R)$ into $C_p^\infty(R)$, usual argument on smooth bounded perturbation proves that $P + B^a$ is a core of the generator of a Markov process on the compact space T, and its semigroup maps $C_p^\infty(R)$ into $C_p^\infty(R)$, and its transition probability has a positive smooth density. Therefore the Markov process corresponding to $P + B^a$ has a unique invariant measure $m_a(x)dx$ which is characterized by (8.11) and (8.12).

[8.9] Let u be in $C_p^2(R)$ and v be the Hilbert transform of u, that is,

$$v(x)=\lim_{N\to\infty}\frac{1}{\pi}\int^*_{[x],|\xi-x|<N}\frac{\xi}{(\xi-x)}u(\xi)d\xi \tag{8.13}$$

$$=\frac{1}{2\pi}\int^*_{[x]}\frac{\sin(\xi-x)}{1-\cos(\xi-x)}u(\xi)d\xi.$$

Then

$$\int_0^{2\pi}(uPf-vf')dx=0 \tag{8.14}$$

for any f in $C_p^2(R)$.

Proof. For $z\in D$ set

$$\bar{u}(z)=\frac{1}{2\pi}\int_0^{2\pi}\tilde{h}_\xi(z)u(\xi)d\xi,$$

$$\bar{v}(z)=\frac{1}{2\pi}\int_0^{2\pi}\tilde{h}_\xi(z)v(\xi)d\xi,$$

$$\bar{f}(z)=\frac{1}{2\pi}\int_0^{2\pi}\tilde{h}_\xi(z)f(\xi)d\xi.$$

Then $\bar{u}$, $\bar{v}$ and $\bar{f}$ are in $C_p^1(D^{[0,\infty)})$ and harmonic in D. $\bar{v}$ is a harmonic conjugate of $\bar{u}$. Noting $\lim_{y\to0}\bar{f}_y(x, y)=Pf(x)$, $\lim_{y\to\infty}\bar{f}_y(x, y)=0$ and $\lim_{y\to\infty}\bar{f}_x(x, y)=0$, we have by [4.4]

$$\frac{1}{2\pi}\int_0^{2\pi}(uPf-vf')dx=\frac{1}{2\pi}\int_0^{2\pi}(\bar{u}\bar{f}_y-\bar{v}\bar{f}_x)(x, y)dx$$

$$=\lim_{y\to\infty}\int_0^{2\pi}(\bar{u}\bar{f}_y-\bar{v}\bar{f}_x)(x, y)dx=0.$$

Therefore (8.14) is proved.

[8.10] Let $m_a(x)$ in $C_p^\infty(R)$ be the function given in [8.8]. Define $m(z)=m_y(x)$ for any $z=(x, y)\in D$. Then $m(z)$ is a positive harmonic function in $C_p(D)$.

Proof. It has been already shown in [8.8] that m is positive and periodic with period 2π. To show m is harmonic in D, it is sufficient to prove

$$m_a(x)=\int_0^{2\pi}\tilde{h}_\xi(x, a-b)m_b(\xi)d\xi \tag{8.15}$$

for any $0<b<a$. Let l_b be the Hilbert transform of m_b and set

$$\bar{m}(z)=\int_0^{2\pi}\tilde{h}_\xi(z-b)m_b(\xi)d\xi,$$

$$\bar{l}(z)=\int_0^{2\pi}\tilde{h}_\xi(z-b)l_b(\xi)d\xi$$

for $z\in D^{(b,\infty)}$. For any $f\in C_p^2(R)$, set $\phi(z)=H^a f(z)$ for $z\in D^a$. Noting $\bar{l}(x, a)$ is the Hilbert transform of $\bar{m}(x, a)$, we have by (8.14), (8.9) and [4.1]

$$\begin{aligned}\int_0^{2\pi}\bar{m}(x, a)(Pf(x)+B^a f(x))dx &= \int_0^{2\pi}(\bar{l}(x, a)f'(x)+\bar{m}(x, a)B^a f(x))dx\\ &= \int_0^{2\pi}(\bar{l}(x, a)\phi_x(x, a)-\bar{m}(x, a)\phi_y(x, a))dx\\ &= \int_0^{2\pi}(\bar{l}(x, b)\phi_x(x, b)-\bar{m}(x, b)\phi_y(x, b))dx\\ &= \int_0^{2\pi}(l_b(x)\phi_x(x, b)+m_b(x)B^b\phi(x, b))dx\\ &= \int_0^{2\pi}m_b(x)(P\phi(x, b)+B^b\phi(x, b))dx\end{aligned}$$

where $P\phi(x, b)=(P\phi(\cdot, b))(x)$ and $B^b\phi(x, b)=(B^b\phi(\cdot, b))(x)$. By definition of $m_b(x)$, the last member of the above equalities is zero. Therefore

$$\int_0^{2\pi}\bar{m}(x, a)(Pf(x)+B^a f(x))dx=0$$

for any f in $C_p^2(R)$. Noting

$$\int_0^{2\pi}\bar{m}(x, a)dx=\int_0^{2\pi}m_b(x)dx=2\pi,$$

and using uniqueness of m_a, we can see $\bar{m}(x, a)=m_a(x)$. Therefore (8.15) is proved.

Combining [8.8] with [8.10], we have:

[8.11] PROPOSITION. *Let $\boldsymbol{P}$ in $\mathfrak{P}$ be given. Then there exists a unique positive harmonic function $m(z)$ in $C_p(D)$ such that*

$$\int_0^{2\pi}m(x, a)(Pf(x)+B^a f(x))dx=0 \tag{8.16}$$

for any $a>0$ and any f in $C_p^2(R)$, and

$$\int_0^{2\pi}m(x, a)dx=2\pi, \tag{8.17}$$

where $B^a f$ is defined in [8.5].

[8.12] DEFINITION. Let $\boldsymbol{P}$ in $\mathfrak{P}$ be given.
(1) For $a>0$ and f in $C_b^2(R)$, $B_P^a f$ is defined by (8.5) in [8.5]. For $a>0$

$A\in\mathscr{B}(R)$ and $x\notin\bar{A}$, $B_P^a(x, A)$ is defined by (8.7) in [8.5].

(2) m_P is defined as the positive harmonic function in D which is in $C_p(D)$ and satisfies (8.16) and (8.17) in [8.11].

(3) μ_P in $M_p(R)$ denotes the supporting measure of m_P on ∂_0, that is,

$$m_P(z)=\int_{[0,2\pi)}\tilde{h}_\xi(z)\mu_P(d\xi). \tag{8.18}$$

[8.13] *Remark.*

(1) The measure μ_P in $M_p(R)$ is uniquely determined for the given $\boldsymbol{P}$ in $\mathscr{P}$, and $[\mu_P]=(1/2\pi)\mu_P([0, 2\pi))=1$.

(2) Let l be a harmonic conjugate of m_P. Then, for any $a>0$, $l(x, a)$ differs from the Hilbert transform of $m_P(x, a)$ only by an additive constant. Therefore it follows from [8.9] that (8.16) in [8.11] is equivalent to

$$\int_0^{2\pi}(m_P(x, a)B^a f(x)+l(x, a)f'(x))dx=0 \tag{8.19}$$

for any f in $C_p^2(R)$.

[8.14] Conditions $[V_r]$ and $[V]$ (*Conditions for variance and Moment*)

Let r be a positive integer. A process $\boldsymbol{P}$ in $\mathscr{P}$ satisfies condition $[V_r]$ if and only if for any $b<a$

$$\sup_x\int_R H_{Pb}^a(x, d\xi)(\xi-x)^{2r}d\xi<\infty.$$

Condition $[V_1]$ is called condition $[V]$.

[8.15] Under condition $[V_r]$, (8.5) and (8.6) in [8.5] and (8.9) in [8.7] remain to be true for all f in $C^2(R)$ satisfying

$$|f(x)|\leqq K(1+|x|^{2r}). \tag{8.20}$$

with a constant K.

Proof. First, we note the following estimates.

$$\frac{1}{a-b}\int_{|x|\geqq N}{}_c^a\Pi_b^a(x, d\xi)\leqq K(c, a)e^{-\pi N/a-c},$$

$$\frac{1}{a-b}\int_{|x|\geqq N}{}_c^a\Pi_b^c(x, d\xi)\leqq K(c, a)e^{-\pi N/a-c},$$

and

$$\int H_c^a(x, d\xi)(1+\xi^{2r})\leqq\int H_c^a(x, d\xi)2^{2r-1}(1+(\xi-x)^{2r}+x^{2r})$$

$$\leqq\tilde{K}(c, a)+2^{2r-1}(1+x^{2r})$$

for $0<c<b<a$ and $N\geqq 1$. Here $K(c, a)$ and $K(c, a)$ are constants depending on c and a. Then, (8.5) and (8.6) for f satisfying (8.20) follow from the equation

$$H_b^a={}_c^a\Pi_b^a+{}_c^a\Pi_b^c H_c^a .$$

(8.9) is a consequence of (8.5).

[8.16] Under condition $[V_r]$,

$$\sup_x\int B^a(x, d\xi)(\xi-x)^{2r}<\infty .$$

Proof. For $0<c<a$,

$$\int_{[0]}^{*} p^{a-c}(x)x^{2r}dx=c_r<\infty,\ \int q^{a-c}(x)x^{2r}dx=c_r'<\infty$$

and

$$\int H_c^a(\eta, d\xi)(\xi-x)^{2r}\leqq 2^{2r-1}\Big(\int H_c^a(\eta, d\xi)(\xi-\eta)^{2r}+(\eta-x)^{2r}\Big).$$

Noting (8.7), we have [8.16].

[8.17] PROPOSITION. *Let* $\boldsymbol{P}$ *in* $\mathfrak{P}$ *be given and assume that* $\boldsymbol{P}$ *satisfies* $[V]$.

(1)

$$k_P=\frac{1}{2\pi}\int_0^{2\pi}m_P(x, a)dx\int_R B_P^a(x, d\xi)(\xi-x)\quad (a>0) \tag{8.21}$$

is independent of a.

(2) *Let* $l_P(z)$ *be the harmonic conjugate of* m_P *satisfying*

$$\frac{1}{2\pi}\int_0^{2\pi}l_P(x, a)dx=-k_P. \tag{8.22}$$

Then

$$\int_0^{2\pi}(m_P(x, a)B_P^a f(x)+l_P(x, a)f'(x))dx=0 \tag{8.23}$$

for any f *in* $C_q^2(R)$.

(3) *Let* m *in* $C_p(D)$ *be positive harmonic in* D *satisfying* $\int_0^{2\pi}m(x, a)dx=2\pi$. *Let* l *be a harmonic conjugate of* m *in* D. *If*

$$\int_0^{2\pi}(m(x, a)B_P^a f(x)+l(x, a)f'(x))dx=0 \tag{8.24}$$

for any f *in* $C_q^2(R)$, *then* $m=m_P$ *and* $l=l_P$.

Proof. First we note that, under $[V]$, $B_P^a f(x)$ is well defined for f in $C_q^2(R)$ and belongs to $C_p(R)$. Let k_a be the right hand side of (8.21) and let

$$l_0(z)=\int_{[0,2\pi)}\tilde{k}_\xi(z)\mu_P(d\xi),$$

which is a harmonic conjugate of $m_P(z)$ such that $\int_0^{2\pi}l_0(x,a)dx=0$. By (8.19)

$$\int_0^{2\pi}(m_P(x,a)B_P^a g(x)+l_0(x,a)g'(x))dx=0 \tag{8.25}$$

for any g in $C_p^2(R)$, since $l_0(x,a)$ is the Hilbert transform of $m_P(x,a)$. For any f in $C_q^2(R)$, set $C_f=(1/2\pi)(f(x+2\pi)-f(x))$ and $\phi(z)=H_P^a f(z)$ for $z\in D^a$. Then, by (h.4) in [2.1],

$$\frac{1}{2\pi}(\phi(z+2\pi)-\phi(z))=\frac{1}{2\pi}\int H_P^a(z,d\xi)(f(\xi+2\pi)-f(\xi))=C_f,$$

and $f-C_f x$ is in $C_p^2(R)$ and $\phi-C_f x$ is in $C_p^2(D^a)$. Therefore by (8.25)

$$\frac{1}{2\pi}\int_0^{2\pi}(m_P(x,a)B_P^a f(x)+l_0(x,a)f'(x))dx=C_f k_a \tag{8.26}$$

and similarly, for any $b<a$,

$$\frac{1}{2\pi}\int_0^{2\pi}(m_P(x,b)B_P^b\phi(x,b)+l_0(x,b)\phi_x(x,b))dx=C_f k_b.$$

On the other hand by [8.7] and [4.1]

$$\begin{aligned}&\int_0^{2\pi}(m_P(x,a)B_P^a f(x)+l_0(x,a)f'(x))dx\\&=\int_0^{2\pi}(-m_P(x,a)\phi_y(x,a)+l_0(x,a)\phi_x(x,a))dx\\&=\int_0^{2\pi}(-m_P(x,b)\phi_y(x,b)+l_0(x,b)\phi_x(x,b))dx\\&=\int_0^{2\pi}(m_P(x,b)B_P^b\phi(x,b)+l_0(x,b)\phi_x(x,b))dx.\end{aligned}$$

Therefore $C_f k_a=C_f k_b$. Choosing $f\equiv x$, we have $C_f=1$ and $k_a=k_b$ which shows $k_P=k_a$ is independent of a. Set $l=l_0-k_P$. Then for any f in $C_q^2(R)$

$$\begin{aligned}&\int_0^{2\pi}(m_P(x,a)B_P^a f(x)+l(x,a)f'(x))dx\\&=\int_0^{2\pi}(m_P(x,a)B_P^a f(x)+l_0(x,a)f'(x))dx-2\pi C_f k_P=0\end{aligned}$$

by (8.26). The assertion (1) and (2) have been proved.

Assume (8.24). Then, for f in $C_p^2(R)$,

$$\int_0^{2\pi}(m(x,\ a)B_P^a f(x)+l_0(x,\ a)f'(x))dx=0.$$

Therefore for any f in $C_p^2(R)$ and $a>0$

$$\int_0^{2\pi}m(x,\ a)(B_P^a f(x)+Pf(x))dx=0$$

by [8.9] and we have $m=m_P$ by [8.11]. Setting $f\equiv x$ in (8.24), we have

$$2\pi k_P+\int_0^{2\pi}l(x,\ a)dx=0,$$

which shows $l=l_P$.

[8.18] DEFINITION. Let $\boldsymbol{P}$ in $\mathcal{P}$ satisfying condition [V] be given. Define

$$k_P=\frac{1}{2\pi}\int_0^{2\pi}m_P(x,\ a)dx\int B_P^a(x,\ d\xi)(\xi-x)$$

and

$$l_P(z)=\int_{[0,2\pi)}\tilde{k}_\xi(z)\mu_P(d\xi)+k_P,$$

where μ_P, m_P and B_P^a are defined in [8.12].

In the remainder of this section we shall proved that $m_P(z)$ in [8.12] is the density of a periodic invariant measure of the process $\boldsymbol{P}$. Note that $\boldsymbol{P}$ need not satisfy the condition [V].

[8.19] For $a>0$, let $\sigma=\sigma_a$ be the hitting time of ∂_a.

(1) For a positive λ,

$$E_z(e^{-\lambda\sigma})=\begin{cases} e^{-\sqrt{2\lambda}(y-a)} & \text{if } y\geqq a, \\ \dfrac{\cosh\sqrt{2\lambda}y}{\cosh\sqrt{2\lambda}a} & \text{if } y\leqq a. \end{cases} \tag{8.27}$$

(2) Let f be in $C_b(R)$ and λ be a positive number and set

$$\phi(z)=E_z(e^{-\lambda\sigma}f(z(\sigma))).$$

Then ϕ, ϕ_x and ϕ_y tend to zero uniformly in x as $y\to\infty$. We have

$$|\phi_y(x,\ b)+B_P^b\phi(x,\ b)|\leqq\sqrt{2\lambda}\tanh(\sqrt{2\lambda}b)\|f\| \tag{8.28}$$

for any $b<a$.

Proof. By [1.6], $E_z(e^{-\lambda\sigma})=E_y^{R,1}(e^{-\lambda\bar{\sigma}_a})$ where $\bar{\sigma}_a$ is the hitting time of $\{a\}$. And by (p.4) [1.1], $\phi(z)=E_z^{B,2}(e^{-\lambda\sigma}f(z(\sigma)))$ for $y>a$. Therefore (1) and the first part of (2) are obvious by properties of Brownian motions. To prove

(8.28), let $\rho=\sigma_b$ be the hitting time of ∂_b and let z be in $D^b(b>a)$. Then $\phi(z)=E_z(e^{-\lambda\rho}\phi(x(\rho), b))$ for $y<b$, and ϕ is in C^∞ in D^a. Therefore,

$$\frac{1}{b-y}(\phi(x, b)-\phi(x, y))$$

$$=\frac{-1}{b-y}(E_z(\phi(x(\rho), b))-\phi(x, b))+\frac{1}{b-y}E_z((1-e^{-\lambda\rho})\phi(x(\rho), b)).$$

By [8.5]

$$\lim_{y\uparrow b}\frac{1}{b-y}(E_z(\phi(x(\rho), b))-\phi(x, b))=(B_P^b\phi(\cdot, b))(x, b)$$

and by (1)

$$\overline{\lim_{y\uparrow b}}\frac{1}{b-y}|E_z(1-e^{-\lambda\rho})\phi(x, b)|$$

$$\leqq\lim_{y\uparrow b}\frac{1}{b-y}E_z(1-e^{-\lambda\rho})\|\phi\|\leqq\sqrt{2\lambda}\tanh(\sqrt{2\lambda}b)\|f\|.$$

Since $\phi_y(x, b)=\lim_{y\uparrow b}\frac{1}{b-y}(\phi(x, b)-\phi(x, y))$, (8.28) is obtained.

[8.20] PROPOSITION. *Let $\boldsymbol{P}$ in $\mathcal{P}$ be given. Set $\tilde{D}=\{z: 0\leqq x<2\pi, 0<y<\infty\}$, and let f be a function in $C_p(D)$ such that*

$$\int_{\tilde{D}}|f(z)|m_P(z)dz<\infty. \tag{8.29}$$

Then, for $\lambda>0$

$$\lambda\int_{\tilde{D}}G_\lambda f(z)m_P(z)dz=\int_{\tilde{D}}f(z)m_P(z)dz, \tag{8.30}$$

$$\int_{\tilde{D}}P_t f(z)m_P(z)dz=\int_{\tilde{D}}f(z)m_P(z)dz, \tag{8.31}$$

where $G_\lambda f(z)=E_z\left(\int_0^\infty e^{-\lambda t}f(z(t))dt\right)$ and $P_t f(z)=E_z(f(z(t)))$.

Proof. Set $m=m_P$. We may assume that f is in $C_p^2(D)$, nonnegative and zero outside $D^{(b,a)}(0<b<a)$. By (p. 4) [1.1]

$$G_\lambda f(z)=E_z^{B,2}\left(\int_0^{\sigma_\varepsilon}e^{-\lambda t}f(z(t))dt\right)+E_z^{B,2}(e^{-\lambda\sigma_\varepsilon}G_\lambda f(z(\sigma_\varepsilon))),$$

where σ_ε is the hitting time of ∂_ε for any $\varepsilon>0$. Therefore $G_\lambda f(z)$ is in $C^2(D)$ and in $C_p(D)$ by (p. 5) [1.1], and $(\lambda-\Delta)G_\lambda f=f$ in D. Hence for $0<\varepsilon<b<a<N$

$$\int_0^{2\pi}dx\int_\varepsilon^N dy(\lambda-\Delta)G_\lambda f(z)m(z) \tag{8.32}$$

$$=\int_0^{2\pi}dx\int_\varepsilon^N dy\, f(z)m(z)=\int_{\tilde{D}}f(z)m(z)dz.$$

By Green's theorem

$$\int_0^{2\pi}dx\int_\varepsilon^N dy\,\Delta G_\lambda f(z)m(z)=\int_0^{2\pi}(m(G_\lambda(f)_y-m_y(G_\lambda f))(x,\ N)dx \tag{8.33}$$

$$+\int_0^{2\pi}(m(G_\lambda f)_y-m_y(G_\lambda f))(x,\ \varepsilon)dx.$$

Since

$$G_\lambda f(z)=\begin{cases} E_z(e^{-\lambda\sigma_a}G_\lambda f(z(\sigma_a)) & \text{if } y>a, \\ E_z(e^{-\lambda\sigma_b}G_\lambda f(z(\sigma_b)) & \text{if } y<b, \end{cases}$$

the first term in the right side in (8.33) tends to zero as $N\to\infty$ by (2) [8.19]. On the other hand, noting remark (2) [8.13], we see that for any harmonic conjugate l of m,

$$\int_0^{2\pi}m_y(G_\lambda f)(x,\ \varepsilon)dx=-\int_0^{2\pi}l_x(G_\lambda f)(x,\ \varepsilon)dx$$

$$=\int_0^{2\pi}l(G_\lambda f)_x(x,\ \varepsilon)dx=-\int_0^{2\pi}mB_P^\varepsilon(G_\lambda f)(x,\ \varepsilon)dx.$$

Therefore, by (8.28),

$$\left|\int_0^{2\pi}(m(G_\lambda f)_y-m_y(G_\lambda f))(x,\ \varepsilon)dx\right|$$

$$=\left|\int_0^{2\pi}m((G_\lambda f)_y+B_P^\varepsilon(G_\lambda f))(x,\ \varepsilon)dx\right|$$

$$\leqq\pi\sqrt{2\lambda}\tanh(\sqrt{2\lambda}\varepsilon)\|G_\lambda f\|.$$

Finally, it holds that

$$\lim_{N\to\infty,\,\varepsilon\to 0}\int_0^{2\pi}dx\int_\varepsilon^N dy\,\Delta G_\lambda f(z)=0.$$

Letting $\varepsilon\to 0$ and $N\to\infty$ in (8.32), we obtain (8.30). Since $P_tf(x)$ is continuous in t by (3) [1.4], (8.31) follows from (8.30).

[8.21] COROLLARY. *Using the notations in* [8.20], *let f be in $C(D)$ such that* $\int_D|f|m_P(z)dz<\infty$. *Then*

$$\lambda\int_D G_\lambda f(z)m_P(z)dz=\int_D f(z)m_P(z)dz, \tag{8.34}$$

$$\int P_tf(z)m_P(z)dz=\int_D f(z)m_P(z)dz. \tag{8.35}$$

Proof. We can assume that f is in $C_K(D)$. Then $\tilde{f}(z)=\sum_{n=-\infty}^{\infty} f(z+2n\pi)$ is in $C_p(D)$, and by (p. 5) [1.1]

$$\int_D f(z)m_P(z)dz=\int_{\tilde{D}}\tilde{f}(z)m_P(z)dz\,,$$

$$\int_D G_\lambda f(z)m_P(z)dz=\int_{\tilde{D}}G_\lambda\tilde{f}(z)m_P(z)dz\,,$$

$$\int_D P_t f(z)m_P(z)dz=\int_{\tilde{D}}P_t\tilde{f}(z)m_P(z)dz\,.$$

Therefore (8.34) and (8.35) are immediate consequence of (8.30) and (8.31).

§ 9. Construction of σ_P and condition [M].

In this section we shall construct, for a given process $\boldsymbol{P}$ in $\mathcal{P}$, a nonnegative harmonic function s_P in D and its supporting measure σ_P on ∂_0. The function $\phi(x)=\int_0^x\sigma_P(d\xi)$ plays a role of harmonic scale on ∂_0. In order to guarantee existence of σ_P or s_P, we need condition [M] and [V]. Later in part II, we shall discuss necessity of conditions [M] and [V].

[9.1] Condition [M] (*Monotonicity Condition*). If f is a non-decreasing function in $C_b(R)$, then $H^a_{Pb}f(x)$ is non-decreasing in x for any a and b with $0<b<a$.

[9.2] *Remark.* Under [M], it is easily seen that, if f is measurable and non-decreasing on R and $H^a_{Pb}f(x)>-\infty$ for any x, then $H^a_{Pb}f(x)$ is non-decreasing in x.

We define classes of harmonic functions related to the process $\boldsymbol{P}$.

[9.3] Definition. For $0<a<\infty$, set

$$H^a=H^a(\boldsymbol{P})=\{\phi\in C_b(D^{(0,a]}):\ \phi=H^a_Pf \text{ for some } f \text{ in } C_b(R)\}.$$

$$H^a_{p,N}=H^a_{p,N}(\boldsymbol{P})=H^a(\boldsymbol{P})\cap C_{p,N}(D^{(0,a]}),\ H^a_p=H^a_{p,1},\ H=\bigcap_{a>0}H^a,$$

$$H_{p,N}=\bigcap_{a>0}H^a_{p,N},\ H_p=\bigcap_{a>0}H^a_p.$$

If condition [V] is satisfied, set

$$H^a_{q,N}=H^a_{q,N}(\boldsymbol{P})$$

$$=\{\phi\in C_{q,N}(D^{(0,a]}):\ \phi=H^a_Pf \text{ for some } f \text{ in } C_{q,N}(R)\}.$$

$H_{q,N}$, H_q^a and H_q are defined similarly.

[9.4] *Remark.* If ϕ belongs to H^a, then ϕ is harmonic in $D^{(0,a)}$ and

$$\phi(z)=\int H_P^b(z, d\xi)\phi(\xi, b)$$

for any $b<a$ and $z\in D^b$.

[9.5] THEOREM. *Assume P in $\mathfrak{P}$ satisfies [V] and [M]. There exists a function u in H_q such that*

$$(9.1)\qquad \begin{cases} u(0, 1)=0, \\ u(x, y) \text{ is increasing in } x, \\ u(z+2\pi)-u(z)=2\pi. \end{cases}$$

Moreover, such a u in H^q is unique. The derivatives $s=u_x$ and $t=-u_y$ of u have the following properties.

(1) *s is positive harmonic in D and belongs to $C_p(D)$.*

(2) *t is a harmonic conjugate of s in D and also belongs to $C_p(D)$.*

(3)

$$(9.2)\qquad \frac{1}{2\pi}\int_0^{2\pi} t(x, y)dx=k_P,$$

where k_P is the constant defined in [8.18].

Proof of (1), (2) *and* (3). Let u in H_q satisfy (9.1). Then (1) and (2) are obvious, for u is harmonic and in $C_q(D)$. Set

$$k=\frac{1}{2\pi}\int_0^{2\pi} t(x, y)dx,$$

which is independent of y by [4.1]. Noting $u(z)=(H^a u(\cdot, a))(z)$ for any $a>0$ and z in D^a, we can see $u_y(x, a)+(B^a u(\cdot, a))(x, a)=0$ by [8.15]. Let m_P and l_P be functions defined in [8.12] and [8.18], respectively. Then, since

$$\frac{1}{2\pi}\int_0^{2\pi} m_P(x, a)dx=1,\ \frac{1}{2\pi}\int_0^{2\pi} l_P(x, a)dx=-k_P,\ \frac{1}{2\pi}\int_0^{2\pi} s(x, a)dx=1,$$

we have

$$\begin{aligned} 2\pi(k-k_P)&=\int_0^{2\pi}(m_P t+l_P s)(x, a)dx \\ &=\int_0^{2\pi}(-m_P u_y+l_P u_x)(x, a)dx \\ &=\int_0^{2\pi}(m_P(B_P^a u)+l_P u_x)(x, a)dx, \end{aligned}$$

by [4.4] and the last member is zero by (8.23) in [8.17], for $u(x, a)$ is in $C_q^2(R)$. Thus (9.2) is proved.

Proof of uniqueness. Let u and $\tilde{u}$ be both functions in H_q satisfying (9.1). Set $v=u-\tilde{u}$. Then v is in H_p. Especially v is a harmonic function in $G_p(D)$, which is represented by

$$v(z)=\int_0^{2\pi}\tilde{h}_\xi(x, y-a)v(\xi, a)d\xi+Cy \tag{9.3}$$

for any $a>0$ and $z\in D^{(a,\infty)}$. By (3) in the theorem,

$$2\pi C=\int_0^{2\pi}v_y(x, y)dx=\int_0^{2\pi}(u_y-\tilde{u}_y)(x, y)dx=0$$

and

$$\lim_{y\to\infty}v(z)=K=\frac{1}{2\pi}\int_0^{2\pi}v(\xi, a)d\xi.$$

For any $a>0$, v is harmonic in $D^{(a,\infty)}$ and $v_y+B_p^a v(\cdot, a)=0$ on ∂_a by [8.7]. Therefore, by the maximum principle, v can not attain strict maximum nor strict minimum in $D^{[a,\infty)}$. Thus $v\equiv K$ in $D^{[a,\infty)}$. Since a is arbitrary and $v(0, 1)=0$, $v=u-\tilde{u}=0$ in D.

Proof of existence.

1° Set $\tilde{u}_a(z)=\int H^a(z, d\xi)\xi$ for any $a>0$ and $z\in D^a$, which is well defined by condition [V]. By definition, $\tilde{u}_a$ is in H_q^a with $\tilde{u}_a(z+2\pi)-\tilde{u}_a(z)=2\pi$. Moreover by condition [M] (see also [9.2]), $\tilde{u}_a(x, y)$ is increasing in x. Set $u_a(z)=\tilde{u}_a(z)-\tilde{u}_a(0.1)$, $s_a=(u_a)_x$ and $t_a=-(u_a)_y$.

2° Obviously $s_a(z)$ is a positive harmonic function in $C_p(D^a)$ with $\int_0^{2\pi}s_a(x, y)dx=2\pi$. Therefore, for $a_0>0$, the integrals of $s_a(z)$ on each fixed compact set in D^{a_0} are bounded with respect to $a\geqq a_0$. Hence we can choose a sequence $\{a_n\}$ with $\lim a_n=\infty$ such that $s_n=s_{a_n}(n=1, 2, \cdots)$ and their derivatives converge uniformly on each compact set in D.

3° Set $u_n=u_{a_n}$ and $t_n=t_{a_n}=-(u_n)_y$. Since

$$u_n(x, 1)=\int_0^x s_n(\xi, 1)d\xi=2k\pi+\int_{2k\pi}^x s_n(\xi, 1)d\xi$$

for $2k\pi\leqq x<2(k+1)\pi$ and $u_n(z)=\int H^1(z, d\xi)u_n(\xi, 1)$, $\{u_n\}$ converges uniformly on ∂_1 and therefore on $D^{(0,1]}$.

4° Let $\tilde{t}_n$ be a harmonic conjugate of s_n such that $\tilde{t}_n(0, 1)=0$. Then

$$\tilde{t}_n(z)=-\int_0^x(s_n)_y(\xi, 1)d\xi+\int_1^y(s_n)_x(x, \eta)d\eta$$

and $\{\tilde{t}_n\}$ converges uniformly in each compact set in D. On the other hand, by 3° $\{t_n\}$ converges in D^1 and $t_n=\tilde{t}_n+k_n$. Therefore $\{k_n\}$ converges, and thus $\{t_n\}$ converges uniformly in each compact set in D. Noting $u_n(1, 0)=0$, we finally see $\{u_n\}$ converges uniformly on each compact set in D.

5° Set $u=\lim_{n\to\infty} u_n$. For any $b>0$, $\{u_n\}$ converges uniformly on ∂_b. Therefore, it follows from $u_n(z)=(H^b u_n(\cdot, b))(z)$ that $u(z)=(H^b u(\cdot, b))(z)$ and we see that u belongs to H_q. Since u_n satisfies (9.1) in D^{a_n}, u also satisfies (9.1) in D. Thus, u is the function just wanted. The proof of the theorem is complete.

[9.6] DEFINITION. Let $\boldsymbol{P}$ in $\mathfrak{P}$ be given and assume that $\boldsymbol{P}$ satisfies the conditions [V] and [M].

(1) u_P is the harmonic function in $H_q(\boldsymbol{P})$ determined by (9.1).

(2) $s_P=(u_P)_x$ and $t_P=-(u_P)_y$.

(3) σ_P is the measure in $M_p(R)$ such that

$$s_P(z)=\int_0^{2\pi}\tilde{h}_\xi(z)\rho_P(d\xi). \tag{9.4}$$

[9.7] *Remark.*

(1) Since s_P is positive harmonic in D and $\lim_{y\to\infty} s_P=1$, existence and uniqueness of σ_P is obvious. Therefore σ_P is uniquely determined for a given $\boldsymbol{P}$ in $\mathfrak{P}$, and $[\sigma_P]=(1/2\pi)\sigma_P([0, 2\pi))=1$.

(2) t_P is the harmonic conjugate of s_P that satisfies

$$\frac{1}{2\pi}\int_0^{2\pi} t_P(x, y)dx=k_P.$$

(3) Since $\lim_{y\to\infty}(u_P)_x(z)=1$ and $\lim_{y\to\infty}(u_P)_y(z)=-k_P$, it is easy to see that

$$w=u_P-x+k_P y \tag{9.5}$$

is a bounded harmonic function in $C_p(D)$.

[9.8] Assume P in $\mathfrak{P}$ satisfies [V] and [M]. Let f be in $C^1(R)$ and $|f(x)|\leqq K(1+|x|^2)$ with a constant K. Let a be a positive number and set $\phi(z)=H_P^a f(z)$. Then we have:

(1) If $f'(x)\leqq Cs_P(x, a)$ for some real C, then $\phi_x(z)\leqq Cs_P(z)$ for z in D^a.

(2) If $|f'(x)|\leqq Cs_P(x, a)$ for some positive C, then

$$|\phi_x(z)|\leqq Cs_P(z) \qquad \text{for } z \text{ in } D^a.$$

Proof. If $f'(x)\leqq Cs_P(x, a)$, then $Cu_P(x, a)-f(x)$ is nondecreasing. Therefore, by condition [M], $Cu_P(x, y)-\phi(x, y)$ is non-decreasing in x, which proves (1). If $|f'(x)|\leqq Cs_P(x, a)$, then $\pm f'(x)\leqq Cs_P(x, a)$ and (2) is obvious from (1).

[9.9] Assume $\boldsymbol{P}$ in $\mathfrak{P}$ satisfies [V] and [M]. For any f in $C_{q,N}$ and $a>0$, set $C_f=1/2N\pi(f(x+2N\pi)-f(x))$ and $\phi(z)=H_P^a f(z)$. Then it holds that, for any $b<a$,

(1) $\frac{1}{2N\pi}\int_0^{2N\pi}\phi_x(x, b)dx=C_f$,

(2) $\int_0^{2N\pi}(m_P\phi_y-l_P\phi_x)(x, b)dx=0$,

and

(3) there exists $K=K(\phi, b)>0$ such that $|\phi_x(z)|\leqq Ks_P(z)$ for all z in D^b.

Proof. By (p. 5) in [1.1], (1) is obvious. Set $\tilde{\phi}(z)=\sum_{k=0}^{N-1}\phi(z+2k\pi)$, then $\tilde{\phi}$ is in $C_q(D^a)$. Therefore, by [8.15] and (8.23) in [8.17],

$$\int_0^{2N\pi}(m_P\phi_y-l_P\phi_x)(x, b)dx=\int_0^{2\pi}(m_P\tilde{\phi}_y-l_P\tilde{\phi}_x)(x, b)dx$$

$$=-\int_0^{2\pi}(m_P(B^b\tilde{\phi})+l_P\tilde{\phi}_x)(x, b)dx=0.$$

Hence (2) is proved. Since s_P is positive harmonic in D, Set

$$K=\sup_{x\in[0, 2N\pi)}\frac{|\phi_x(x, b)|}{s_P(x, b)}<\infty.$$

Then $|\phi_x(x, b)|\leqq Ks_P(x, b)$. Since $\phi(z)=(H^b\phi(\cdot, b))(z)$ for $z\in D^b$, (3) is a consequence of (2) in [9.8].

§ 10. Construction of U_P, $U_P(\phi)$ and p_P.

Throughout this section, we shall fix a process P in $\mathscr{P}$ and assume [V] and [M] for P. First, in order to apply (8.23) in [8.17] to this section, we rewrite it in a slightly more general form.

[10.1] For any f in $C_{q,N}^2(R)$ and $a>0$,

$$\int_0^{2N\pi}(m_P(x, a)B_P^af(x)+l_P(x, a)f'(x))dx=0.$$

Proof. Let $\tilde{f}(x)=\sum_{n=0}^{N-1}f(x+2n\pi)$. Then $\tilde{f}$ is in $C_q^2(R)$ and satisfies (8.23). Noting that m_P and l_P are in $C_p(R)$, we can easily show [10.1].

[10.2] Notations. For f and g in $C_{q,N}^2(R)$ and $a>0$, set

$$\rho_{f,g}(x, \xi)=\int_x^\xi g'(t)(f(t)-f(x))dt \tag{10.1}$$

$$=\int_x^\xi(g(\xi)-g(t))f'(t)dt,$$

(10.2) $$\boldsymbol{B}^{\alpha}_{P,N}(f, g)=\int_0^{2N\pi} m_P(x, \alpha)dx\int B^{\alpha}_P(x, d\xi)\rho_{f,g}(x, \xi)$$

and

(10.3) $$\hat{\boldsymbol{B}}^{\alpha}_{P,N}(f, g)=\frac{1}{2}(\boldsymbol{B}^{\alpha}_{P,N}(f, g)+\boldsymbol{B}^{\alpha}_{P,N}(g, f)).$$

For f and g in $C^2_q(R)$ and $\alpha>0$, set

(10.4) $$\boldsymbol{B}^{\alpha}_P(f, g)=\boldsymbol{B}^{\alpha}_{P,1}(f, g)$$

and

(10.5) $$\hat{\boldsymbol{B}}^{\alpha}_P(f, g)=\hat{\boldsymbol{B}}^{\alpha}_{P,1}(f, g).$$

[10.3] *Remark.* It holds that

(10.6) $$\frac{1}{2}(\rho_{f,g}(x, \xi)+\rho_{g,f}(x, \xi))=\frac{1}{2}(f(\xi)-f(x))(g(\xi)-g(x)),$$

and especially

(10.7) $$\rho_{f,f}(x, \xi)=\frac{1}{2}(f(\xi)-f(x))^2.$$

[10.4] LEMMA. *For any f in $C^2_{p,N}(R)$, g in $C^2_{q,N}(R)$ and $\alpha>0$,*

(10.8) $$\boldsymbol{B}^{\alpha}_{P,N}(f, g)=-\int_0^{2N\pi} f(x)(m_P(x, \alpha)B^{\alpha}_P g(x)+l_P(x, \alpha)g'(x))dx.$$

Proof. Set $F(x)=\int_0^x f(t)g'(t)dt$, then F is in $C^2_{q,N}(R)$ and by [10.1] it holds that

$$0=\int_0^{2N\pi}(m_P(x, \alpha)B^{\alpha}_P F(x)+l_P(x, \alpha)F'(x))dx$$

$$=\int_0^{2N\pi} m_P(x, \alpha)dx\int B^{\alpha}_P(x, d\xi)(F(\xi)-F(x))+\int_0^{2N\pi} l_P(x, \alpha)f(x)g'(x)dx.$$

Add the right side to the right side of (10.8). Then the right side of (10.8)

$$=\int_0^{2N\pi} m_P(x, \alpha)dx\int B^{\alpha}_P(x, d\xi)\left\{\int_x^{\xi} f(t)g'(t)dt-f(x)(g(\xi)-g(x))\right\}$$

$$=B^{\alpha}_{P,N}(f, g).$$

For any g in $C^2_{q,N}(R)$, set

(10.9) $$\tilde{V}_{\alpha}(g)(x)=\int_0^x(m_P(t, \alpha)B^{\alpha}_P g(t)+l_P(t, \alpha)g'(x))dt.$$

Then, by [10.1], $\tilde{V}_a(g)$ is in $C^1_{p,N}(R)$. If f is in $C^2_{p,N}(R)$ and g is in $C^2_{q,N}(R)$, then we have, by [10.4] and by integration by part,

$$\boldsymbol{B}^a_{P,N}(f, g)=\int_0^{2N\pi} f'(x)\tilde{V}_a(g)(x)dx. \tag{10.10}$$

[10.5] For any g in $C^2_{q,N}(R)$, there exists a unique function $V_a(g)$ in $C^1_{p,N}(R)$ such that

$$V_a(g)'(x)=m_P(x, a)B^a_P g(x)+l_P(x, a)g'(x) \tag{10.11}$$

and

$$\boldsymbol{B}^a_{P,N}(f, g)=\int_0^{2N\pi} f'(x)V_a(g)(x)dx \tag{10.12}$$

for any f in $C^2_{q,N}(R)$.

Proof. It is sufficient to prove that there exists a unique constant $c=c(a, g)$ such that (10.12) holds with $V_a(g)=\tilde{V}_a(g)+c$ for all f in $C^2_{q,N}(R)$. Set

$$2N\pi c=\boldsymbol{B}^a_{P,N}(x, g)-\int_0^{2N\pi}\tilde{V}_a g(x)dx.$$

For any f in $C^2_{q,N}(R)$ with $2N\pi c_f=\int_0^{2N\pi} f'(x)dx$, $f-c_f x$ belongs in $C^2_{p,N}(R)$. Therefore by (10.10)

$$\begin{aligned}\boldsymbol{B}^a_{P,N}(f, g)&=\boldsymbol{B}^a_{P,N}(f-c_f x, g)+c_f\boldsymbol{B}^a_{P,N}(x, g)\\&=\int_0^{2N\pi}(f'(x)-c_f)\tilde{V}_a(g)dx+c_f\boldsymbol{B}^a_{P,N}(x, g)\\&=\int_0^{2N\pi} f'(x)\tilde{V}_a(g)dx+c_f\left(\boldsymbol{B}^a_{P,N}(x, g)-\int_0^{2N\pi}\tilde{V}_a(g)(x)dx\right)\\&=\int_0^{2N\pi} f'(x)(\tilde{V}_a(g)+c)dx,\end{aligned}$$

which shows that $V_a(g)$ satisfies (10.12). On the other hand, c is uniquely determined by the equality

$$\boldsymbol{B}^a_{P,N}(x, g)=\int_0^{2N\pi}\tilde{V}_a(g)(x)dx+2N\pi c.$$

Let N_1, N_2 and M be positive integers with $N_2=N_1M$ and let g be a function in $C_{q,N_1}(R)$. Then, g is also in $C_{q,N_2}(R)$. Set $V^j_a(g)(j=1, 2)$ be functions satisfying (10.11) and (10.12) for $N=N_j(j=1, 2)$. Then by (10.11) $V^1_a(g)'=V^2_a(g)'$, or $V^2_a(g)=V^1_a(g)+c$. Since

$$\boldsymbol{B}^a_{P,N_2}(x, g)=M\boldsymbol{B}^a_{P,N_1}(x, g)=M\int_0^{2N_1\pi}V^1_a(g)(x)dx$$

and

$$B^a_{P,N_2}(x, g)=\int_0^{2N_2\pi} V^2_a(g)(x)dx=\int_0^{2N_2\pi}(V^1_a(g)(x)+c)dx$$

$$=M\int_0^{2N_1\pi} V^1_a(g)(x)dx+2N_2\pi c,$$

we have $c=0$, that is, $V^1_a(g)=V^2_a(g)$. Therefore we have the following remark:

[10.6] *Remark.* In the above sense, $V_a(g)$ is independent of the positive integer N used in its definition.

[10.7] Let g and h be in $C^2_{q,N}(R)$ and a be positive.
(1) The mapping $g \mapsto V_a(g)$ from $C^2_{q,N}(R)$ into $C^1_{p,N}(R)$ is linear.
(2) $V_a(g)$ is nonnegative if g is nondecreasing.
(3) If h is nondecreasing and $|g'| \leqq ch'$, then

$$|V_a(g)| \leqq cV_a(h).$$

Proof. Since $V_a(g)$ is uniquely determined by (10.11) and (10.12), (1) is obvious. Let g be nondecreasing. For any nonnegative f in $C^2_{p,N}(R)$, set $F(x)=\int_0^x f(t)dt$. Then F is in $C^2_{q,N}(R)$ and nondecreasing, and $\rho_{F,g}(x, \xi) \geqq 0$. Therefore

$$\int_0^{2N\pi} f(x)V_a(g)(x)dx=B^a_P(F, g) \geqq 0,$$

which proves $V_a(g) \geqq 0$ and (2) is verified, (3) is an immediate consequence of (2).

Let ϕ be in the class $H^a_{q,N}=H^a_{q,N}(P)$ defined in [9.3]. In the following we shall fix a solution $\tilde{U}(\phi)$ of the equation

$$\text{(10.13)} \qquad \begin{cases} \tilde{U}(\phi)_x=-m_P\phi_y+l_P\phi_x, \\ \tilde{U}(\phi)_y=m_P\phi_x+l_P\phi_y \qquad \text{in } D^a. \end{cases}$$

For any $b<a$

$$\tilde{U}(\phi)_x(x, b)=(-m_P\phi_y+l_P\phi_x)(x, b)$$

$$=m_P B^a_P\phi(\cdot, b)(x)+l_P(x, b)\phi_x(x, b$$

$$=V_b(\phi(\cdot, b))'(x)$$

by [8.15]. Therefore we can see:

[10.8] For any $0<b<a$ and ϕ in $H^a_{q,N}$

$$V_b(\phi(\cdot, b))(x)=\tilde{U}(\phi)(x, b)+c_b(\phi),$$

where $V_b(\cdot)$ is defined in [10.5] and $c_b(\phi)$ is a constant depending on b and ϕ. In the following, we shall show that $c_b(\phi)$ is independent of b.

[10.9] Let f be in $C_{q,N}(R)$. Then, for $0<b<a$, the function $H_b^a f^2-(H_b^a f)^2$ is in $C_{p,N}(R)$.

Proof. Since $|f(x)|\leqq K+L|x|$ with constants k and L, $H_b^a f^2$ is well defined by [V]. Noting $f-H_b^a f$ is in $C_{p,N}(R)$, we can easily show

$$H_b^a f^2(x)-(H_b^a f(x))^2=\int H_b^a(x,\, d\xi)(f(\xi)-H_b^a f(x))^2$$

is in $C_{p,N}(R)$.

[10.10] For $0<c<b<a$ and ϕ in $H_{q,N}^a$,

$$\boldsymbol{B}_{P,N}^b(\phi(\cdot,\, b),\, \phi(\cdot,\, b))-\boldsymbol{B}_{P,N}^c(\phi(\cdot,\, c),\, \phi(\cdot,\, c))$$
$$=\int_0^{2N\pi}dx\int_c^b m_P(\phi_x^2+\phi_y^2)dy\,.$$

Proof. By definition, $\phi=H^a f$ with some f in $C_{q,N}(R)$. Set $\psi=H^a f^2$. Then by [10.9] $\psi-\phi^2$ is in $C_{p,N}(R)$ and by [10.1]

$$\int_0^{2N\pi} m_P(x,\, b)B_P^b(\phi^2(\cdot,\, b))(x)dx$$
$$=\int_0^{2N\pi} m_P(x,\, b)B_P^b(\psi(\cdot,\, b))(x)dx$$
$$+\int_0^{2N\pi} l_P(x,\, b)(\psi_x(x,\, b)-(\phi^2)_x(x,\, b))dx\,.$$

By [8.15]

$$B_P^b(\psi(\cdot,\, b))(x)=-\psi_y(x,\, b) \quad\text{and}\quad B_P^b(\phi(\cdot,\, b))(x)=-\phi_y(x,\, b).$$

Therefore, noting $(m_P)_y=-(l_P)_x$, we have

$$2\boldsymbol{B}_{P,N}^b(\phi(\cdot,\, b),\, \phi(\cdot,\, b))$$
$$=\int_0^{2N\pi} m_P(x,\, b)dx\int B_P^b(x,\, d\xi)(\phi(\xi,\, b)-\phi(x,\, b))^2$$
$$=\int_0^{2N\pi} m_P(x,\, b)(B_P^b(\phi^2(\cdot,\, b))(x)-2\phi(x,\, b)B_P^b(\phi(\cdot,\, b))(x))dx$$
$$=\int_0^{2N\pi} m_P(x,\, b)(B_P^b(\psi(\cdot,\, b))(x)-2\phi(x,\, b)B_P^b(\phi(\cdot,\, b))(x))dx$$
$$+\int_0^{2N\pi} l_P(x,\, b)(\psi-\phi^2)_x(x,\, b)dx$$

$$=-\int_0^{2N\pi} m_P(x,\, b)(\phi_y-2\phi\phi_y)(x,\, b)dx$$

$$-\int_0^{2N\pi}(l_P)_x(x,\, b)(\phi-\phi^2)(x,\, b)dx$$

$$=-\int_0^{2N\pi}\{m_P(\phi-\phi^2)_y-(m_P)_y(\phi-\phi^2)\}(x,\, b)dx\,.$$

The preceeding equalities also hold for $b=c$. Therefore Green's theorem proves that

$$2\boldsymbol{B}^b_{P,N}(\phi(\cdot,\, b),\, \phi(\cdot,\, b))-2\boldsymbol{B}^c_{P,N}(\phi(\cdot,\, c),\, \phi(\cdot,\, c))$$

$$=-\int_0^{2N\pi}dx\int_c^b m_P\Delta(\phi-\phi^2)dy=2\int_0^{2N\pi}dx\int_c^b m_P(\phi_x^2+\phi_y^2)dy\,.$$

[10.11] the constant $c_b(\phi)$ in [10.8] is independent of b.

Proof. For any $0<c<b<a$, by (10.12) and [10.8] we have

(10.14) $$\boldsymbol{B}^b_{P,N}(\phi(\cdot,\, b),\, \phi(\cdot,\, b))-\boldsymbol{B}^c_{P,N}(\phi(\cdot,\, c),\, \phi(\cdot,\, c))$$

$$=\int_0^{2N\pi}\phi_x(x,\, b)V_b(\phi(\cdot,\, b))dx-\int_0^{2N\pi}\phi_x(x,\, c)V_c(\phi(\cdot,\, c))dx$$

$$=\int_0^{2N\pi}\phi_x\tilde{U}(\phi)(x,\, b)dx-\int_0^{2N\pi}\phi_x\tilde{U}(\phi)(x,\, c)dx$$

$$+2N\pi c_\phi(c_b(\phi)-c_c(\phi))\,.$$

Here $2N\pi c_\phi=\phi(x+2N\pi,\, y)-\phi(x,\, y)$, which is independent of $(x,\, y)$, for ϕ is in $H^a_{q,N}$. Setting $f=\phi_x$, $g=-\phi_y$ and $u=m_P$ in (4.7) of [4.5], we have

(10.15) $$\int_0^{2N\pi}\phi_x\tilde{U}(\phi)(x,\, b)dx-\int_0^{2N\pi}\phi_x\tilde{U}(\phi)(x,\, c)dx$$

$$=\int_0^{2\pi}dx\int_c^b m_P(\phi_x^2+\phi_y^2)dy\,.$$

By [10.10], (10.14) and (10.15), it holds that

$$c_\phi(c_b(\phi)-c_c(\phi))=0\,.$$

Therefore $c_b(\phi)=c_c(\phi)$ if $c_\phi\neq 0$. If $c_\phi=0$, choose $u=u_P$ defined in [9.6]. Then $c_u=c_{\phi+u}=2\pi$. Noting $\tilde{U}(\phi+u)-\tilde{U}(\phi)-\tilde{U}(u)$ is constant in D^a and $g\rightarrow V_b(g)$ is linear, we can see by [10.7]

$$c_b(\phi+u)-c_b(\phi)-c_b(u)=(\tilde{U}(\phi+u)-\tilde{U}(\phi)-\tilde{U}(u))(x,\, b)$$

$$=(\tilde{U}(\phi+u)-\tilde{U}(\phi)-\tilde{U}(u))(x,\, c)$$

$$=c_c(\phi+u)-c_c(\phi)-c_c(u)\,.$$

Since $c_b(\phi+u)=c_c(\phi+u)$ and $c_b(u)=c_c(u)$, it is shown that $c_b(\phi)=c_c(\phi)$.

[10.12] THEOREM. *For any* ϕ *in* $H^a_{q,N}(0<a\leqq\infty)$, *there exists a unique* $U(\phi)(z)(z\in D^a)$ *such that* $U(\phi)$ *satisfies* (10.13) *and*

$$\boldsymbol{B}^b_{P,N}(f,\phi(\cdot,b))=\int_0^{2N\pi}f'(x)U(\phi)(x,b)dx \tag{10.16}$$

for any f *in* $C^2_{q,N}(R)$ *and* $0<b<a$.

Proof. Set $U(\phi)(x,b)=V_b(\phi(\cdot,b))$. Then, by [10.8], and [10.11], $U(\phi)$ satisfies (10.13), for $\tilde{U}(\phi)$ satisfies (10.13). This proves existence of $U(\phi)$, since $V_b(\phi(\cdot,b))$ satisfies (10.16). On the other hand, if $U(\phi)$ satisfies (10.13) and (10.16), set $\tilde{V}_b(\phi)(x)=U(\phi)(x,b)$. Then $\tilde{V}_b(\phi)$ satisfies (10.11) and (10.12) and $\tilde{V}_b(\phi)=V_b(\phi(\cdot,b))$. Therefore, uniqueness of $U(\phi)$ follows.

In the proof of theorem [10.12], we have seen that $U(\phi)(x,b)=V_b(\phi(\cdot,b))$. Therefore, by [10.5] and [10.7], it is easy to show:

[10.13] PROPOSITION. *Let* ϕ *and* ψ *be in* $H^a_{q,N}(0<a\leqq\infty)$.

(1) *The mapping* $\phi\mapsto U(\phi)$ *from* $H^a_{q,N}$ *into the set of periodic harmonic functions with period* $2N\pi$ *in* D^a *is linear.*

(2) *If* $\phi_x\geqq 0$, *then* $U(\phi)\geqq 0$. *Especially* $U(c)=0$ *for a constant function* c.

(3) *If* $|\phi_x|\leqq c\psi_x$ *for a nonnegative constant* c, *then* $|U(\phi)|\leqq cU(\psi)$.

[10.14] DEFINITION.

(1) For ϕ in $H^a_{q,N}(\boldsymbol{P})(0<a\leqq\infty)$, $U_P(\phi)=U(\phi)$ is the function which is uniquely determined by theorem [10.12].

(2) Especially we set $U_P=U(u_P)$, where u_P is the function given in [9.6].

(3) Set $p_P(a)=\int_0^{2\pi}U_P(x,a)s_P(x,a)dx$ and $p_P=\lim_{a\to 0}p_P(a)$, where s_P is given in [9.6].

[10.15] *Remark.*

(1) By [10.13] U_P is a nonnegative periodic harmonic function with period 2π in D.

(2) $U_P(\phi)$ is the solution of (10.13) determined by

$$\int_0^{2N\pi}U_P(\phi)(x,b)dx=\boldsymbol{B}^b_{P,N}(x,\phi(\cdot,b)), \tag{10.17}$$

if ϕ is in $H^a_{q,N}$.

For $0<b<a$

$$p_P(a)-p_P(b)=\int_0^{2\pi}dx\int_b^a m_P(s_P^2+t_P^2)dy$$

by (4.7) of [4.5]. Therefore $p_P(a)$ is nonincreasing in a and nonnegative, and p_P always exists.

Since U_P is nonnegative solution of (10.13) for $\phi_x=s_P$ and $\phi_y=-t_P$, we see that $\{\mu_P, \sigma_P, k_P\}$ defined in [8.12], [8.18] and [9.6] satisfies the condition [P^*] and therefore $\{\mu_P, \sigma_P\}$ satisfies the condition [P] by theorem [5.11]. By the construction in § 8, § 9 and § 10, we can easily show:

[10.16] THEOREM. *Let* $\boldsymbol{P}$ *in* $\mathcal{P}$ *satisfying* [V] *and* [M] *be given. Then* $\boldsymbol{B}_P=\{\mu_P, \sigma_P, k_P, p_P\}$ *is in the class* $\mathcal{B}$. *Moreover* $s_P=s(\boldsymbol{B}_P)$, $t_P=t(\boldsymbol{B}_P)$, $m_P=m(\boldsymbol{B}_P)$, $l_P=l(\boldsymbol{B}_P)$, $\mu_P=\mu(\boldsymbol{B}_P)$ *and* $U_P=U(\boldsymbol{B}_P)$(†).

For later use we note:

[10.17] For ϕ in $H^a_{q,N}$ and $0<b<a$, there exists a constant $K=K(\phi, b)$ such that

$$|U_P(\phi)|\leqq KU_P \quad \text{in } D^b. \tag{10.18}$$

Proof. By (3) in [9.9] there exists a constant $K=K(\phi, b)$ such that $|\phi_x(z)|\leqq K(u_P)_x$. (10.18) is an immediate consequence of (3) in proposition [10.13].

[10.18]

(1) For f and g in $C^1_{q,N}(R)$, $\boldsymbol{B}^a_P(f, g)$ and $\hat{\boldsymbol{B}}^a_P(f, g)$ are well defined.
(2) For f in $C^1_{q,N}(R)$, ϕ in $H^a_{q,N}$ and $b<a$, (10.16) still holds.

Proof. By the explicit form of $B^a_P(x, d\xi)$ in (8.7) of [8.5] and condition [V], (1) is obvious. To prove (2), approximate f uniformly together with its first derivative by a sequence in $C^2_{q,N}(R)$. (10.16) for f follows from (10.16) for each element of the approximating sequence by virtue of the bounded convergence theorem.

§ 11. Condition [L] and B_P-processes.

In this section, we shall also fix a process $\boldsymbol{P}$ in $\mathcal{P}$, which satisfies [V] and [M]. In § 10, we have seen $\boldsymbol{B}_P=\{\sigma_P, \mu_P, k_P, p_P\}$ is in the class $\mathcal{B}$. Noting [9.9], we can easily see:

[11.1] PROPOSITION.

$$H^a_{q,N}(P)\subset D^a_{q,N}(\boldsymbol{B}_P), \tag{11.1}$$

where $D^a_{q,N}$ *is defined in* [4.13].

In general $\boldsymbol{P}$ is not a $\boldsymbol{B}_P$-process. In this section, however, we shall prove

(†) See definition [4.12].

that $\boldsymbol{P}$ is a $\boldsymbol{B_P}$-process under the condition $[L]$ defined below. Later we shall see, $[L]$ is the condition which implies continuity of the process $\boldsymbol{P}$ on the boundary $\partial_0=\{y=0\}$. In the following, for simplicity, we shall suppress the suffix $\boldsymbol{P}$ for quantities in § 8, § 9 and § 10. That is, we shall write $\sigma=\sigma_{\boldsymbol{P}}$, $\mu=\mu_{\boldsymbol{P}}$, $k=k_{\boldsymbol{P}}$, $p=p_{\boldsymbol{P}}$, $u=u_{\boldsymbol{P}}$, $B^a(x, d\xi)=B^a_{\boldsymbol{P}}(x, d\xi)$, $U=U_{\boldsymbol{P}}$ etc.

[11.2] For any $a>0$ and z in D^a, there exists a constant M_a such that
(1) $0\leqq U(z)\leqq M_a$,
(2) $\boldsymbol{B}^b(x, u(\cdot, b))\leqq 2\pi M_a$ for $0<b<a$.

Proof. (1) is a special case of [5.18]. Since $B^b(x, u(\cdot, b))=\int_0^{2\pi} U(x, b)dx$ by (10.16), (2) is obvious.

[11.3] For f and g in $C^1_{q,N}(R)$ and $a>0$, $|\boldsymbol{B}^a_N(f, g)|$ and $2|\hat{\boldsymbol{B}}^a_N(f, g)|$ are bounded by

$$2\left(\left\|\frac{f'}{s(\cdot, a)}\right\|\left\|\frac{g'}{s(\cdot, a)}\right\|\int_0^{2N\pi}|f'|dx\int_0^{2N\pi}|g'|dx\right)^{1/2}\|U(\cdot, a)\|.$$

Proof. Set $\bar{f}(x)=\int_0^x|f'(t)|dt$ and $\bar{g}(x)=\int_0^x|g'(t)|dt$. Then $\bar{f}$ and $\bar{g}$ are in $C^1_{q,N}(R)$ and nondecreasing. Since

$$|\rho_{f,g}(x, \xi)|\leqq|(\bar{f}(\xi)-\bar{f}(x))(\bar{g}(\xi)-\bar{g}(x))|,$$

we have

$$\frac{1}{2}|\boldsymbol{B}^a_N(f, g)|,\ |\hat{\boldsymbol{B}}^a_N(f, g)|\leqq\boldsymbol{B}^a_N(\bar{f}, \bar{g})\leqq(\hat{\boldsymbol{B}}^a_N(\bar{f}, \bar{f})\hat{\boldsymbol{B}}^a_N(\bar{g}, \bar{g}))^{1/2}. \tag{11.2}$$

Since, by (10.1)

$$\begin{aligned}\rho_{\bar{f},\bar{f}}(x, \xi)&=\int_x^\xi(\bar{f}(\xi)-\bar{f}(t))|f'(t)|dt\\&=\left\|\frac{f'}{s(\cdot, a)}\right\|\int_x^\xi(\bar{f}(\xi)-\bar{f}(t))s(t, a)dt\\&=\rho_{\bar{f},u(\cdot,a)}(x, \xi)\left\|\frac{f'}{s(\cdot, a)}\right\|,\end{aligned}$$

it follows from [10.18] that

$$\begin{aligned}\hat{\boldsymbol{B}}^a_N(\bar{f}, \bar{f})=\boldsymbol{B}^a_N(\bar{f}, \bar{f})&\leqq\boldsymbol{B}^a_N(\bar{f}, u(\cdot, a))\left\|\frac{f'}{s(\cdot, a)}\right\|\\&=\int_0^{2N\pi}|f'(x)|U(x, a)dx\left\|\frac{f'}{s(\cdot, a)}\right\|\\&\leqq\left\|\frac{f'}{s(\cdot, a)}\right\|\|U(\cdot, a)\|\int_0^{2N\pi}|f'(x)|dx.\end{aligned}$$

A similar inequality holds for $\hat{\boldsymbol{B}}_N^a(\bar{g}, \bar{g})$. Therefore, [11.3] is proved.

[11.4] Condition [L] (Locality condition). For positive ε, set

$$\boldsymbol{B}^a(u\,;\varepsilon)=\boldsymbol{B}_{\boldsymbol{P}}^a(u\,;\varepsilon)=\int_0^{2\pi} m_{\boldsymbol{P}}(x, a)dx\int_{|\xi-x|\geqq\varepsilon}\boldsymbol{B}_{\boldsymbol{P}}^a(x, d\xi)(u_{\boldsymbol{P}}(\xi, a)-u_{\boldsymbol{P}}(x, a))^2.$$

Then $\boldsymbol{P}$ satisfies the condition [L] if and only if

$$\lim_{a\to 0}\boldsymbol{B}^a(u\,;\varepsilon)=0 \quad \text{for any positive } \varepsilon.$$

For any h in $C_{p,N}(R)$, set

$$\boldsymbol{B}_N^a[h]=\frac{1}{2}\int_0^{2N\pi} m(x, a)dx\int B^a(x, d\xi)h(x)(u(\xi, a)-u(x, a))^2. \tag{11.3}$$

[11.5] For any f and g in $C_{p,N}(R)$ and $z=(x, y)$, set

$$F(z)=\int_0^x f(t)s(t, y)dt \quad \text{and} \quad G(z)=\int_0^x g(t)s(t, y)dt.$$

Then, under condition [L],

$$\lim_{a\to 0}\{\boldsymbol{B}_N^a(F(\cdot, a), G(\cdot, a))-\boldsymbol{B}_N^a[fg]\}=0.$$

Proof. By definition (10.1)

$$\begin{aligned}&\boldsymbol{B}_N^a(F(\cdot, a), G(\cdot, a))\\&=\int_0^{2N\pi} m(x, a)dx\int B^a(x, d\xi)\int_x^\xi g(\alpha)s(\alpha, a)d\alpha\int_x^\alpha f(\beta)s(\beta, a)d\beta.\end{aligned}$$

Therefore

$$\begin{aligned}&|\boldsymbol{B}_N^a(F(\cdot, a), G(\cdot, a))-\boldsymbol{B}_N^a[fg]|\\&\leqq\int_0^{2N\pi} m(x, a)dx\Big(\int B^a(x, d\xi)\int_x^\xi|g(\alpha)-g(x)|s(\alpha, a)d\alpha\int_x^\alpha|f(\beta)|s(\beta, a)d\beta\\&\qquad+\int_x^\xi|g(x)|s(\alpha, a)d\alpha\int_x^\alpha|f(\beta)-f(x)|s(\beta, a)d\beta\Big).\end{aligned}$$

Set $\varepsilon(f)=\sup_{|\xi-x|\leqq\varepsilon}|f(\xi)-f(x)|$ and $\varepsilon(g)=\sup_{|\xi-x|\leqq\varepsilon}|g(\xi)-g(x)|$ for any positive ε. Then

$$\begin{aligned}&|\boldsymbol{B}_N^a(F(\cdot, a), G(\cdot, a))-\boldsymbol{B}_N^a[fg]|\\&\leqq 4\|f\|\|g\|N\boldsymbol{B}^a(u\,;\varepsilon)+(\|f\|\varepsilon(g)+\|g\|\varepsilon(f))\boldsymbol{B}_N^a(u(\cdot, a), u(\cdot, a)).\end{aligned}$$

Since $\boldsymbol{B}_N^a(u(\cdot, a), u(\cdot, a))$ is bounded for $a\leqq 1$ by [10.10], we get [11.15] by uniform continuity of f and g and the condition [L].

[11.6] Let a be positive and $z=(x, y)$. Let ϕ be in $H^a_{q,N}$ and g be in $C_{p,N}(R)$. Set

$$H(z)=\int_0^x g(t)\phi_x(t, y)dt \quad \text{and} \quad G(z)=\int_0^x g(t)s(t, y)dt.$$

Then, under the condition [L],

$$\lim_{b\to 0}\{\boldsymbol{B}^b_N(G(\cdot, b), \phi(\cdot, b))-\boldsymbol{B}^b_N(H(\cdot, b), u(\cdot, b))\}=0 \tag{11.4}$$

and

$$\lim_{b\to 0}\int_0^{2N\pi} g(x)\{U(\phi)(x, b)s(x, b)-U(x, b)\phi_x(x, b)\}dx=0. \tag{11.5}$$

For f in $C_{q,N}(R)$, set $\phi=H^a f$. Then, by [11.1] ϕ is in $D^a_{q,N}$. Therefore, if we obtain [11.6], then, by definitions [4.16] and [4.19], we conclude that $\boldsymbol{P}$ is a $\boldsymbol{B_P}$-process.

[11.7] THEOREM. *If* $\boldsymbol{P}$ *satisfies* [V], [M'] *and* [L], *then* $\boldsymbol{P}$ *is a* $\boldsymbol{B_P}$*-process.*

Proof of [11.6]. By (10.16) in [10.12], equivalence of (11.4) and (11.5) is obvious. We shall prove (11.4). First, for any fixed $a_0<a$ and $z\in D^{a_0}$, $|\phi_x(z)|\leqq Ks(z)$ holds for some positive K and ϕ_x can be represented in form

$$\phi_x(z)=\int^{a_0}\pi^{a_0-y}(\xi-x)\phi_x(\xi, a_0)d\xi+\int^{a_0}\pi^y(\xi-x)d\sigma_\phi(\xi)$$

and $d|\sigma_\phi|\leqq Kd\sigma$.

1° Set

$$\phi_{1x}(z)=\int\pi^y(\xi-x)d\sigma_\phi(\xi), \qquad \phi_1=\int_0^x\phi_{1x}(t, y)dt$$

and $H_1(z)=\int_0^x g(t)\phi_{1x}(t, y)dt$. Then it holds that

$$\begin{cases} \lim_{b\to 0}\boldsymbol{B}^b_N(G(\cdot, b), \phi_1(\cdot, b)-\phi(\cdot, b))=0, \\ \lim_{b\to 0}\boldsymbol{B}^b_N(H_1(\cdot, b)-H(\cdot, b), u(\cdot, b))=0. \end{cases} \tag{11.6}$$

Proof of 1° First we note;

$$\left\|\frac{G_x(\cdot, b)}{s(\cdot, b)}\right\|\leqq\|g\| \quad \text{and} \quad \int_0^{2N\pi}|G_x(x, b)|dx\leqq 2N\pi\|g\|.$$

For $b<a_0$

$$\left\|\frac{\phi_{1x}(\cdot, b)}{s(\cdot, b)}\right\|\leqq\sup_x\left|\frac{\int\pi^y(\xi-x)d\sigma_\phi(\xi)}{\int\pi^y(\xi-x)d\sigma(\xi)}\right|\leqq K$$

and

$$\left\|\frac{\phi_x(\cdot, b)}{s(\cdot, b)}\right\| \leqq K.$$

Further we have for $b < a_0$

$$\|\phi_{1x}(\cdot, b) - \phi_x(\cdot, b)\| \leqq K_1 b$$

with some positive K_1, since $\phi_{1x} - \phi_x$ is a periodic harmonic function in D^a and has null boundary value on ∂_0. Therefore by [11.3]

$$\begin{aligned}
&|\boldsymbol{B}_N^b(G(\cdot, b), \phi_1(\cdot, b) - \phi(\cdot, b))| \\
&\leqq 2\Big(\left\|\frac{G_x(\cdot, b)}{s(\cdot, b)}\right\| \left\|\frac{\phi_{1x}(\cdot, b) - \phi_x(\cdot, b)}{s(\cdot, b)}\right\| \\
&\quad \times \int_0^{2N\pi} |G_x(x, b)|\, dx \int_0^{2N\pi} |\phi_{1x}(x, b) - \phi_x(x, b)|\, dx\Big)^{1/2} \sup_{z \in D}{}_{a_0} |U(z)| \\
&\leqq 2(\|g\| \cdot 2K \cdot 2N\pi \|g\| \cdot 2N\pi K_1 b)^{1/2} M_{a_0},
\end{aligned}$$

where M_{a_0} is given in (1) [11.2]. Letting b tend to 0, we get the first part of (11.6). Noting $u_x(x, b) = s(x, b)$,

$$\int_0^{2N\pi} u_x(t, b)\, dt = 2N\pi,$$

$$\left\|\frac{H_{1x}(\cdot, b)}{s(\cdot, b)}\right\| \leqq K\|g\|, \qquad \left\|\frac{H_x(\cdot, b)}{s(\cdot, b)}\right\| \leqq K\|g\|$$

and

$$\begin{aligned}
\|H_{1x}(\cdot, b) - H_x(\cdot, b)\| &\leqq \|g\| \|\phi_{1x}(\cdot, b) - \phi_x(\cdot, b)\| \\
&\leqq K_1 \|g\| b,
\end{aligned}$$

we obtain the second part of (11.6) in the similar way.

2° For any $\varepsilon > 0$, there exists a function f in $C_{p,N}(R)$ such that

$$\text{(11.7)} \qquad \begin{cases} |\boldsymbol{B}_N^b(G(\cdot, b), \phi_2(\cdot, b) - \phi_1(\cdot, b))| < \varepsilon. \\ |\boldsymbol{B}_N^b(H_2(\cdot, b) - H_1(\cdot, b), u(\cdot, b))| < \varepsilon \end{cases}$$

for $b < a_0$, where

$$\phi_{2,x}(z) = \int \pi^y(\xi - x) f(\xi)\, d\sigma(\xi), \qquad \phi_2(z) = \int_0^x \phi_{2x}(t, y)\, dt$$

and

$$H_2(z) = \int_0^x g(t) \phi_{2x}(t, y)\, dt,$$

Proof of 2° Since $d|\sigma_\phi| \leqq K d\sigma$, there exists a bounded periodic measurable function $\tilde{f}(x)$ with period $2N\pi$ such that $d\sigma_\phi=\tilde{f}d\sigma$. For any positive ε_1, choose f in $C_{p,N}(R)$ such that $\int_0^{2N\pi}|f(x)-\tilde{f}(x)|d\sigma(x)<\varepsilon_1$. We can assume $\|\tilde{f}\|$ and $\|f\|$ are less than or equal to K. Then

$$\left\|\frac{\phi_{2x}(\cdot,b)}{s(\cdot,b)}\right\|\leqq K,\qquad \left\|\frac{H_{2x}(\cdot,b)}{s(\cdot,b)}\right\|\leqq K\|g\|,$$

$$\int_0^{2N\pi}|\phi_{2x}(x,b)-\phi_{1x}(x,b)|dx\leqq\int_0^{2N\pi}dx\int\pi^y(\xi-x)|f(\xi)-\tilde{f}(\xi)|d\sigma(\xi)$$

$$\leqq\int_0^{2N\pi}|f(\xi)-\tilde{f}(\xi)|d\sigma(\xi)<\varepsilon_1$$

and

$$\int_0^{2N\pi}|H_{2x}(x,b)-H_{1x}(x,b)|dx\leqq\|g\|\int_0^{2N\pi}|\phi_{2x}(x,b)-\phi_{1x}(x,b)|dx$$

$$<\|g\|\varepsilon_1.$$

Therefore, by [11.3] we have

$$|\boldsymbol{B}_N^b(G(\cdot,b),\phi_2(\cdot,b)-\phi_1(\cdot,b))|<2(\|g\|\cdot 2K\cdot 2N\pi\cdot\|g\|\cdot\varepsilon_1)^{1/2}M_{a_0},$$

$$|\boldsymbol{B}_N^b(H_2(\cdot,b)-H_1(\cdot,b),u(\cdot,b))|<2(2K\cdot\|g\|\cdot 2N\pi\cdot\|g\|\cdot\varepsilon_1)^{1/2}M_{a_0}.$$

Since ε_1 can be arbitrarily small, we have (11.7).

3° For f in 2°, set $\phi_3(z)=\int_0^x f(t)s(t,y)dt$ and $H_3(z)=\int_0^x f(t)g(t)s(t,y)dt$. Then

$$\text{(11.8)}\qquad\begin{cases}\lim_{b\to 0}B_N^b(G(\cdot,b),\phi_3(\cdot,b)-\phi_2(\cdot,b))=0,\\ \lim_{b\to 0}B_N^b(H_3(\cdot,b)-H_2(\cdot,b),u(\cdot,b))=0.\end{cases}$$

Proof of 3°. First, we note

$$\left\|\frac{\phi_{3x}(\cdot,b)}{s(\cdot,b)}\right\|\leqq\|f\|\leqq K\quad\text{and}\quad\left\|\frac{H_{3x}(\cdot,b)}{s(\cdot,b)}\right\|\leqq\|fg\|\leqq K\|g\|.$$

Since

$$\int_0^{2N\pi}|\phi_{3x}(x,b)-\phi_{2x}(x,b)|dx\leqq\int_0^{2N\pi}dx\int\pi^b(\xi-x)|f(x)-f(\xi)|d\sigma(\xi)$$

$$\leqq\int_0^{2N\pi}d\sigma(\xi)\int\pi^b(\xi-x)|f(x)-f(\xi)|dx$$

and $\int\pi^b(\xi-x)|f(x)-f(\xi)|dx$ tens to zero uniformly in x as $b\to 0$ for f in $C_{p,N}(R)$, we have

$$\lim_{b\to 0}\int_0^{2N\pi}|\phi_{3x}(x, b)-\phi_{2x}(x, b)|dx=0.$$

Similarly, we have

$$\lim_{b\to 0}\int|H_{3x}(x, b)-H_{2x}(x, b)|dx\leqq\lim_{b\to 0}\|g\|\int|\phi_{3x}(x, b)-\phi_{2x}(x, b)|dx$$
$$=0.$$

Using [11.3], we get (11.8).

4° Since f and g are in $C_{p,N}(R)$, we have

$$\text{(11.9)}\quad\begin{cases}\lim_{b\to 0}|\boldsymbol{B}_N^b(G(\cdot, b), \phi_3(\cdot, b))-\boldsymbol{B}_N^b[fg]|=0,\\ \lim_{b\to 0}|\boldsymbol{B}_N^b(H_3(\cdot, b), u(\cdot, b))-\boldsymbol{B}_N^b[fg]|=0,\end{cases}$$

by [11.5]. Combining (11.6), (11.7), (11.8) and (11.9), we see

$$\overline{\lim_{b\to 0}}|\boldsymbol{B}_N^b(G(\cdot, b), \phi(\cdot, b))-\boldsymbol{B}_N^b(H(\cdot, b), u(\cdot, b))|\leqq\varepsilon.$$

Since ε can be taken arbitrarily small, (11.4) is now proved.

In the remainder of this section, we shall give sufficient conditions for the condition [L].

[11.8] Condition [L^*]. For positive ε, set

$$\boldsymbol{B}^a(\varepsilon)=\boldsymbol{B}_P^a(\varepsilon)=\int_0^{2\pi}m(x, a)dx\int_{|\xi-x|\geqq\varepsilon}\boldsymbol{B}_P^a(x, d\xi)(\xi-x)^2.$$

Then $\boldsymbol{P}$ satisfies the condition [L^*] if and only if

$$\lim_{a\to 0}\boldsymbol{B}_P^a(\varepsilon)=0\quad\text{for every }\varepsilon.$$

[11.9] LEMMA. *Let ρ be in $M_{p,N}(R)$ and positive on any open set. Then, for any $\varepsilon>0$, $\inf_x\rho((x-\varepsilon, x+\varepsilon))>0$.*

Proof. If there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty}\rho((x_n-\varepsilon, x_n+\varepsilon))=0$, we may assume that x_n's are in $[0, 2N\pi]$ and $\lim_{n\to\infty}x_n=x$. Then

$$0=\lim_{n\to\infty}\rho((x_n-\varepsilon, x_n+\varepsilon))\geqq\rho\left(\left(x-\frac{\varepsilon}{2}, x+\frac{\varepsilon}{2}\right)\right),$$

which contradicts the assumption.

[11.10]
(1) The condition [L^*] implies the condition [L].
(2) If σ_P is positive on any open set, then [L] implies [L^*].

Proof. Since $u(x, a)$ is nondecreasing and in $C_{q,N}(R)$, we have

$$|u(\xi, a)-u(x, a)|\leqq 2\pi+|\xi-x|\leqq\left(\frac{2\pi}{\varepsilon}+1\right)|\xi-x|$$

if $|\xi-x|\geqq\varepsilon$. Hence (1) is obvious. If σ is positive on any open set, then $\delta(\varepsilon)=\inf_x \sigma((x-\varepsilon/2, x+\varepsilon/2))$ is positive by [11.9]. For $\xi-x\geqq\varepsilon$ we have

$$u(\xi, a)-u(x, a)=\int_x^\xi s(t, a)dt=\int_x^\xi dt\int\pi^a(\zeta-t)\sigma(d\zeta)$$
$$=\int du\, \pi^a(u)\int_{u+x}^{u+\xi}\sigma(d\zeta)\geqq\delta(\varepsilon).$$

Therefore, if $|\xi-x|\geqq\varepsilon$, then

$$|\xi-x|\leqq 2\pi+|u(\xi, a)-u(x, a)|$$
$$\leqq\left(\frac{2\pi}{\delta(\varepsilon)}+1\right)|u(\xi, a)-u(x, a)|.$$

Now (2) is obvious by definitions of $\boldsymbol{B}^a(u; \varepsilon)$ and $\boldsymbol{B}^a(\varepsilon)$.

[11.11] PROPOSITION. *If*

$$\lim_{a\to 0}\sup_x \frac{1}{a}\int_{|\xi-x|>\varepsilon} H_a^{2a}(x, d\xi)(\xi-x)^2=0 \tag{11.10}$$

for any positive ε, *then the conditions* $[L^*]$ *and* $[L]$ *hold.*

Proof. By (8.7) in [8.5]

$$B^{2a}(x, d\xi)=p^a(\xi-x)d\xi+\int q^a(\eta-x)H_a^{2a}(\eta, d\xi)d\eta.$$

By explicit forms of functions p^a and q^a in § 0, 8° we have

$$\begin{cases} p(a)=\int p^a(x)x^2dx=Ka, \\ q_1(a)=\int q^a(x)dx=\dfrac{L_1}{a}, \\ q_2(a)=\int q^a(x)x^2dx=L_2a, \\ q_3(\varepsilon, a)=\int_{|x|>\varepsilon} q^a(x)dx=\dfrac{L_3}{a^3}e^{-\pi(\varepsilon/a)} \end{cases} \tag{11.11}$$

where K, L_1, L_2 and L_3 are absolute positive constants. Set

$$h(\varepsilon, a)=\sup_x\int_{|\xi-x|>\varepsilon} H_a^{2a}(x, d\xi)(\xi-x)^2.$$

Then

$$\lim_{a\to 0}\frac{1}{a}h(\varepsilon, a)=0 \tag{11.12}$$

by the assumption and we have

$$\int H_a^{2a}(x, d\xi)(\xi-x)^2 \leqq \varepsilon^2+h(\varepsilon, a).$$

Noting $H_a^{2a}(x, R)=1$, we have

$$\int_{|\xi-x|>2\varepsilon} B^{2a}(x, d\xi)(\xi-x)^2$$

$$\leqq \int p^a(\xi-x)(\xi-x)^2 d\xi$$

$$+2\int_{|\xi-x|>2\varepsilon} q^a(\eta-x)H_a^{2a}(\eta, d\xi)\{(\xi-\eta)^2+(\eta-x)^2\}d\eta$$

$$\leqq p(a)+2q_2(a)+2\int_{\{|\xi-\eta|>\varepsilon\} \text{ or } \{|\eta-x|>\varepsilon\}} q^a(\eta-x)H_a^{2a}(\eta, d\xi)(\xi-\eta)^2$$

$$\leqq p(a)+2q_2(a)+2q_1(a)h(\varepsilon, a)+q_3(\varepsilon, a)(\varepsilon^2+h(\varepsilon, a)).$$

By (11.11) and (11.12), the last member of the above inequalities goes to 0 uniformly in x as $a\to 0$. Therefore, the condition [L^*] is satisfied.

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DEPARTMENT OF INFORMATION SCIENCES
TOKYO DENKI UNIVERSITY
HATOYAMA-MACHI, HIKI-GUN
SAITAMA 350-03, JAPAN