T. KATO
KODAI MATH. J.
12 (1989), 72—91

SUBSPACES OF TRIGONAL RIEMANN SURFACES

Dedicated to Professor Kôtaro Oikawa on his sixtieth birthday

BY TAKAO KATO

1. Introduction.

In the present paper, we shall study on relations between subspaces of the space of trigonal Riemann surfaces of genus $g \geqq 5$.

Recently, the author and Horiuchi [6] have studied on the Weierstrass gap sequences at the ramification points of trigonal Riemann surfaces. It was also studied by Coppens [1, 2, 3]. Coppens' study in [2] depends upon the fact that any trigonal Riemann surface lies on a rational normal scroll. On the other hand, the author and Horiuchi's study depends upon the fact that any trigonal Riemann surface is defined by an algebraic equation in x and y whose degree is three with respect to y. They determined a canonical equation of a trigonal Riemann surface of genus g and of the n-th kind and gave the necessary and sufficient condition for determining the types of ramification points in terms of zeros of the discriminant of the defining equation.

At first, we shall give an algebraic equation:

$$y^3+Q(x)y+R(x)=0.$$

Let S be the trigonal Riemann surface defined by the equation. We shall decide the genus and the kind of S and the types of the ramification points.

Using this result, we obtain incidence relations between $M_{g,3,n}(\rho_1, \rho_2, \rho_3, \rho_4)$'s. The definition of $M_{g,3,n}(\rho_1, \rho_2, \rho_3, \rho_4)$ will be given later.

Let S be a trigonal Riemann surface of genus g and let $x: S \rightarrow \mathbf{P}^1$ be a trigonal covering. Following Coppens [1] we say that S is of the n-th kind if $l(nD)=n+1$ and $l((n+1)D) \geqq n+3$, where $D=(x)_\infty$ is the polar divisor of x, $l(nD)$ (resp. $l((n+1)D)$ is the affine dimension of the space of meromorphic functions on S whose divisors are multiples of nD (resp. $(n+1)D$) and n satisfies $(g-1)/3 \leqq n \leqq g/2$.

By definition, a point P on S is a total (resp. an ordinary) ramification point if the ramification index of x at P is equal to three (resp. two). We say that P is a total ramification point of type I (resp. type II) if the gap sequence at P is equal to

$$(1, 2, 4, 5, \cdots, 3n-2, 3n-1, 3n+1, 3n+4, \cdots, 3(g-n-1)+1),$$

Received January 28, 1988, Revised August 1, 1988

(resp. $(1, 2, 4, 5, \cdots, 3n-2, 3n-1, 3n+2, 3n+5, \cdots, 3(g-n-1)+2)$).

We say that P is an ordinary ramification point of type I (resp. type II) if the gap sequence at P is equal to

$$(1, 2, 3, \cdots, 2n-1, 2n, 2n+1, 2n+3, \cdots, 2g-2n-1),$$

$$(\text{resp.}\ (1, 2, 3, \cdots, 2n-1, 2n, 2n+2, 2n+4, \cdots, 2g-2n)).$$

In both total and ordinary cases, each ramification point is of type I or type II [1, 2, 5, 6].

Let $M_{g,3,n}(\rho_1, \rho_2, \rho_3, \rho_4)$ be the set of trigonal Riemann surfaces of genus g, and of the n-th kind which have ρ_1 total ramification points of type I, ρ_2 total ramification points of type II, ρ_3 ordinary ramification points of type I and ρ_4 ordinary ramification points of type II. In [6], we have proved that if $3n-g+1-\rho_2-\rho_4=0$, then $M_{g,3,n}(\rho_1, \rho_2, \rho_3, \rho_4)$ is not empty.

In [3], Coppens studied on the structure of these spaces in the algebraic moduli space. For example, he proved that both $M_{g,3,n}(1, 0)$ and $M_{g,3,n}(0, 1)$ are irreducible and unirational, where $M_{g,3,n}(\rho_1, \rho_2)$ is the set of trigonal Riemann surfaces of genus g, and of the n-th kind which have ρ_1 total ramification points of type I and ρ_2 total ramification points of type II.

Concerning the incidence relations, he proved the following. If $(g-1)/3<n<g/2$, then $M_{g,3,n}(0, 1)$ is included in the closure of $M_{g,3,n}(1, 0)$. If $(g-1)/3\leqq n<g/2$, then $M_{g,3,n}(1, 0)$ is included in the closure of $M_{g,3,n+1}(1, 0)$. If $(g-1)/3\leqq n\leqq g/2-2$, then $M_{g,3,n}(1, 0)$ is included in the closure of $M_{g,3,n+1}(0, 1)$. In this paper, we shall consider the sets $M_{g,3,n}(\rho_1, \rho_2, \rho_3, \rho_4)$'s as subsets of the Teichmüller space and prove some incidence relations of them under the assumption $3n-g+1-\rho_2-\rho_4=0$.

The author would like to express his thanks to the referee for his valuable comments and suggestion.

2. Kind of trigonal Riemann surface.

In this section, we shall show how to decide the genus, the kind and the types of the ramification points for a given equation. To do this, we first show some theorems which have been already proved in [6].

Following the proofs of Lemmas 3-5 in [6], we know that these lemmas remain valid without assuming S is of the n-th kind. Hence, we have the following:

THEOREM A. *Let S be a trigonal Riemann surface defined by an algebraic equation*

$$y^3+Q(x)y+R(x)=0. \tag{1}$$

Here, $Q(x)$ and $R(x)$ are polynomials in x, $\deg Q=2n+2$, $\deg R=3n+3$ *and*

$\deg(4Q^3+27R^2)=6n+6$. *Furthermore, assume there is no common zero* α *of* $Q(x)$ *and* $R(x)$ *such that the order of zero of* $Q(x)$ *at* $x=\alpha$ *is greater than one and that of* $R(x)$ *is greater than two.*

Let β *be an arbitrary complex number and let* μ, ν *and* λ *be the orders of zeros of* $Q(x)$, $R(x)$ *and* $4Q(x)^3+27R(x)^2$ *at* $x=\beta$, *respectively. Then, we have:*

i) *There is a total ramification point over* $x=\beta$, *if* $\mu\geqq\nu=1$ *or* $\mu\geqq\nu=2$,

ii) *There is an ordinary ramification point over* $x=\beta$, *if* $\mu=\nu=0$ *and* λ *is odd or* $\nu>\mu=1$,

iii) *There is no ramification point over* $x=\beta$, *otherwise.*

Remark. In Theorem A, there is no ramification point over $x=\infty$. To see this, take a complex number α so that $Q(\alpha)R(\alpha)(4Q(\alpha)^3+27R(\alpha)^2)\neq 0$. Let $t=1/x$, $Y=t^{n+1}y$, $Q_1(t)=t^{2n+2}Q(1/t+\alpha)$ and $R_1(t)=R(1/t+\alpha)$. Then, we have

$$Y^3+Q_1(t)Y+R_1(t)=0.$$

Evidently, $Q_1(t)$, $R_1(t)$ and $4Q_1(t)^3+27R_1(t)^2$ have no zero at $t=0$. We can apply Theorem A again.

In the case that S is of the n-th kind, the author and Horiuchi [6] have proved the following:

THEOREM B. *In the preceding theorem, assume* S *is of the n-th kind. Then, we have:*

i) *There is a total ramification point of type* I *over* $x=\beta$, *if* $\mu\geqq\nu=1$,

ii) *There is a total ramification point of type* II *over* $x=\beta$, *if* $\mu\geqq\nu=2$,

iii) *There is an ordinary ramification point of type* I *over* $x=\beta$, *if* $\mu=\nu=0$ *and* λ *is odd,*

iv) *There is an ordinary ramification point of type* II *over* $x=\beta$, *if* $\nu>\mu=1$,

v) *There is no ramification point over* $x=\beta$, *otherwise.*

To decide the Weierstrass gap sequences at ramification points, we need the following:

THEOREM C. *Let* S *be a trigonal Riemann surface defined by* (1). *Then, every holomorphic differential on* S *is given by*

$$\Omega(D, E)=\frac{D(x)y+E(x)}{3y^2+Q(x)}dx,$$

where $D(x)$ *and* $E(x)$ *are suitable polynomials in* x *and* $\deg D\leqq n-1$ *and* $\deg E\leqq 2n$.

Henceforth, we demand the following hypotheses on (1):

i) $Q(x)$ and $R(x)$ are polynomials in x,

ii) $\deg Q=2n+2$, $\deg R=3n+3$ and $\deg(4Q^3+27R^2)=6n+6$ for some positive integer n,

iii) There is no common zero α of Q and R such that the order of zero of Q at $x=\alpha$ is greater than one and that of R is greater than two.

Then, we have the following equations by suitable polynomials Γ_i $(i=1, 2, 3)$ and Π_j $(j=1, 2, 3, 4)$ in x;

$$Q(x)=\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x), \tag{2}$$

$$R(x)=\Gamma_2(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)^2 \tag{3}$$

and

$$\Pi_3(x)\Gamma_3(x)^2=4\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27\Gamma_2(x)^2\Pi_4(x). \tag{4}$$

Here, $\Gamma_1\Pi_1\Pi_2$ and $\Gamma_2\Pi_4$ have no common zero,

$$\Pi_j(x)=\prod_{i=1}^{\rho_j}(x-a_{i,j}), \qquad (j=1, \cdots, 4) \tag{5}$$

for nonnegative integer ρ_j $(j=1, \cdots, 4)$ and mutually distinct complex numbers $a_{i,j}$ $(i=1, \cdots, \rho_j,\ j=1, \cdots, 4)$,

$$\deg\Gamma_1=2n+2-\rho_1-2\rho_2-\rho_4, \tag{6}$$

$$\deg\Gamma_2=3n+3-\rho_1-2\rho_2-2\rho_4 \tag{7}$$

and

$$2\deg\Gamma_3=6n+6-2\rho_1-4\rho_2-\rho_3-3\rho_4. \tag{8}$$

Then, we have:

LEMMA 1. *Assume S is a trigonal Riemann surface defined by* (1), *where $Q(x)$ and $R(x)$ in* (1) *satisfy* (2)-(8). *If a differential*

$$\Omega(D, E)=\frac{D(x)y+E(x)}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}dx$$

is holomorphic on S, then $\Pi_2(x)\Pi_4(x)$ is a factor of $E(x)$.

Proof. Assume α is a zero of $\Pi_2(x)$. By Theorem A, there is a total ramification point P over $x=\alpha$. Let t be a local parameter at P so that $x-\alpha=t^3$. Note that $\Gamma_2(\alpha)\Pi_1(\alpha)\Pi_4(\alpha)\neq 0$ by (3) and (4). Hence, the order of the zero of y at P is two and those of $3y^2+Q(x)$ and dx are four and two, respectively. Therefore, $E(x)$ must have a zero at $x=\alpha$.

Assume α is a zero of $\Pi_4(x)$. By Theorem A, there is an ordinary ramification point P over $x=\alpha$. Let s be a local parameter at P so that $x-\alpha=s^2$. Note that $\Gamma_1(\alpha)\Pi_1(\alpha)\Pi_2(\alpha)\neq 0$ by (2) and (4). Hence, the order of the zero of y at P is one or two and those of $3y^2+Q(x)$ and dx are three and one, respectively. Therefore, $E(x)$ must have a zero at $x=\alpha$.

THEOREM 1. *Assume that S, Q and R are as in Lemma* 1. *Then,*
i) *The genus g of S is given by*

$$\frac{2\rho_1+2\rho_2+\rho_3+\rho_4-4}{2}. \tag{9}$$

ii) *If* $g\geqq 5$ *and* $2n\leqq g\leqq 3n+1$, *then* S *is of the* n*-th kind and has* ρ_1 *total ramification points of type* I, ρ_2 *those of type* II, ρ_3 *ordinary ones of type* I *and* ρ_4 *those of type* II.

iii) *If* $g\geqq 5$ *and* $(3n-3\rho-1)/2\leqq g\leqq 2n-2\rho$, *where* $\rho=\deg \Gamma_3$, *then* S *is of the* $(g-n+\rho)$*-th kind and has* ρ_2 *total ramification points of type* I, ρ_1 *those of type* II *and* $\rho_3+\rho_4$ *ordinary ones of type* I. *In this case, there is no ordinary ramification point of type* II.

Proof. Theorem A implies that S has $\rho_1+\rho_2$ total ramification points and $\rho_3+\rho_4$ ordinary ramification points. Hence, by the Riemann-Hurwitz formula the genus g of S is given by (9).

Assume that $2n\leqq g\leqq 3n+1$ and S is of the m-th kind. Since $\deg(y)_\infty=3n+3$, we have $m\leqq n$.

Assume Γ_3 has a zero of order λ at $x=\alpha$, i.e. $\Gamma_3(x)=a(x-\alpha)^\lambda+\cdots$ $(\lambda>0, a\neq 0)$ near $x=\alpha$.

If $\Pi_3(\alpha)\neq 0$, then there is a branch of y, say y_1, so that

$$3y_1(x)^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)=\beta(x-\alpha)^\lambda+\cdots,$$

for some $\beta\neq 0$. If $\Pi_3(\alpha)=0$, then there is a branch of y, say y_1, and a local parameter $s=\sqrt{x-\alpha}$ at the ordinary ramification point over $x=\alpha$, so that

$$3y_1(x)^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)=\beta s^{2\lambda+1}+\cdots,$$

for some $\beta\neq 0$. Hence, $D(x)y_1(x)+E(x)$ in Theorem C must have a zero of order at least λ at $x=\alpha$.

Put

$$\omega_k=\frac{x^{k-1}\Pi_2(x)\Pi_4(x)\Gamma_3(x)dx}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}, \qquad (k=1,\cdots,g-n).$$

By the preceding discussion, these differentials are holomorphic on S. Moreover, it is easy to see that every holomorphic differential ω of the form

$$\omega=\frac{E(x)dx}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}$$

is a linear combination of ω_k's $(k=1,\cdots,g-n)$.

Assume that $\rho_2\neq 0$. Let P be a total ramification point over a zero of Π_2, say $a_{1,2}$. Then, for $l=1,\cdots,g-n$, the differential

$$\begin{aligned}\omega_l'&=\sum_{k=0}^{l-1}\binom{l-1}{k}(-a_{1,2})^{l-k-1}\omega_{k+1}\\&=\frac{(x-a_{1,2})^{l-1}\Pi_2(x)\Pi_4(x)\Gamma_3(x)dx}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}\end{aligned}$$

has a zero of order $3l-2$ at P.

If there were a holomorphic differential

$$\omega=\frac{D(x)y+E_1(x)\Pi_2(x)\Pi_4(x)}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}dx,$$

which has a zero of order $3(g-n)+1$ at P. Then, $D(x)$ had a zero of order at least $g-n+1(\geqq n+1)$ at $x=a_{1,2}$. Hence, by Theorem C, $D(x)$ would be identically zero. Therefore, ω would be a linear combination of ω_k's $(k=1,\cdots,g-n)$. Then, the order of zero at P were at most $3l-2$. This is absurd. Hence, $m=n$.

Assume that $\rho_2=0$ and $\rho_4\neq 0$. Let P be an ordinary ramification point over a zero of Π_4, say $a_{1,4}$. For $l=1,\cdots,g-n$, the differential

$$\omega_l'=\sum_{k=0}^{l-1}\binom{l-1}{k}(-a_{1,4})^{l-k-1}\omega_{k+1}$$
$$=\frac{(x-a_{1,4})^{l-1}\Pi_2(x)\Pi_4(x)\Gamma_3(x)dx}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}$$

has a zero of order $2l-1$ at P. In a similar way as above, we obtain that there is no holomorphic differential on S which has a zero of order $2(g-n)+1$ at P. Hence, $m=n$.

Assume that $\rho_2=0$, $\rho_4=0$ and $\rho_1\neq 0$. Let P be a total ramification point over a zero of Π_1, say $a_{1,1}$. For $l=1,\cdots,g-n$, the differential

$$\omega_l'=\sum_{k=0}^{l-1}\binom{l-1}{k}(-a_{1,1})^{l-k-1}\omega_{k+1}$$
$$=\frac{(x-a_{1,1})^{l-1}\Pi_2(x)\Pi_4(x)\Gamma_3(x)dx}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}$$

has a zero of order $3l-3$ at P. Again, we obtain that there is no holomorphic differential on S which has a zero of order $3(g-n)$ at P. Hence, $m=n$.

Finally, assume that $\rho_1=\rho_2=\rho_4=0$ and $\rho_3\neq 0$. Again, let P be an ordinary ramification point over a zero of Π_3, say $a_{1,3}$. For $l=1,\cdots,g-n$, the differential

$$\omega_l'=\sum_{k=0}^{l-1}\binom{l-1}{k}(-a_{1,3})^{l-k-1}\omega_{k+1}$$
$$=\frac{(x-a_{1,3})^{l-1}\Pi_2(x)\Pi_4(x)\Gamma_3(x)dx}{3y^2+\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)}$$

has a zero of order $2l-2$ at P. We shall show that there is no holomorphic differential on S which has a zero of order $2(g-n)$ at P. Then, as above, we obtain that $m=n$.

To prove the preceding fact, assume $\Gamma_3(x)$ has a zero of order λ at $x=a_{1,3}$. Put $\Pi_1(x)\Gamma_3(x)^2=27(x-a_{1,3})^{2\lambda+1}C_1(x)$. Let u and v be multivalued meromorphic functions on S which satisfy

$$u^3=(-R(x)+s^{2\lambda+1}\sqrt{C_1(x)})/2, \tag{10}$$

$$v^3=(-R(x)-s^{2\lambda+1}\sqrt{C_1(x)})/2 \tag{11}$$

and

$$uv=-Q(x)/3, \tag{12}$$

where s is a branch of $\sqrt{x-a_{1,3}}$. Then, there exist functions $U(x)$ and $V(x)$ which are holomorphic at $x=a_{1,3}$ and satisfy $U(a_{1,3})\neq 0$, $V(a_{1,3})\neq 0$ and

$$u=U(x)+s^{2\lambda+1}V(x).$$

Using (10), (11) and (12), we have

$$v=U(x)-s^{2\lambda+1}V(x).$$

Choose a branch of y which corresponds to the ordinary ramification point:

$$y=\omega u+\omega^2 v=-U(x)+(\omega-\omega^2)s^{2\lambda+1}V(x).$$

Then we have

$$3y^2+Q(x)=-6(\omega-\omega^2)s^{2\lambda+1}U(x)V(x)-6x^{2\lambda+1}V(x)^2.$$

Hence, $dx/(3y^2+Q(x))$ has a pole of order 2λ at P.

Assume that $\omega=\Omega(D, E)=(D(x)y+E(x))dx/(3y^2+Q(x))$ has a zero of order $2(g-n)$ at P. Put

$$D(x)=d_0+d_1s^2+d_2s^4+\cdots$$

$$E(x)=e_0+e_1s^2+e_2s^4+\cdots$$

and

$$y=b_0+b_2s^2+\cdots+b_{2\lambda}s^{2\lambda}+b_{2\lambda+1}s^{2\lambda+1}+\cdots,$$

where $b_0\neq 0$ and $b_{2\lambda+1}\neq 0$. By the preceding discussion, $D(x)y+E(x)$ must have a zero of order $2(g-n)+2\lambda$ at P. Hence, $b_0d_0+e_0=0$. If $d_0\neq 0$, then $d_0b_{2\lambda+1}$ would not be zero. Thus, $D(x)y+E(x)$ must have a zero of order at most $2\lambda+1$ at P. This is impossible. Hence, $d_0=e_0=0$.

In a similar way, we have

$$d_\iota=e_\iota=0 \qquad (\iota=0, \cdots, g-n-1).$$

Since $\deg D\leqq n-1\leqq g-n-1$, we have $D(x)$ is identically zero. Since such a holomorphic differential is a linear combination of ω_k's ($k=1, \cdots, g-n$), there is no holomorphic differential on S which has a zero of order $2(g-n)$ at P.

In each case, we have $m=n$. Hence, by Theorem B, we have the desired result.

Next, assume $(3n-3\rho-1)/2\leqq g\leqq 2n-2\rho$. Taking the birational transformation

$$(X,\ Y)=\left(x,\ \frac{(3\Gamma_2(x)\Pi_4(x)+\Gamma_1(x)y)\Pi_1(x)\Pi_2(x)}{3y}\right),$$

we have S is conformally equivalent to the surface defined by

$$Y^3-Q_1(X)\Pi_1(X)^2\Pi_2(X)Y+R_1(X)\Pi_1(X)^2\Pi_2(X)=0, \tag{13}$$

where

$$Q_1(X)=\Gamma_1(X)^2\Pi_2(X)/3$$

and

$$R_1(X)=(2/27)\Gamma_1(X)^3\Pi_1(X)\Pi_2(X)^2+\Gamma_2(X)^2\Pi_4(X).$$

The discriminant of the equation (13) is

$$\begin{aligned}-4(Q_1\Pi_1^2\Pi_2)^3+27(R_1\Pi_1^2\Pi_2)^2&=\Gamma_2^2\Pi_1^4\Pi_2^2\Pi_4(4\Gamma_1^3\Pi_1\Pi_2^2+27\Gamma_2^2\Pi_4)\\&=(\Gamma_2\Gamma_3)^2\Pi_1^4\Pi_2^2\Pi_3\Pi_4.\end{aligned}$$

By (6), (7), (8) and (9), we have

$$\begin{aligned}\deg Q_1&=2\deg\Gamma_1+\rho_2\\&=2(g-n+\rho)+2-2\rho_1-\rho_2,\\\deg(\Gamma_1^3\Pi_1\Pi_2^2)&=3(g-n+\rho)+3-2\rho_1-\rho_2,\\\deg(\Gamma_2^2\Pi_4)&=6n+6-2\rho_1-4\rho_2-3\rho_4\\&=3(g-n+\rho)+3-2\rho_1-\rho_2\end{aligned}$$

and

$$\deg\Gamma_2\Gamma_3=3(g-n+\rho)-g+1-\rho_1.$$

By the assumption, $2(g-n+\rho)\leqq g\leqq 3(g-n+\rho)+1$. If $\deg R_1=3(g-n+\rho)+3-2\rho_1-\rho_2$, then this case reduces to the case $2n\leqq g\leqq 3n+1$ and we have the desired result.

Assume that $\deg R_1<3(g-n+\rho)+3-2\rho_1-\rho_2$. Let α be a complex number such that $Q_1(\alpha)R_1(\alpha)\Pi_1(\alpha)\Pi_2(\alpha)\neq 0$. Take the birational transformation

$$(\xi,\ \eta)=(1/(X-\alpha),\ Y/(X-\alpha)^{g-n+\rho+1}).$$

Using the same discussion as in Remark following Theorem A, we have the desired result.

3. Incidence relations.

We would like to consider incidence relations between $M_{g,3,n}(\rho_1,\ \rho_2,\ \rho_3,\ \rho_4)$'s in the Teichmüller space. We first show that, roughly speaking, if two trigonal Riemann surfaces whose branch loci is close, then the Teichmüller distance of corresponding points is also close.

In the sequel we shall state our situation precisely. Let S be a Riemann surface of genus g having a trigonal covering $x: S \to \mathbf{P}^1$. Let $A=\{a_1, \cdots, a_m\} \subset \mathbf{P}^1$ be the projection of the set of ramification points of x. Without loss of generality, we may assume that $A \subset \mathbf{C}$ and $|a_i - a_j| > 2$ if $i \neq j$. Take a point $z \in \boldsymbol{C}$ so that $|z-a_i|>1$ for $i=1, \cdots, m$. Take an arbitrary ε $(0<\varepsilon<1)$. For each $i=1, \cdots, m$, let l_i' be a curve joining z and a_i in $\boldsymbol{C}-\bigcup_{j\neq i}\{z: |z-a_j|<1\}$ and let l_i be the curve starting at z and traveling along l_i' to the circle of radius ε with center a_i, then surrounding the circle and returning to z along l_i'. Let $x^{-1}(z)=\{Q_1, Q_2, Q_3\}$. Then, each l_i induces a permutation of $\{Q_1, Q_2, Q_3\}$. Then, the lemma is stated as follows:

LEMMA 2. *Assume S, A, ε, z, l_i are defined as above. Let S^* be another trigonal Riemann surface of genus g having a trigonal covering $x^*: S^* \to \mathbf{P}^1$. Assume the projection of the set of ramification points of x^* is included in $\bigcup_{i=1}^m$ (ε-neighborhood of a_i). Let $x^{*-1}(z)=\{Q_1^*, Q_2^*, Q_3^*\}$. Assume l_i induces the same permutation of $\{Q_1^*, Q_2^*, Q_3^*\}$ as that of $\{Q_1, Q_2, Q_3\}$ for each $i=1, \cdots, k$. Then, the Teichmüller distance of those points corresponding to S and S^* is at most $O(\varepsilon^{2/3})$.*

Proof. (The author is indebted to Professor A. Yamada who showed him the proof of this version, cf. Gardiner [4]). By virtue of the triangle inequality, it is enough to prove the case that only one ramification point, say $P_1 \in x^{-1}(a_1)$, varies and all the other ramification points remain fixed. Without loss of generality, we may assume $a_1=0$.

Assume P_1 is a total ramification point and there are two ordinary ramification points P_1^* and P_1^{**} over $\{|x^*|<\varepsilon\}$. Let $\alpha=x^*(P_1^*)$ and $\beta=x^*(P_1^{**})$. Let t be a local parameter at P_1 such that $x=t^3$, $|t|<3/2$. Let $t=\varphi(s)=s(1+As^{-2}+Bs^{-3})^{1/3}$, where

$$A=-3\left(\frac{\alpha-\beta}{4}\right)^{2/3} \quad \text{and} \quad B=\frac{\alpha+\beta}{2}.$$

If ε is sufficiently small, then, for some positive number δ, φ is a univalent map of $\{1-\delta<|s|<1+\delta\}$ onto a domain D which is contained in $\{1/2<|t|<3/2\}$. Let $\gamma=\varphi(\{|s|=1\})$ and let D_1 be the interior of γ.

Construct a new Riemann surface $\tilde{S}$ as follows. As a set, $\tilde{S}=(S-D_1)\cup\{|s|<1\}$, welding $\{|s|=1\}$ and γ in such a way that s is identified with $t=\varphi(s)$. A system of charts for $\tilde{S}$ is given by those for S on $S-\bar{D}_1$ and s itself on $\{|s|<1\}$. We can still take s as a chart on the set $(D-D_1)\cup\{1-\delta<|s|<1\}$.

Define a function $\tilde{x}$ on $\tilde{S}$ so that $\tilde{x}(P)=x(P)$ if $P\in(S-D_1)$ and $\tilde{x}(P)=s(P)^3+As(P)+B$ if $P\in\{|s|<1\}$. Then, $\tilde{x}$ gives a trigonal covering $\tilde{x}: \tilde{S}\to\mathbf{P}^1$ and there are two ordinary ramification points whose projection on the $\tilde{x}$-plane are α and β. Hence, $\tilde{S}$ is conformally equivalent to S^*.

Let $\psi(s)=s(1+A\bar{s}^2+B\bar{s}^3)^{1/3}$. Since $\varphi(s)=\psi(s)$ on $\{|s|=1\}$, $\psi(s)$ is a quasiconformal mapping of $\{|s|<1\}$ onto D_1. Evidently, the complex dilatation of ψ is bounded by $O(\varepsilon^{2/3})$.

The extension $\tilde{\psi}$ of ψ, defined by $\tilde{\psi}(P)=P$ if $P\in S-D_1$ and $\tilde{\psi}(P)=\psi(s(P))$ if

$P \in \{|s|<1\}$, is a quasiconformal mapping of $\tilde{S}$ onto S. The dilatation of $\tilde{\psi}$ is also bounded by $O(\varepsilon^{2/3})$.

A similar argument is applicable to both the cases that P_1 is total and there is one total ramification point P_1^* over $\{|x^*|<\varepsilon\}$ and that P_1 is ordinary. In each of these cases, we have a quasiconformal mapping of S^* onto S whose dilatation is bounded by $O(\varepsilon)$.

Hence, the Teichmüller distance between S and S^* is bounded by $O(\varepsilon^{2/3})$.

This completes the proof of the lemma.

In the following, we shall prove several lemmas related to the distribution of zeros of polynomials.

LEMMA 3. *Let $P(x)$ and $Q(x)$ be polynomials satisfying the following*:

i) *there is no common zero of $P(x)$ and $Q(x)$,*

ii) *every zero of $P(x)+Q(x)$ is simple,*

iii) $Q(0)\neq 0$.

Let $x_1, \cdots, x_n$ be the zeros of $P(x)+Q(x)$. Let k be an arbitrary positive integer. Let ε be a sufficiently small positive number. For an arbitrary $\alpha \in \mathbf{C}$, $|\alpha|<\varepsilon$, let $y_1, \cdots, y_{n+k}$ be the zeros of

$$x^k P(x)+(x-\alpha)^k Q(x).$$

Then, every y_i is simple, i.e. $y_i \neq y_j$ if $i\neq j$, and $|y_i-x_i|=O(\varepsilon^{1/k})$, $i=1, \cdots, n$ and $|y_i|=O(\varepsilon^{1/k})$, $i=n+1, \cdots, n+k$, suitably renumbering the suffixes of y, if necessary.

Proof. Since there is no common zero of $P(x)$ and $Q(x)$, $Q(x_i)\neq 0$, $i=1, \cdots, n$. Take the circles $C(x_i; r)=\{|x-x_i|=r\}$ and $C(0; r)=\{|x|=r\}$ so that these are mutually disjoint. Let $M=\max|Q(x)|$ and $m=\min|P(x)+Q(x)|$, where x runs over the sets $\cup C(x_i; r)$ and $C(0; r)$. Let $M_1=\max\{|x_1|, \cdots, |x_n|\}+2$. If $|\alpha|<\min\{1, mr^k/kMM_1^{k-1}\}$, then

$$\begin{aligned}|((x-\alpha)^k-x^k)Q(x)| &< |\alpha| kMM_1^{k-1} \\ &< mr^k \\ &< |x^k(P(x)+Q(x))|,\end{aligned}$$

on $C(0; r)\cup\bigcup C(x_i; r)$. Applying Rouché's theorem to $x^k(P(x)+Q(x))$ and $x^kP(x)+(x-\alpha)^kQ(x)$, we have the desired result.

Remark. Without assuming the condition ii) of this lemma, a similar result holds. However, the proof becomes slightly complicated and the exponent of ε in the estimate of y may be changed.

LEMMA 4. *Let $P(x)$ and $Q(x)$ be polynomials satisfying the following*

i) *there is no common zero of $P(x)$ and $Q(x)$,*

ii) *every zero of $P(x)+Q(x)$ is simple,*

iii) $P(0)+Q(0)\neq 0$ *and* $P(0)\neq 0$.
Let $x_1, \cdots, x_n$ *be the zeros of* $P(x)+Q(x)$. *Let* k *be a positive integer. Let* $y_1, \cdots, y_{n+k+1}$ *be the zeros of*

$$(x-\alpha)^{k+1}P(x)+x(x-\beta)^k Q(x).$$

Then, for sufficiently small positive number ε, *there are distinct* α *and* β *such that* $0<|\alpha|<\varepsilon$, $0<|\beta|<\varepsilon$ *and* y_i, $i=1, \cdots, n+k-1$ *are simple,* $y_{n+k}=y_{n+k+1}$ *and* $|y_i-x_i|=O(\varepsilon^{1/(k+1)})$, $i=1, \cdots, n$ *and* $|y_i|=O(\varepsilon^{1/(k+1)})$, $i=n+1, \cdots, n+k$.

Proof. Consider the polynomial

$$f(x, \alpha, \beta)=(x-\alpha)^{k+1}P(x)+x(x-\beta)^k Q(x)$$

in three variables. The intersection V of two surfaces $f(x, \alpha, \beta)=0$ and $\partial f(x, \alpha, \beta)/\partial x=0$ contains a curve through the point $(x, \alpha, \beta)=(0, 0, 0)$. Hence, for an arbitrary $\varepsilon>0$, there is a point $(x_1, \alpha_1, \beta_1)\in V$ such that $0<|x_1|+|\alpha_1|+|\beta_1|<\varepsilon$. It is sufficient to find such an $(x_1, \alpha_1, \beta_1)\in V$ that $\alpha_1\beta_1(\alpha_1-\beta_1)\neq 0$.

If ε is sufficiently small, applying Rouché's theorem to a pair of functions $f(x, \alpha_1, \beta_1)$ and $f(x, 0, 0)$, we have exactly $k+1$ zeros of $f(x, \alpha_1, \beta_1)$, counting multiplicity, in a neighborhood of $x=0$ and any other zero of $f(x, \alpha_1, \beta_1)$ is simple.

Assume that $\beta_1=0$. Applying Lemma 3, we have no double zero of $f(x, \alpha_1, 0)$. This is a contradiction.

We shall show that $V\cap\{\alpha\neq 0\}$ is not empty. Since $(\alpha, \alpha, \alpha)\in V$ for $k\geqq 2$, it is evident in this case. Assume that $k=1$. Fix an arbitrary x and eliminate β from $f(x, \alpha, \beta)=0$ and $\partial f(x, \alpha, \beta)/\partial x=0$. Then, we have the quadratic equation in α:

$$\begin{aligned}(P(x)Q(x)&+x(P(x)Q'(x)-P'(x)Q(x)))\alpha^2\\&-2x^2(P(x)Q'(x)-P'(x)Q(x))\alpha\\&+x^2(x(P(x)Q'(x)-P'(x)Q(x))-(P(x)+Q(x))Q(x))=0.\end{aligned}$$

If there were no nonzero solution of α for any x, comparing the coefficients, we have $Q(x)\equiv cP(x)$ for some constant c. This is a contradiction.

To show that $V\not\subset\{\alpha=\beta\}$, we fix an arbitrary $x_1\neq 0$ $(|x_1|<\varepsilon)$ and find a point $(\alpha, \beta)\neq(x_1, x_1)$ such that $(x_1, \alpha, \beta)\in V\cap\{\alpha\neq 0\}\cap\{\alpha\neq\beta\}$.

For simplicity's sake, replace $\alpha-x_1$ (resp. $\beta-x_1$) by α (resp. β) and denote $P(x_1)$, $P'(x_1)$, $Q(x_1)$, $Q'(x_1)$ by P, P', Q, Q', respectively. Then, we have the following equations:

$$-P\alpha^{k+1}+x_1Q\beta^k=0, \tag{14}$$

and

$$-P'\alpha^{k+1}+(k+1)P\alpha^k+(Q+x_1Q')\beta^k-kx_1Q\beta^{k-1}=0. \tag{15}$$

Substituting (14) into (15), we have

$$kx_1Q\beta^{k-1}=R\alpha^{k+1}+(k+1)P\alpha^k, \tag{16}$$

where $R=(P/x_1)+(PQ'/Q)-P'$. Taking the k-th powers of the both sides of (16), we have

$$\alpha^{k^2-1}(\alpha(R\alpha+(k+1)P)^k-k^kx_1QP^{k-1})=0. \tag{17}$$

Since $R=O(|x_1|^{-1})$, there exists a non-zero solution α_1 of the equation (17) so that $\alpha_1=O(|x_1|^t)$, where $t=(m+k+1)/(k+1)$ and m is the order of zero of $Q(x)$ at $x=0$. Substituting α_1 into (14), we have a solution $\beta=\beta_1=O(|x_1|)$.

Assume that $\alpha_1=\beta_1$. Then, by (14), we have $x_1Q=P\alpha_1$. By (16) we have

$$kx_1Q=(k+1)P\alpha_1+R\alpha_1^2,$$

hence, $P+R\alpha_1=0$. Therefore, we have $P^2+x_1QR=0$, that is

$$\frac{x_1PQ'+QP-x_1QP'}{P^2}=-1.$$

Since x_1 is arbitrarily chosen, we have $(xQ(x)/P(x))'=-1$. Then, $P(x)+Q(x)\equiv0$. This is a contradiction.

The rest of the proof is similar to that of Lemma 3.

LEMMA 5. *Let $P(x)$ and $Q(x)$ be as in Lemma 4. Let $y_1, \cdots, y_{n+2}$ be the zeros of*

$$(x-\alpha)(x-\beta)P(x)+x^2Q(x).$$

Then, for sufficiently small positive number ε, there are distinct α and β such that $0<|\alpha|<\varepsilon$, $0<|\beta|<\varepsilon$ and y_i, $i=1, \cdots, n$ are simple, $y_{n+1}=y_{n+2}$ and $|y_i-x_i|=O(\varepsilon^{1/2})$, $i=1, \cdots, n$ and $|y_{n+1}|=O(\varepsilon^{1/2})$.

Proof. Similar to the preceding lemma.

LEMMA 6. *Let $P(x)$ and $Q(x)$ be polynomials satisfying the following:*

i) *there is no common zero of $P(x)$ and $Q(x)$,*

ii) *every zero of $P(x)+Q(x)$ is simple,*

iii) *$P(0)+Q(0)\neq0$, $Q(0)\neq0$ and $P(x)$ has a simple zero at $x=0$.*

Let $y_1, \cdots, y_{n+4}$ be the zeros of

$$x^2(x-\alpha)^2P(x)+(x-\beta)^2(x-\gamma)^2Q(x).$$

Then, for sufficiently small positive number ε, there are distinct α, β and γ such that $0<|\alpha|<\varepsilon$, $0<|\beta|<\varepsilon$, $0<|\gamma|<\varepsilon$ and y_i, $i=1, \cdots, n$ are simple, $y_{n+1}=y_{n+2}\neq y_{n+3}=y_{n+4}$ and $|y_i-x_i|=O(\varepsilon^{1/4})$, $i=1, \cdots, n$ and $0\neq|y_i|=O(\varepsilon^{1/4})$, $i=n+1, \cdots, n+4$.

Proof. By the hypothesis, we can choose a neighborhood U of $x=0$ satisfying that

$$P(x)Q(x)(P(x)+Q(x))\neq 0 \qquad \text{for} \quad x\in U-\{0\}$$

and there is an $\varepsilon_0>0$ such that

$$x^2(x-\alpha)^2P(x)+(x-\beta)^2(x-\gamma)^2Q(x)$$

has exactly four zeros in U if $|\alpha|, |\beta|, |\gamma|<\varepsilon_0$. Then, we have a single valued branch of $\sqrt{P(x)/x}$ (resp. $\sqrt{-Q(x)}$) in U which we denote by

$$A(x)=a+a_1x+\cdots, \qquad (\text{resp. } B(x)=b+b_1x+\cdots).$$

Let $U'=U-\{\text{real negative}\}-\{0\}$. In $U'^2\times U^3$, consider two functions

$$f(x, \alpha, \beta, \gamma)=x^{3/2}(x-\alpha)A(x)-(x-\beta)(x-\gamma)B(x), \tag{18}$$

$$g(y, \alpha, \beta, \gamma)=y^{3/2}(x-\alpha)A(y)+(y-\beta)(y-\gamma)B(y). \tag{19}$$

Here, we choose suitable branches of $x^{3/2}$ and $y^{3/2}$, for instance, $\mathrm{Re}\, x^{1/2}>0$ and $\mathrm{Re}\, y^{1/2}>0$.

Then, we have

$$f(x, \alpha, \beta, \gamma)g(x, \alpha, \beta, \gamma)=x^2(x-\alpha)^2P(x)+(x-\beta)^2(x-\gamma)^2Q(x).$$

Consider the system of equations:

$$f(x, \alpha, \beta, \gamma)=0, \tag{20}$$

$$\frac{\partial}{\partial x}f(x, \alpha, \beta, \gamma)=0, \tag{21}$$

$$g(y, \alpha, \beta, \gamma)=0, \tag{22}$$

$$\frac{\partial}{\partial y}g(y, \alpha, \beta, \gamma)=0. \tag{23}$$

Then, we have an analytic variety $V=V(f, \partial f/\partial x, g, \partial g/\partial y)$, in the $(x, y, \alpha, \beta, \gamma)$-space, whose dimension is at least one. We shall show that, for an arbitrary $\varepsilon>0$, $(\varepsilon_0>\varepsilon)$ there is a point $(x_1, y_1, \alpha_1, \beta_1, \gamma_1)\in V$ such that

$$0<|x_1|+|y_1|+|\alpha_1|+|\beta_1|+|\gamma_1|<\varepsilon.$$

Let $F(x)=\dfrac{A(x)}{B(x)}=c_0+c_1x+\cdots$ and let $G(x)=x^{3/2}F(x)$, $\mathrm{Re}\, x^{1/2}>0$.

From the equation (20)-(23), we have

$$G(x)\alpha-x(\beta+\gamma)+\beta\gamma=xG(x)-x^2, \tag{24}$$

$$G(y)\alpha+y(\beta+\gamma)-\beta\gamma=yG(y)+y^2, \tag{25}$$

$$G'(x)\alpha-(\beta+\gamma)=G(x)+xG'(x)-2x, \tag{26}$$

$$G'(y)\alpha+(\beta+\gamma)=G(y)+yG'(y)+2y. \tag{27}$$

By (26) and (27), we have

(28) $$(G'(x)+G'(y))\alpha=G(x)+G(y)+xG'(x)+yG'(y)-2(x-y)$$

and

$$(xG'(x)+yG'(y))\alpha-(x-y)(\beta+\gamma)$$
$$=xG(x)+yG(y)+x^2G'(x)+y^2G'(y)-2(x^2-y^2).$$

From (24) and (25) we have

$$(G(x)+G(y))\alpha-(x-y)(\beta+\gamma)=xG(x)+yG(y)-2(x^2-y^2).$$

Hence, we have

(29) $$((xG'(x)+yG'(y))-(G(x)+G(y)))\alpha=x^2G'(x)+y^2G'(y)-(x^2-y^2).$$

Eliminating α from (28) and (29), we have

$$H(x, y)=(G(x)+G(y))^2-2(x-y)(G(x)+G(y))$$
$$+(x-y)^2(G'(x)G'(y)+G'(x)-G'(y))=0.$$

$H(x, y)$ is holomorphic in $(\{|x|<\varepsilon^2\}-\{x\leqq 0\})\times(\{|y|<\varepsilon^2\}-\{y\leqq 0\})$. Putting $x=s^2$ and $y=t^2$ and noting that $G'(x)=x^{1/2}(3F(x)/2+xF'(x))$, we have

$$h(s, t)=H(s^2, t^2)$$
$$=(s^3F(s^2)+t^3F(t^2))^2-2(s^2-t^2)(s^3F(s^2)+t^3F(t^2))$$
$$+\frac{1}{2}(s^2-t^2)^2(s(3F(s^2)+2s^2F'(s^2))-t(3F(t^2)+2t^2F'(t^2))$$
$$+st(3F(s^2)+2s^2F'(s^2))((3/2)F(t^2)+t^2F'(t^2))).$$

Then, $h(s, t)$ is holomorphic in $\{|s|<\varepsilon\}\times\{|t|<\varepsilon\}$ and $h(\cdot, 0)$ has a zero of order 5 at $s=0$. Therefore, $h(\cdot, t)$ has five zeros (counting multiplicity) near $s=0$ for any t.

Since

$$-2(\lambda^2-1)(\lambda^3+1)+\frac{3}{2}(\lambda^2-1)^2(\lambda-1)=-\frac{1}{2}(\lambda-1)(\lambda+1)^4,$$

we have

$$h(\lambda t, t)=-\frac{1}{2}(\lambda-1)(\lambda+1)^4c_0t^5(1+O(t)).$$

Hence, for a sufficiently small arbitrary t, there is a λ sufficiently close by 1 so that $h(\lambda t, t)=0$. Hence, there is a pair of (x, y) such that $\mathrm{Re}\, x>0$, $\mathrm{Re}\, y>0$ and $H(x, y)=0$.

Hence, there is a point $(x_1, y_1, \alpha_1, \beta_1, \gamma_1)\in V$ such that

$$0<|x_1|+|y_1|+|\alpha_1|+|\beta_1|+|\gamma_1|<\varepsilon.$$

Assume $x_1=y_1$. By (20) and (22), we have

$$x_1^{3/2}(x_1-\alpha_1)A(x_1)=0 \quad \text{and} \quad (x_1-\beta_1)(x_1-\gamma_1)B(x_1)=0.$$

Since x_1 is a double zero of

$$x^{3/2}(x-\alpha_1)A(x)\pm(x-\beta_1)(x-\gamma_1)B(x),$$

x_1 is also a double zero of $x^{3/2}(x-\alpha_1)A(x)$. Hence, we have

$$x_1=y_1=\alpha_1=\beta_1=\gamma_1=0.$$

This is a contradiction.

If $\beta_1=0$ or $\gamma_1=0$, then $x_1=0$ (resp. $y_1=0$) is a double zero of f (resp. g). Hence, $x_1=y_1=0$. Again a contradiction.

If $\alpha_1=\beta_1$ or $\alpha_1=\gamma_1$, then $x_1=y_1=\alpha_1$. This is also a contradiction.

Assume that $\beta_1=\gamma_1$ for every $(x_1, y_1, \alpha_1, \beta_1, \gamma_1)\in V$. Then, instead of (20)-(23), we have

$$x^{3/2}(x-\alpha)A(x)-(x-\beta)^2B(x)=0,$$

$$x^{1/2}(x-\alpha)\Big(\frac{3}{2}A(x)+xA'(x)\Big)+x^{3/2}A(x)-2(x-\beta)B(x)-(x-\beta)^2B'(x)=0,$$

$$y^{3/2}(y-\alpha)A(y)+(y-\beta)^2B(y)=0,$$

$$y^{1/2}(y-\alpha)\Big(\frac{3}{2}A(y)+yA'(y)\Big)+y^{3/2}A(y)+2(y-\beta)B(y)+(y-\beta)^2B'(y)=0.$$

Eliminating α from these equations, we have

$$x-y=\frac{(x-\beta)^2B(x)}{x^{3/2}A(x)}+\frac{(y-\beta)^2B(y)}{y^{3/2}A(y)}, \tag{30}$$

$$\frac{(x-\beta)^2B(x)}{xA(x)}=\frac{2(x-\beta)B(x)+(x-\beta)^2B'(x)-x^{3/2}A(x)}{3A(x)/2+xA'(x)}, \tag{31}$$

$$\frac{(y-\beta)^2B(y)}{yA(y)}=\frac{2(y-\beta)B(y)+(y-\beta)^2B'(y)+y^{3/2}A(y)}{3A(y)/2+yA'(y)}. \tag{32}$$

From (31) and (32), we have

$$\Big(\frac{3ab}{2}+O(\eta)\Big)(x-\beta)^2-2(ab+O(\eta))x(x-\beta)+(a^2+O(\eta))x^{3/2}=0,$$

$$\Big(\frac{3ab}{2}+O(\eta)\Big)(y-\beta)^2-2(ab+O(\eta))y(y-\beta)-(a^2+O(\eta))y^{3/2}=0,$$

where $\eta=|x|+|y|+|\beta|$. Hence, we have

$$\frac{ab}{2}(1+O(\eta))(x-\beta)(x+(3+O(\eta))\beta)=(a^2+O(\eta))x^{5/2} \tag{33}$$

and

$$\frac{ab}{2}(1+O(\eta))(y-\beta)(y+(3+O(\eta))\beta)=-(a^2+O(\eta))y^{5/2}. \tag{34}$$

From (33), $x-\beta=O(x^{3/2})$ or $x+3\beta=O(x^{3/2})$ and from (34), $y-\beta=O(x^{3/2})$ or $y+3\beta=O(x^{3/2})$.

Assume that $x-\beta=Mx^{3/2}$ and $y-\beta=Ny^{3/2}$. Then, by (33) and (34), we have

$$M=\frac{a}{2b}+o(1) \quad \text{and} \quad N=-\frac{a}{2b}+o(1).$$

Substituting them into (30), we have

$$M-N+o(1)=\left(\frac{b}{a}+o(1)\right)(M^2+N^2+o(1)).$$

Hence, $a/b+o(1)=a/(2b)+o(1)$. A contradiction.

Assume that $x+3\beta=Mx^{3/2}$ and $y-\beta=Ny^{3/2}$. Then,

$$x-y=-4\beta(1+o(1))$$

and

$$\frac{(x-\beta)^2B(x)}{x^{3/2}A(x)}+\frac{(y-\beta)^2B(y)}{y^{3/2}A(y)}=\frac{16\beta^2(b+o(1))}{(-3\beta)^{3/2}(a+o(1))}+\frac{\beta^3(a^2/(4b^2))}{\beta^{3/2}(a+o(1))}.$$

Hence, $b+o(1)=o(1)$. Contradiction.

Assume that $x+3\beta=Mx^{3/2}$ and $y+3\beta=Ny^{3/2}$. Then, by (33) and (34), we have

$$M=\frac{3a}{2b}+o(1) \quad \text{and} \quad N=-\frac{3a}{2b}+o(1).$$

Hence,

$$x-y=\left(\frac{3a}{b}+o(1)\right)((-3\beta)^{3/2}+o(1)$$

and

$$\frac{(x-\beta)^2B(x)}{x^{3/2}A(x)}+\frac{(y-\beta)^2B(y)}{y^{3/2}A(y)}=\frac{16\beta^2(b+o(1))}{(-3\beta)^{3/2}(2a+o(1))}.$$

Hence, $b+o(1)=o(1)$. Contradiction.

Thus, there is a point $(x_1, y_1, \alpha_1, \beta_1, \gamma_1)\in V$ such that $x_1\neq y_1$ and $\alpha_1, \beta_1, \gamma_1$ are mutually distinct.

The rest of the proof is similar to that of Lemma 3.

Let $S\in M_{g,3,n}(\rho_1, \rho_2, \rho_3, \rho_4)$ be defined by the equation (1). There are polynomials Γ_i $(i=1, 2, 3)$ and Π_j $(j=1, 2, 3, 4)$ in x which satisfy (2)-(8).

Then we have:

THEOREM 2. *Assume that $3n-g+1-\rho_2-\rho_4=0$ i.e. Γ_3 is constant and $2n< g\leqq 3n-1$. Then, S is included in the closure of any one of*

i) $M_{g,3,n}(\rho_1-1, \rho_2, \rho_3+2, \rho_4)$, *if* $\rho_1>0$.

ii) $M_{g,3,n}(\rho_1, \rho_2-1, \rho_3+1, \rho_4+1)$, *if* $\rho_2>0$.

iii) $M_{g,3,n}(\rho_1+1, \rho_2-1, \rho_3, \rho_4)$, *if* $\rho_2>0$.
iv) $M_{g,3,n}(\rho_1, \rho_2-1, \rho_3+2, \rho_4)$, *if* $\rho_2>0$.
v) $M_{g,3,n}(\rho_1, \rho_2, \rho_3+1, \rho_4-1)$, *if* $\rho_4>0$.
vi) $M_{g,3,n+1}(\rho_1-1, \rho_2, \rho_3, \rho_4+2)$, *if* $\rho_1>0$.
vii) $M_{g,3,n+1}(\rho_1-1, \rho_2+1, \rho_3, \rho_4)$, *if* $\rho_1>0$.

Proof. Without loss of generality, we may assume that $|a_{i,j}-a_{k,l}|>2$ if $(i, j)\neq(k, l)$.

Next, we shall show that we may assume that $\Gamma_k(a_{i,j})\neq 0$ for $k=1, 2, i=1, \cdots, \rho_j, j=1, \cdots, 4$. Assume that $\Gamma_k(a_{i,j})=0$ for some k, i, j. Let ε be an arbitrary positive number. Then, there is an $\varepsilon^*\neq 0$ such that

$$\Pi_3^*(x)=4\Gamma_1^*(x)^3\Pi_1(x)\Pi_2(x)^2+27\Gamma_2^*(x)^2\Pi_4(x)$$

has ρ_3 simple zeros $a_{i,3}^*(i=1, \cdots, \rho_3)$ and that $|a_{i,3}^*-a_{i,3}|<\varepsilon$. Here, $\Gamma_k^*(x)=\Gamma_k(x)+\varepsilon^*$ $k=1, 2$. Let S^* be defined by

$$y^{*3}+\Gamma_1^*(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)y^*+\Gamma_2^*(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)^2=0.$$

Obviously, we can assume that $\Gamma_k^*(a_{i,j})\neq 0$ for each k, i, j.

Let $\Delta_{i,j}=\{|x-a_{i,j}|<1\}$ and let Δ be a closed disk such that $\Delta_{i,j}\subset\Delta$ for every i, j. Then, for every $x\in\Delta-\cup\Delta_{i,j}$, in the equation (1), y takes three distinct values. It is easy to see that $m=\inf_{1\leq i\leq j\leq 3}|y_i(x)-y_j(x)|>0$, where $y_1(x)$, $y_2(x)$, $y_3(x)$ are three branches of y and x runs over $\Delta-\cup\Delta_{i,j}$. If Γ_k changes continuously, then y_i also varies continuously. Hence, if ε^* is sufficiently small, then for each $i=1, 2, 3$, there is a branch $y_i^*(x)$ of y^* so that $|y_i^*(x)-y_i(x)|<m/2$ on $\Delta-\cup\Delta_{i,j}$. Therefore, for any closed curve γ in $\Delta-\cup\Delta_{i,j}$, the continuations of y and y^*, respectively, along γ induce the same permutations of branches. Hence, S and S^* satisfies the assumption of Lemma 2. Therefore, S can be approximated by such an S^*.

Assume that $\rho_1>0$. Without loss of generality we may assume that $a_{1,1}=0$. Let $x\Pi(x)=\Pi_1(x)$ and let S_α be the Riemann surface defined by

$$y^3+x\Gamma_1(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)y+(x-\alpha)\Gamma_2(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)^2=0.$$

By assumption, every zero of $4\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27\Gamma_2(x)^2\Pi_4(x)$ is simple. Hence, by Lemma 3, for sufficiently small α,

$$\begin{aligned}&4x^3\Gamma_1(x)^3\Pi(x)\Pi_2(x)^2+27\Gamma_2(x)^2(x-\alpha)^2\Pi_4(x)\\&=4x^2\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27(x-\alpha)^2\Gamma_2(x)^2\Pi_4(x)\end{aligned}$$

has ρ_3+2 simple zeros.

By Theorem 1, every S_α corresponds to an element of $M_{g,3,n}(\rho_1-1, \rho_2, \rho_3+2, \rho_4)$. By Lemma 2, S_α tends to S as α tends to 0. This is the case i).

Again, let $x\Pi(x)=\Pi_1(x)$ and let $S_{\alpha,\beta}$ be the Riemann surface defined by

$$y^3+x(x-\alpha)(x-\beta)\Gamma_1(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)y$$
$$+(x-\alpha)^2(x-\beta)^2\Gamma_2(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)^2=0.$$

Using Lemma 5, we can choose sufficiently small α and β so that

$$4x^3\Gamma_1(x)^3\Pi(x)\Pi_2(x)^2+27\Gamma_2(x)^2(x-\alpha)(x-\beta)\Pi_4(x)$$
$$=4x^2\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27(x-\alpha)(x-\beta)\Gamma_2(x)^2\Pi_4(x)$$

has ρ_3 simple zeros and exactly one double zero near $x=0$.

By Theorem 1, every $S_{\alpha,\beta}$ corresponds to an element of $M_{g,3,n+1}(\rho_1-1, \rho_2, \rho_3, \rho_4+2)$. Since $S_{0,0}$ is defined by

$$y^3+x^3\Gamma_1(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)y+x^4\Gamma_2(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)^2$$
$$=y^3+x^2\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)y+x^3\Gamma_2(x)\Pi_1(x)\Pi_2(x)^2\Pi_4(x)^2=0,$$

it is equivalent to S. Again, by Lemma 2, we have the case vi).

Let $x\Pi(x)=\Pi_1(x)$ and let $S_{\alpha,\beta,\gamma}$ be the Riemann surface defined by

$$y^3+x(x-\alpha)^2\Gamma_1(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)y$$
$$+(x-\alpha)^2(x-\beta)(x-\gamma)\Gamma_2(x)\Pi(x)\Pi_2(x)^2\Pi_4(x)^2=0.$$

Since $4\Gamma_1^3\Pi_1\Pi_2^2+27\Gamma_2^2\Pi_4$ has exactly ρ_3 simple zeros, using Lemma 6, we can choose sufficiently small α, β and γ so that

$$4x^3\Gamma_1(x)^3\Pi(x)(x-\alpha)^2\Pi_2(x)^2+27(x-\beta)^2(x-\gamma)^2\Gamma_2(x)^2\Pi_4(x)$$
$$=4x^2(x-\alpha)^2\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27(x-\beta)^2(x-\gamma)^2\Gamma_2(x)^2\Pi_4(x)$$

has ρ_3 simple zeros and two double zeros. By Theorem 1, every $S_{\alpha,\beta,\gamma}$ corresponds to an element of $M_{g,3,n+1}(\rho_1-1, \rho_2+1, \rho_3, \rho_4)$. Hence, we have the case vii).

Assume that $\rho_2>0$. Again without loss of generality, we may assume that $a_{1,2}=0$.

Let $x\Pi(x)=\Pi_2(x)$ and let S_α be the Riemann surface defined by

$$y^3+x(x-\alpha)\Gamma_1(x)\Pi_1(x)\Pi(x)^2\Pi_4(x)y+(x-\alpha)^2\Gamma_2(x)\Pi_1(x)\Pi(x)^2\Pi_4(x)^2=0.$$

By a similar argument as above, we have S_α is an element of $M_{g,3,n}$ $(\rho_1, \rho_2-1, \rho_3+1, \rho_4+1)$. Hence, we have the case ii).

Again, let $x\Pi(x)=\Pi_2(x)$ and let $S_{\alpha,\beta}$ be the Riemann surface defined by

$$y^3+x(x-\beta)\Gamma_1(x)\Pi_1(x)\Pi(x)^2\Pi_4(x)y$$
$$+(x-\alpha)(x-\beta)\Gamma_2(x)\Pi_1(x)\Pi(x)^2\Pi_4(x)^2=0.$$

Using Lemma 4 for $k=1$, we can find arbitrary small α and β so that

$$4x^3\Gamma_1(x)^3(x-\beta)\Pi_1(x)\Pi(x)^2+27(x-\alpha)^2\Gamma_2(x)^2\Pi_4(x)$$
$$=4x(x-\beta)\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27(x-\alpha)^2\Gamma_2(x)^2\Pi_4(x)$$

has one double zero and ρ_3 simple zeros. Hence, we have the case iii).

Let $x\Pi(x)=\Pi_2(x)$ and let $S_{\alpha,\beta}$ be the Riemann surface defined by

$$y^3+x(x-\beta)\Gamma_1(x)\Pi_1(x)\Pi(x)^2\Pi_4(x)y+(x-\alpha)^2\Gamma_2(x)\Pi_1(x)\Pi(x)^2\Pi_4(x)^2=0.$$

Using Lemma 4 for $k=2$, we can find arbitrary small α and β so that

$$4x^3\Gamma_1(x)^3(x-\beta)^3\Pi_1(x)\Pi_2(x)^2+27(x-\alpha)^4\Gamma_2(x)^2\Pi_4(x)$$
$$=4x(x-\beta)^3\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27(x-\alpha)^4\Gamma_2(x)^2\Pi_4(x)$$

has one double zero and ρ_3+2 simple zeros. Hence, we have the case iv).

Finally, assume that $\rho_4>0$ and assume that $a_{1,4}=0$. Let $x\Pi(x)=\Pi_4(x)$ and let $S_{\alpha,\beta}$ be the Riemann surface defined by

$$y^3+(x-\alpha)\Gamma_1(x)\Pi_1(x)\Pi_2(x)^2\Pi(x)y+x(x-\beta)\Gamma_2(x)\Pi_1(x)\Pi_2(x)^2\Pi(x)^2=0.$$

Using Lemma 4, we can choose sufficiently small α and β so that

$$4(x-\alpha)^3\Gamma_1(x)^3\Pi_1(x)\Pi(x)^2+27x^2(x-\beta)^2\Gamma_2(x)^2\Pi(x)$$
$$=4(x-\alpha)^3\Gamma_1(x)^3\Pi_1(x)\Pi_2(x)^2+27x(x-\beta)^2\Gamma_2(x)^2\Pi_4(x)$$

has one double zero and ρ_3+1 simple zeros. Hence, we have the case v).

This completes the proof.

Making use of Theorem 1 iii) instead of ii), we have another sort of inclusion relation.

THEOREM 3. *If $\rho_1\leqq 2g-3n+1$, $\rho_2\leqq 3n-g+1$, $\rho_1+\rho_2\leqq n-1$ and $2n\leqq g<3n+1$, then there exists an $S\in M_{g,3,n}(\rho_1, \rho_2, \rho_3, 0)$ so that S is included in the closure of any one of*

i) $M_{g,3,n}(\rho_1-1, \rho_2, \rho_3+2, 0)$, *if* $\rho_1>0$.
ii) $M_{g,3,n}(\rho_1, \rho_2-1, \rho_3+2, 0)$, *if* $\rho_2>0$.
iii) $M_{g,3,n+1}(\rho_1-1, \rho_2+1, \rho_3, 0)$, *if* $\rho_1>0$ *and* $2n+2\leqq g$.
iv) $M_{g,3,n+1}(\rho_1, \rho_2, \rho_3, 0)$, *if* $2n+2\leqq g$.
v) $M_{g,3,n+1}(\rho_1+1, \rho_2-1, \rho_3, 0)$, *if* $\rho_2>0$ *and* $2n+4\leqq g$.

Proof. Let $\rho_4=2g-3n+1-\rho_1$ and $m=g-n$. Then, we have

$$\rho_2+2\rho_1+\rho_4\leqq 2m+2,$$
$$\rho_2+2\rho_1+2\rho_4\leqq 3m+3,$$
$$\rho_1+\rho_4=3m-g+1.$$

As is stated in the section 1, we have proved in [6] that there is a trigonal Riemann surface of genus g defined by an equation such as (1) satisfying (2)-

(8), where n, ρ_1, ρ_2 and ρ_3 are replaced by m, ρ_2, ρ_1 and $\rho_3-\rho_4$, respectively. Since $n=g-m$ and $3m-g+1-\rho_1-\rho_4=0$, by Theorem 1, we have that $S\in M_{g,3,n}(\rho_1, \rho_2, \rho_3, 0)$. The rest of the proof is done by the same procedure as that of Theorem 2, i.e. the cases i), ii), iii), iv) and v) correspond to the cases ii), i), iv), v) and vii), respectively. We omit the details.

REFERENCES

[1] M. COPPENS, The Weierstrass gap sequences of the total ramification points of trigonal covering of $\mathbf{P}^1$, Indag. Math., **47** (1985), 245-270.
[2] M. COPPENS, The Weierstrass gap sequences of the ordinary ramification points of trigonal covering of $\mathbf{P}^1$; Existence of a kind of Weierstrass gap sequence, J. pure appl. Algebra, **43** (1986), 11-25.
[3] M. COPPENS, Weierstrass points on trigonal curves I: The ramification points, Preprint R.U. Utrecht, **430** (1986).
[4] F.P. GARDINER, Schiffer's interior variation and quasiconformal mapping, Duke Math. J., **42** (1975), 371-380.
[5] T. KATO, On Weierstrass points whose first non-gaps are three, J. reine angew. Math., **316** (1980), 99-109.
[6] T. KATO AND R. HORIUCHI, Weierstrass gap sequences at the ramification points of trigonal Riemann surfaces, J. pure appl. Algebra, **50** (1988), 271-285.

DEPARTMENT OF MATHEMATICS
YAMAGUCHI UNIVERSITY
YOSHIDA, YAMAGUCHI 753, JAPAN