J.M. MØLLER
KODAI MATH. J.
11 (1988), 372—378

SELF-MAPS ON TWISTED EILENBERG-MACLANE SPACES

BY JESPER MICHAEL MØLLER

1. Introduction.

To any (based) space X is associated the monoid $(\sigma(X, *))$ $\sigma(X)$ of (based) homotopy classes of (based) self-maps of X. This monoid contains as its group of units the group $(\varepsilon(X, *))$ $\varepsilon(X)$ of (based) homotopy classes of (based) homotopy equivalences of X.

Let π be any group, A a $\boldsymbol{Z}(\pi)$-module, and denote by $L:=L(A, n)$, $n\geqq 2$, the unique homotopy type with $\pi_1(L)=\pi$, $\pi_n(L)=A$, $\pi_i(L)=0$ for $i\neq 1, n$, that realizes A as a $\pi_1(L)=\pi$-module and has k-invariant $k=0\in H^{n+1}(\pi, A)$.

The purpose of this note is to determine $\sigma(L, *)$ and $\sigma(L)$ explicitly in terms of group theoretic invariants, see Theorems 3.2 and 3.4.

The monoid $\sigma(L, *)$, or rather its subgroup of units $\varepsilon(L, *)$, has attracted some interest in recent years [7], [9], [10], [1], [2] but as far as I know, no explicit formula has been given, at least not in the case of a non-abelian fundamental group.

Throughout this note, I use the notation of [8]: If (X, A) is a pair of spaces, $p: Y\to B$ a fibration, and $u: X\to Y$ a continuous map, then $F_u(X, A; Y, B)$ is the space, equipped with the compactly generated topology associated to the compact-open topology, of all maps $v: X\to Y$ such that $v|A=u|A$ and $pv=pu$. An empty space in the A-entry or a one-point space in the B-entry will be omitted; thus e.g. $F_1(X, *; X)$ is the space of all based self-maps of X.

2. Strategy of proof.

Let $\omega: E\pi\to B\pi$ be a universal numerable principal π-bundle. Then $E\pi$ is a contractible free right π-space. Moreover, E and B are functors: For any group endomorphism $\alpha: \pi\to\pi$, denote by $E\alpha: E\pi\to E\pi$ and $B\alpha: B\pi\to B\pi$ the induced maps. Equip $E\pi$ and $B\pi$ with base points $e_0\in E\pi$, $b_0=\omega(e_0)\in B\pi$ fixed by all the maps $E\alpha$ and $B\alpha$, respectively.

Let π be any group and A a $\boldsymbol{Z}(\pi)$-module. Realize the Eilenberg-MacLane space $K(A, n)$, $n\geqq 2$, as a strictly associative H-space with strict unit $0\in K(A, n)$. Since A is a π-module, π acts from the left on $K(A, n)$ by topological group homeomorphisms. As a model for $L(A, n)$, take the total space [5] of the associated fibre bundle

Received February 22, 1988

$$L(A, n)=E\pi\times_{\pi}K(A, n)\xrightarrow{p}B\pi.$$

Since $0\in K(A, n)^{\pi}$, there is a section s given by $s(e\pi)=(e, 0)\pi$, $e\in E\pi$.

In the following, I use the abbreviations $E=E\pi$, $B=B\pi$, $K=K(A, n)$, and $L=L(A, n)$.

Let $(\mathcal{F}_1(L\,;*;L))\ \mathcal{F}_1(L\,;L)$ be the space of all (based) fibre maps of L; i.e. (based) maps $\bar{u}: L\to L$ such that $p\bar{u}=up$ for some (based) map $u: B\to B$. Then there are pull back diagrams

$$\begin{array}{ccc} \mathcal{F}_1(L\,;L) & \longrightarrow & F_1(L\,;L) \\ \downarrow & & \downarrow{\scriptstyle \underline{p}} \\ F_1(B\,;B) & \xrightarrow{\bar{p}} & F_p(L\,;B) \end{array} \qquad \begin{array}{ccc} \mathcal{F}_1(L, *;L) & \longrightarrow & F_1(L, *;L) \\ \downarrow & & \downarrow{\scriptstyle \underline{p}} \\ F_1(B, *;B) & \xrightarrow{\bar{p}} & F_p(L, *;B) \end{array}$$

where $\underline{p}$ is post-composition and $\bar{p}$ pre-composition with p. Since $\underline{p}$ is a fibration and $\bar{p}$ a weak homotopy equivalence [3], [6] the left hand vertical maps, $\bar{u}\to u$, are fibrations and the inclusions $\mathcal{F}_1(L\,;L)\subset F_1(L\,;L)$ and $\mathcal{F}_1(L, *;L)\subset F_1(L, *;L)$ are weak homotopy equivalences and morphisms of topological monoids. Consequently

$$\sigma(L)=\pi_0\mathcal{F}_1(L\,;L),\quad \sigma(L, *)=\pi_0\mathcal{F}_1(L, *;L)$$

as monoids.

By composing, in the case of free maps, with the evaluation fibration $F_1(B\,;B)\to B$, we obtain a third fibration of the form

$$\mathcal{F}_1(L, *;L)\longrightarrow \mathcal{F}_1(L\,;L)\longrightarrow B$$

and thus $\sigma(L)=\sigma(L, *)/\pi$ is known once $\sigma(L, *)$ is known as a monoid with π-action. But $\sigma(L, *)$ is actually easily determined since each component of the base space $F_1(B, *;B)$ is weakly contractible [3], [6] and the π-action will follow from Lemma 2.1 below.

For any group element $\eta\in\pi$, $\bar{\eta}\in\mathrm{Aut}(\pi)$ will denote conjugation by η, $\bar{\eta}(\zeta)=\eta\zeta\eta^{-1}$ for $\zeta\in\pi$.

LEMMA 2.1. *There exist maps $\bar{\mu}: E\times E\to E$ and $\mu: E\times B\to B$ such that $\omega\bar{\mu}=\mu(1\times\omega)$ and*

(1) $$\bar{\mu}(e_0\eta, e)=(E(\bar{\eta})e)\eta,\ \bar{\mu}(e, e_0\eta)=e\eta$$

(2) $$\bar{\mu}(e_1\eta, E(\bar{\eta})^{-1}e_2)=\bar{\mu}(e_1, e_2)\eta$$

(3) $$\bar{\mu}(e_1, e_2\eta)=\bar{\mu}(e_1, e_2)\eta$$

for all $e, e_1, e_2\in E$ and $\eta\in\pi$.

Proof. Consider the maps

$$\bar{\mu}: (E\times e_0\pi)\cup(e_0\pi\times E) \longrightarrow E$$

$$\mu': (E\times b_0)\cup(e_0\pi\times B) \longrightarrow B$$

given by $\bar{\mu}(e, e_0\eta)=e\eta$, $\bar{\mu}(e_0\eta, e)=(E(\bar{\eta})e)\eta$, $\mu'(e, b_0)=\omega(e)$, and $\mu'(e_0\eta, b)=B(\bar{\eta})b$. Note that $\bar{\mu}$ and μ' are well defined and that $\omega\bar{\mu}=\mu'(1\times\omega)$. Equip $E\times B$ with the free right π-action

$$(e, b)\eta=(e\eta, B(\bar{\eta})^{-1}b) \qquad e\in E,\ b\in B,\ \eta\in\pi.$$

Note that $(E\times b_0)\cup(\pi\times B)$ is a π-invariant subspace and μ' a π-invariant map. Let μ be the map induced by μ' on the orbits. Then the diagram

$$\begin{array}{ccc}
(E\times\pi)\cup(\pi\times E) & \xrightarrow{\ \bar{\mu}\ } & E \\
\downarrow{\scriptstyle 1\times\omega} & & \downarrow{\scriptstyle \omega} \\
(E\times b_0)\cup(\pi\times B) & \xrightarrow{\ \mu'\ } & B \\
\downarrow & & \\
(E\times B)/\pi\supset(E\times b_0\cup\pi\times B)/\pi & &
\end{array}$$

commutes and so does the induced diagram

$$\begin{array}{ccccc}
\pi_2((E, \pi)\times(E, \pi)) & \xrightarrow{\ \partial_3\ } & \pi_1(E\times\pi\cup\pi\times E) & \xrightarrow{\ \bar{\mu}_*\ } & \pi_1(E)=1 \\
\cong\downarrow & & \downarrow & & \omega_*\downarrow \\
\pi_2((E, \pi)\times(B, b_0)) & \xrightarrow{\ \partial_2\ } & \pi_1(E\times b_0\cup\pi\times B) & \xrightarrow{\ \mu'_*\ } & \pi_1(B) \\
\cong\downarrow & & \downarrow & & \\
\pi_2(((E, \pi)\times(B, b_0))/\pi) & \xrightarrow{\ \partial_1\ } & \pi_1((E\times b_0\cup\pi\times B)/\pi) & &
\end{array}$$

where ∂_3, ∂_2, ∂_1 denote boundary maps. The left hand vertical maps are all isomorphisms and hence $\mu_*\partial_1$ is trivial. But the inclusion

$$\pi_1((E\times b_0\cup\pi\times B)/\pi) \longrightarrow \pi_1((E\times B)/\pi)$$

is an epimorphism since

$$\pi_1(((E, \pi)\times(B, b_0))/\pi)\cong\pi_1((E, \pi)\times(E, \pi))\cong\pi_0(E\times\pi\cup\pi\times E)=1$$

and thus $\mu_*\partial_1$ is the obstruction to extending μ. Hence μ extends to $(E\times B)/\pi$.

Let now also μ denote an extension of μ. By covering space theory there exists a (unique) based lift $\bar{\mu}: E\times E\to E$ of $\mu(1\times\omega)$ which extends the given life $\bar{\mu}$ on $(E\times\pi)\cup(\pi\times E)$. It is not hard to see that $\bar{\mu}$ has the properties (1)-(3). □

COROLLARY 2.2. *The equivariant self-map on* E *given by* $e\to(E(\bar{\zeta})e)\zeta$, $\zeta\in\pi$, *is* π*-homotopic to the identity map.*

Proof. $\bar{\mu}(e_0\zeta, e)=(E(\bar{\zeta})e)\zeta$, $\bar{\mu}(e_0, e)=e$, e_0 and $e_0\zeta$ can be connected by a path. □

3. Construction of self-maps.

For any group G, let $\mathrm{End}(G)$ denote the monoid (under composition) of group endomorphisms of G. For $\alpha\in\mathrm{End}(\pi)$, let

$$\mathrm{End}(A)_\alpha=\{\varphi\in\mathrm{End}(A)\mid\forall\zeta\in\pi,\ a\in A:\varphi(\zeta a)=\alpha(\zeta)\varphi(a)\}$$

$$F_0(E,e_0;K)_\alpha=\{x:(E,e_0)\longrightarrow(K,0)\mid\forall\zeta\in\pi,\ e\in E:x(e\zeta)=\alpha(\zeta)^{-1}x(e)\}$$

$\mathrm{End}(A)_\alpha$ is viewed as a discrete space and $F_0(E,e_0;K)_\alpha$ as a subspace of $F_0(E,e_0;K)$. Algebraically $\mathrm{End}(A)_\alpha$ is an abelian group under pointwise addition and $\mathrm{End}_\pi(A)=\mathrm{End}(A)_1$ is also a monoid under composition of maps.

Consider the disjoint union

$$\sum(L;*):=\bigcup_{\alpha\in\mathrm{End}(\pi)}F_0(E,e_0;K)_\alpha\times\mathrm{End}(A)_\alpha.$$

equipped with the product

$$(x,\varphi)_\alpha\cdot(u,\phi)_\beta=(x\circ E\beta+\varphi y,\varphi\phi)_{\alpha\beta}$$

where a typical element of $\sum(L,*)$ is denoted by $(x,\varphi)_\alpha$ for

$$\alpha\in\mathrm{End}(\pi),\quad\varphi\in\mathrm{End}(A)_\alpha,\quad x\in F_0(E,e_0;K)_\alpha.$$

This product is associative and $(0,1)_1$ is a unit element so $(\sum(L,*),\cdot)$ is a topological monoid.

Define a map $F:\sum(L,*)\rightarrow\mathcal{F}_1(L,*;L)$ by the formula

$$F((x,\varphi)_\alpha)((e,k)\pi)=(E\alpha(e),x(e)+\varphi(k))\pi$$

for $(x,\varphi)_\alpha\in\sum(L,*)$, $e\in E$, and $k\in K$. (Since $K(-,n)$ has a functorial construction, we may confuse $\varphi\in\mathrm{End}(A)_\alpha$ with the induced map $K(\varphi,n)=\varphi$; $+$ refers to the H-space structure of K.)

LEMMA 3.1. *F is a morphism of topological monoids and $\pi_0(F):\pi_0\sum(L,*)\rightarrow\pi_0\mathcal{F}_1(L,*;L)=\sigma(L,*)$ is an isomorphism of monoids.*

Proof. A direct verification shows that F respects the monoid structures. As each component of $F_1(B,*;B)$ is weakly contractible and $\pi_0=\mathrm{End}(\pi)$, one of the fibrations of the preceeding section shows that, as a set,

$$\pi_0\mathcal{F}_1(L,*;L)=\bigcup_{\alpha\in\mathrm{End}(\pi)}\pi_0F_{s\circ B\alpha\circ p}(L,*;L,B)$$

Furthermore, the restriction of $\pi_0(F)$,

$$\pi_0F_0(E,e_0;K)_\alpha\times\mathrm{End}(A)_\alpha\longrightarrow\pi_0F_{s\circ B\alpha\circ p}(L,*;L,B)$$

is bijective according to the split exact sequence of ([5], p. 4)

$$\begin{array}{ccccc} \pi_0 F_{s\circ B\alpha}(B, *; L, B) & \underset{\bar{s}}{\overset{\bar{p}}{\rightleftarrows}} & \pi_0 F_{s\circ B\alpha\circ p}(L, *; L, B) & & \\ \| & & \| & & \\ 0 \longrightarrow \bar{H}^n(B; \alpha^*A) & \underset{s^*}{\overset{p^*}{\rightleftarrows}} & \bar{H}^n(L; \alpha^*A) & \overset{i^*}{\longrightarrow} \mathrm{End}(A)_\alpha & \longrightarrow 0 \end{array}$$

combined with the facts that $p: L \to B$ classifies cohomology with local coefficients [1] and ([4], Theorem 4.8.1) $F_0(E, e_0; K)_\alpha = F_{S\circ B\alpha}(B, *; L, B)$. □

A typical element of

$$\pi_0 \Sigma(L, *) = \bigcup_{\alpha \in \mathrm{End}(\pi)} H^n(B; \alpha^*A) \times \mathrm{End}(A)_\alpha$$

will, by a slight abuse of notation, also be denoted by $(x, \varphi)_\alpha$ where $\alpha \in \mathrm{End}(\pi)$, $x \in H^n(B; \alpha^*A)$, and $\varphi \in \mathrm{End}(A)_\alpha$. Note that if $\alpha, \beta \in \mathrm{End}(\pi)$ and $\varphi \in \mathrm{End}(A)_\alpha$, composition with φ induces a coefficient group homomorphism $\varphi_*: H^n(B; \beta^*A) \to H^n(B; (\alpha\beta)^*A)$.

An immediate corollary of Lemma 3.1 is

THEOREM 3.2. *The monoid $\sigma(L, *)$ of based homotopy classes of based self-maps of L is isomorphic to*

$$(\pi_0 \Sigma(L, *), \cdot)$$

where $(x, \varphi)_\alpha \cdot (y, \psi)_\beta = (B\beta^*(x) + \varphi_*(y), \varphi\psi)_{\alpha\beta}$. In particular, there exists a short exact sequence

$$1 \longrightarrow \mathrm{Ext}^n_\pi(\boldsymbol{Z}, A) \rtimes \mathrm{End}_\pi(A) \longrightarrow \sigma(L, *) \longrightarrow \mathrm{End}(\pi) \longrightarrow 1$$

of monoids.

The next goal is to describe the monoid of free maps $\sigma(L)$.

For $\eta \in \pi$ and

$$(x, \varphi)_\alpha \in \Sigma(L, *), \quad \text{let}$$

$$\eta(x, \varphi)_\alpha = (\eta x, \eta\varphi)_{\bar{\eta}\alpha}$$

this defines a left π-action on $\Sigma(L, *)$ which doesn't respect the monoid structure, though. Indeed one easily verifies

PROPOSITION 3.3. *Suppose $(x, \varphi)_\alpha, (y, \eta)_\beta \in \Sigma(L, *)$ and $\eta, \zeta \in \pi$. Then*

$$(\eta(x, \varphi)_\alpha) \cdot (\zeta(y, \psi)_\beta) = \eta\alpha(\zeta)(\alpha(\zeta)^{-1} x \circ E(\bar{\zeta}\beta) + \varphi y, \varphi\psi)_{\alpha\bar{\zeta}}$$

in the monoid $\Sigma(L, *)$.

There is, however, an induced π-action on $\pi_0\Sigma(L, *)$ given by

$$\eta(x, \varphi)_\alpha=(\eta_*(x), \eta\varphi)_{\bar{\eta}\alpha}$$

where $\eta_*: H^n(B; \alpha^*A)\to H^n(B; (\bar{\eta}\alpha)^*A)$ is the coefficient group homomorphism induced by $\eta\in\mathrm{End}(A)_{\bar{\eta}}$. Since, in the situation of Proposition 3.3,

$$\alpha(\zeta)^{-1}x(E(\bar{\zeta}\beta)(e)=x(E(\bar{\zeta}\beta)(e)\zeta)$$

for any $e\in E$, Corollary 2.2 implies that the formula

$$(\eta(x, \varphi)_\alpha)\cdot(\zeta(y, \psi)_\beta)=\eta\alpha(\zeta)((x, \varphi)_\alpha\cdot(y, \psi)_\beta)$$

does hold in the monoid $\pi_0\Sigma(L, *)$ of components. Hence the monoid structure on $\pi_0\in(L, *)$ descends to one on the orbit set $\pi_0\Sigma(L, *)/\pi$.

THEOREM 3.4. *The monoid $\sigma(L)$ of free homotopy classes of free self maps of L is isomorphic to $\pi_0\Sigma(L, *)/\pi$. In particular there exists a short exact sequence of monoids*

$$1 \longrightarrow \mathrm{Ext}^n_\pi(\boldsymbol{Z}, A)\rtimes\mathrm{End}_\pi(A)/Z \longrightarrow \sigma(L) \longrightarrow \mathrm{End}(\pi)/\mathrm{Inn}(\pi) \longrightarrow 1$$

where $Z=\{(0, \varphi)\,|\,\varphi(a)=za$ for some $z\in Z(\pi)\}$, $Z(\pi)$ the center of π, and $\mathrm{Inn}(\pi)$ *is the group of inner automorphisms of π.*

Proof. Extend F to a B-map

$$\begin{array}{ccc} E\times_\pi\Sigma(L, *) & \xrightarrow{F} & \mathcal{F}_1(L; L) \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

by the formula

$$F((e_1, (x, \varphi)_\alpha)\pi)((e, k)\pi)=(\bar{\mu}(e_1, E\alpha(e)), x(e)+\varphi(k))\pi$$

where e_1, $e\in E$, $(x, \varphi)_\alpha\in\Sigma(L, *)$, $k\in K$, and $\bar{\mu}$ is the H-space structure on E from Lemma 2.1.

Note that F is well defined and that $F((e_1, (x, \varphi)_\alpha)\pi)$ is really a fiber map. F induces a map F_* between the homotopy sequences of the two fibrations and since $F_*: \pi_0\Sigma(L, *)\to\pi_0\mathcal{F}_1(L, *; L)$ is an isomorphism of monoids by Theorem 3.2, it follows that also

$$F_*: \pi_0(E\times_\pi\Sigma(L, *))=\pi_0\Sigma(L, *)/\pi \longrightarrow \pi_0\mathcal{F}_1(L; L)=\sigma(L)$$

is an isomorphism of monoids.

The epimorphism $\pi_0\Sigma(L, *)/\pi\to\mathrm{End}(\pi)/\mathrm{Inn}(\pi)$ which takes $\pi(x, \varphi)_\alpha$ to α has kernel equal to the orbit set of

$$I=\bigcup_{\alpha\in\mathrm{Inn}(\pi)} H^n(B; \alpha^*A)\times\mathrm{End}(A)_\alpha$$

and the epimorphism $H^n(B; A) \rtimes \mathrm{End}_\pi(A) \to I/\pi$ given by $(x, \varphi) \to \pi(x, \varphi)_1$ for $x \in H^n(B; A)$, $\varphi \in \mathrm{End}_\pi(A)$, has kernel Z. □

Extraction of units yields

COROLLARY 3.5. *The group* $\varepsilon(L, *)$ *is isomorphic to the set*

$$\bigcup_{\alpha \in \mathrm{Aut}(\pi)} H^n(B; \alpha^* A) \times \mathrm{Aut}(A)_\alpha$$

equipped with the product of Theorem 3.2, $\varepsilon(L) = \varepsilon(L, *)/\pi$, *and there are short exact sequences of groups*

$$1 \longrightarrow \mathrm{Ext}^n_\pi(\boldsymbol{Z}, A) \rtimes \mathrm{Aut}_\pi(A) \longrightarrow \varepsilon(L, *) \longrightarrow \mathrm{Aut}(\pi) \longrightarrow 1$$

$$1 \longrightarrow \mathrm{Ext}^n_\pi(\boldsymbol{Z}, A) \rtimes \mathrm{Aut}_\pi(A)/Z \longrightarrow \varepsilon(L) \longrightarrow \mathrm{Out}(\pi) \longrightarrow 1$$

where $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ *is the group of outer automorphisms of* π.

REFERENCES

[1] Y. ANDO AND K. YAMAGUCHI, On Homotopy Self-Equivalences of the Product A×B. Proc. Japan Acad. 58, Ser. A (1982), 323-325.

[2] G. DIDIERJEAN, Homotopie de l'espace des équivalences d'homotopie fibrées. Ann. Inst. Fourier, Grenoble **35** (1985), 33-47.

[3] D.H. GOTTLIEB, Covering transformations and universal fibrations. Illinois J. Math. **13** (1969), 432-437.

[4] D. HUSEMOLLER, Fibre Bundles, Second Edition. Graduate Texts in Mathematics 20, Springer-Verlag, Berlin-Heidelberg-New York 1975.

[5] J.F. MCCLENDON, Obstruction Theory in Fiber Spaces. Math. Z. **120** (1971), 1-17.

[6] J.M. MØLLER, Spaces of sections of Eilenberg-MacLane spaces. Pacific J. Math. **130** (1987), 171-186.

[7] W. SHIH, On the group $\varepsilon(X)$ of homotopy equivalence maps. Bull. Amer. Math. Soc. **492** (1964), 361-365.

[8] R.M. SWITZER, Counting elements in homotopy sets. Math. Z. **178** (1981), 527-554.

[9] K. TSUKIYAMA, Note on self-maps inducing the identity automorphism of homotopy groups. Hiroshima Math. J. **5** (1975), 215-222.

[10] K. TSUKIYAMA, Self-homotopy-equivalences of a space with two nonvanishing homotopy groups. Proc. Amer. Math. Soc. **79** (1980), 134-138.

[11] G.W. WHITEHEAD, Elements of Homotopy Theory. Graduate Texts in Mathematics 61, Springer-Verlag, Berlin-Heidelberg-New York 1978.

MATHEMATICAL INSTITUTE
UNIVERSITETSPARKEN 5
DK-2100 KØBENHAVN Ø
DENMARK