S. KODANI
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AN ESTIMATE ON THE VOLUME OF METRIC BALLS

BY SHIGERU KODANI

1. Introduction.

Let M be a complete Riemannian manifold of dimension n. We denote by $i(M)$ the injectivity radius of M, by $B(p, r)$ the metric ball in M of radius $r \leqq i(M)$ centered at $p \in M$ and by $\mathrm{vol}(B(p, r))$ the volume of $B(p, r)$. Furthermore we denote by $\alpha(n)$ the volume of the round sphere S^n of sectional curvature 1. M. Berger and J. Kazdan [3] showed that if M is closed then the volume $\mathrm{vol}(M)$ of M satisfies

$$\mathrm{vol}(M) \geqq \alpha(n)(i(M)/\pi)^n, \tag{1}$$

where the equality holds if and only if M is a round sphere of constant sectional curvature $(\pi/i(M))^2$. Later, C. B. Croke [6] showed that if M is closed then for $r \in [0, i(M)]$,

$$\operatorname*{Ave}_{x \in M} \mathrm{vol}(B(x, r)) \geqq \alpha(n)(r/\pi)^n. \tag{2}$$

Here the equality holds if and only if $r = i(M)$ and M is a round sphere. Here $\operatorname*{Ave}_{x \in M} f(x)$, for any function f on M, means $\frac{1}{\mathrm{vol}(M)}\int_M f(x)dx$. But it is believed that for any point $p \in M$ and for $r \in [0, i(M)]$,

$$\mathrm{vol}(B(p, r)) \geqq \alpha(n)(r/\pi)^n. \tag{3}$$

Here the equality holds if and only if $r = i(M)$, $B(p, i(M)) = M$ and M is a round sphere. As partial results on this problem, not sharp lower bounds are already known ([1], [2] for $n = 2, 3$ and [4] for all n). And under some restriction on the metric form, a sharp one is obtained by C. B. Croke [5]. Especially C. B. Croke [4] showed the following remarkable inequality,

$$\mathrm{vol}(B(p, r)) \geqq \left[\frac{\pi\alpha(n-1)}{n\alpha(n)}\right]^n \alpha(n)\left[\frac{r}{\pi}\right]^n. \tag{4}$$

Here

$$\left[\frac{\pi\alpha(n-1)}{n\alpha(n)}\right]^n \approx \left[\frac{\pi}{2n}\right]^{n/2}, \qquad n \to \infty. \tag{5}$$

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In this paper, we will improve the Croke's constant under an additional curvature condition on the metric balls.

2. Result.

Let k be the infimum of the sectional curvature of the metric ball $B(p, r)$ in a complete Riemannian manifolds M. If the radius r is not greater than the injectivity radius $i(M)$ of M, then by Myers' theorem (cf. [7]), we see $k \leqq (\pi/r)^2$, and by maximal diameter theorem (cf. [7]), the equality holds if and only if $r=i(M)$, $M=B(p, i(M))$ and M is a round sphere. Our result is the following.

THEOREM. *Let $B(p, r)$ be a metric ball of radius r centered at p in a complete Riemannian manifold M. Let k be the infimum of the sectional curvature of the metric ball $B(p, r)$. Then, for $r \leqq i(M)$, there exist an increasing function $f: [-\infty, (\pi/r)^2] \to [0, 1]$ such that*

$$f((\pi/r)^2)=1, \qquad f(0)>(2/3)^n, \qquad f(-\infty)=0$$

and

$$\mathrm{vol}(B(p, r)) \geqq \frac{(n+3)}{6(n+1)} f(k)\alpha(n)\left[\frac{r}{\pi}\right]^n. \tag{6}$$

Before we prove the theorem, we need some definitions. For $x \in B(p, r)$, put

$$E(x, r)=\exp_x^{-1}(B(x, r) \cap B(p, r)),$$

and define $\mathrm{vol}(E(x, r))$ as the euclidiean volume in the tangent space T_xM at x. As a special case of the inequality in theorem A of [6], we get

$$\operatorname*{Ave}_{x \in B(p,r)} \mathrm{vol}(B(x, r) \cap B(p, r)) \geqq \frac{(n+3)\alpha(n)}{6(n+1)\pi^n \beta(n)} \cdot \operatorname*{Ave}_{x \in B(p,r)} \mathrm{vol}(E(x, r)), \tag{7}$$

where $\beta(n)$ is the volume of the standard disk of radius 1 in Euclidiean space R^n. Evidently,

$$\mathrm{vol}(B(p, r)) \geqq \operatorname*{Ave}_{x \in B(p,r)} \mathrm{vol}(B(x, r) \cap B(p, r)). \tag{8}$$

Let M_k be an n-dimensional simply connected space form of sectional curvature k and $B_k(q, r)$ be a metric ball in M_k at $q \in M_k$. Then the following lemma holds.

LEMMA. *For all $p \in M$, $q \in M_x$ and $0<r \leqq i(M)$,*

$$\operatorname*{Ave}_{x \in B(p,r)} \mathrm{vol}(E(x, r)) \geqq \operatorname*{Ave}_{y \in B_k(q,r)} \mathrm{vol}(E_k(y, r)). \tag{9}$$

Proof. Fix an isometry $I_p: T_pB(p, r)\to T_qB_k(q, r)$. For $x\in B(p, r)$, we put $y=\exp_q\circ I_p\circ\exp_p^{-1}(x)$ and put $s=d(p, x)=d_k(q, y)$, where d (d_k, resp.) is the distance on $B(p, r)(B_k(q, r)$, resp.). Then we get an isometry $I_x=\tau_q^y\circ I_p\circ\tau_x^p$: $T_xB(p, r)\to T_yB_k(q, r)$, where $\tau_{x'}^x$ is the parallel translation from $T_{x'}M$ to T_xM. For a unit vector $v\in T_xM$, let $l(v)$ denote the length of the geodesic segment γ_v emanating from x with the velocity vector $\dot{\gamma}(0)=v$ and hitting the boundary of $B(p, r)\cap B(x, r)$ at $\gamma_v(l(v))$. Similarly for $v'=I_x(v)\in T_yM$, we define $\gamma_{v'}\subset B(q, r)$ and $l(v')$.

Let $\xi_x\subset B(p, r)$ ($\xi_y\subset B_k(q, r)$, resp.) be the distance minimizing geodesic segment from p (q, resp.) to x (y, resp.). Since I_x is isometry, we see

$$\langle v, -\dot{\xi}_x(s)\rangle=\langle v', -\dot{\xi}_y(s)\rangle .$$

By Toponogov's triangle comparison theorem (cf. [7]), we obtain

$$d(p, \gamma_v(t))\leqq d_k(q, \gamma_{v'}(t)),$$

for all $0\leqq t\leqq \min(l(v), l(v'))$. Therefore if $l(v)<r$ then γ_v hits the boundary of $B(p, r)$, and so,

$$d(p, \gamma_v(l(v)))=d_k(q, \gamma_{v'}(l(v')))=r .$$

On the other hand, if $l(v)=r$ then $l(v')\leqq r=l(v)$. Therefore we always find

$$l(v)\geqq l(v'), \tag{10}$$

and so,

$$\operatorname{vol}(E(x, r))=\int_0^{l(v)}\int_{S^{n-1}} r^{n-1}drdv=\int_{S^{n-1}}\frac{l(v)^n}{n}dv \tag{11}$$

$$\geqq\int_{S^{n-1}}\frac{l(v')^n}{n}dv'=\operatorname{vol}(E_k(y, r)):=V(r).$$

Evidently, $dV/ds(s)=\dot{V}(s)<0$.

By Bishop's inequality (cf. [8]), we have, for $0\leqq s\leqq s'\leqq r$,

$$\frac{\operatorname{vol}(B(p, s))}{\operatorname{vol}(B(p, s'))}\geqq\frac{\operatorname{vol}(B_k(q, s))}{\operatorname{vol}(B_k(q, s'))}, \tag{12}$$

and by (11), we get

$$\operatorname*{Ave}_{x\in B(p,r)}\operatorname{vol}(E(x, r))=\frac{\int_{B(p,r)}\operatorname{vol}(E(x, r))dx}{\operatorname{vol}(B(p, r))}\geqq\frac{\int_0^r\int_{\partial B(p,s)}V(s)dsdx}{\operatorname{vol}(B(p, r))}$$

$$=\frac{\int_0^r\operatorname{vol}(\partial B(p, s))V(s)ds}{\operatorname{vol}(B(p, r))}$$

$$\geqq\left[\frac{\operatorname{vol}(B(p, s))V(s)}{\operatorname{vol}(B(p, r))}\right]_0^r-\frac{\int_0^r\operatorname{vol}(B(p, s))\dot{V}(s)ds}{\operatorname{vol}(B(p, r))}$$

$$\geqq \left[\frac{\mathrm{vol}(B_k(q, s))V(s)}{\mathrm{vol}(B_k(q, r))} \right]_0^r - \frac{\int_0^r \mathrm{vol}(B_k(q, s))\dot{V}(s)ds}{\mathrm{vol}(B_k(q, r))}$$

$$= \underset{y \in B_k(q, r)}{\mathrm{Ave}} \mathrm{vol}(E_k(y, r)). \qquad \text{q. e. d.}$$

Proof of Theorem. Put

$$f(k) = \frac{1}{\beta(n) r^n} \underset{y \in B_k(q, r)}{\mathrm{Ave}} \mathrm{vol}(E_k(y, r)).$$

We can easily verify that f is strictly increasing and $f(\pi/r)^2 = 1$, $f(-\infty) = 0$. By combining (7) with (8) and (9), we get (6). $f(0)$ is calculated as follows:

$$\begin{aligned} f(0) &= \frac{1}{\beta(n)} \underset{y \in B_0(q, 1)}{\mathrm{Ave}} \mathrm{vol}(E_0(y, 1)) \\ &= \frac{1}{\beta(n)^2} \int_0^1 \alpha(n-1) r^{n-1} 2 \int_{r/2}^1 \beta(n-1)(1-s^2)^{n-1/2} ds dr \\ &= \frac{2\alpha(n-1)\beta(n-1)}{\beta(n)^2} \left\{ \int_0^1 r^{n-1} \int_{1/2}^1 (1-s^2)^{n-1/2} ds dr \right. \\ &\qquad \left. + \int_0^1 r^{n-1} \int_{r/2}^{1/2} (1-s^2)^{n-1/2} ds dr \right\}. \end{aligned}$$

Here we have

$$\int_0^1 r^{n-1} \int_{1/2}^1 (1-s^2)^{n-1/2} ds dr = \frac{1}{n} \int_0^{\pi/3} \sin^n \vartheta d\vartheta.$$

By Fubini's theorem, we have

$$\begin{aligned} \int_0^1 r^{n-1} \int_{r/2}^{1/2} (1-s^2)^{n-1/2} ds dr &= \int_0^{1/2} (1-s^2)^{n-1/2} \int_0^{2s} r^{n-1} dr ds \\ &= \frac{1}{n} \int_0^{1/2} (1-s^2)^{n-1/2} (2s)^n ds \\ &= \frac{1}{n} \int_{\pi/3}^{\pi/2} (2 \sin\vartheta \cos\vartheta)^n d\vartheta \\ &= \frac{1}{2n} \int_{2\pi/3}^{\pi} \sin^n \vartheta \, d\vartheta = \frac{1}{2n} \int_0^{\pi/3} \sin^n \vartheta \, d\vartheta. \end{aligned}$$

Hence, we get

$$\begin{aligned} f(0) &= \frac{3n}{n-1} \frac{\alpha(n-2)}{\alpha(n-1)} \int_0^{\pi/3} \sin^n \vartheta d\vartheta \\ &> \frac{3n}{n-1} \frac{\alpha(n-2)}{\alpha(n-1)} \left\{ \frac{1}{\alpha(n)} \frac{\alpha(n+1)}{2} \left(\frac{2\pi/3}{\pi} \right)^{n+1} \right\} = \left(\frac{2}{3} \right)^n, \end{aligned}$$

where we have used the relations

$$n\beta(n)=\alpha(n-1)\,,\qquad \alpha(n+2)=2\pi\alpha(n)/(n+1)\,,$$

and replaced the volume of the spherical cup of radius $\pi/3$ in the unit $(n+1)$-sphere by the volume of the hemisphere of $(n+1)$-dimensional round sphere with diameter $2\pi/3$. q. e. d.

Remark. Put

Table 1.

n	$c_1(n)$	$c_3(n)$
2	.616849	.1234568
3	.296296	.0740741
4	.120394	.0460905
5	.043151	.0292638
6	.013989	.0188125
7	4.17219×10^{-3}	.0121933
10	7.45077×10^{-5}	3.41576×10^{-3}
15	3.48113×10^{-8}	4.28182×10^{-4}
...	...	...
$n\to+\infty$	$\approx\left[\frac{\pi}{2n}\right]^{n/2}$	$\approx\frac{1}{6}\left[\frac{2}{3}\right]^{n}$

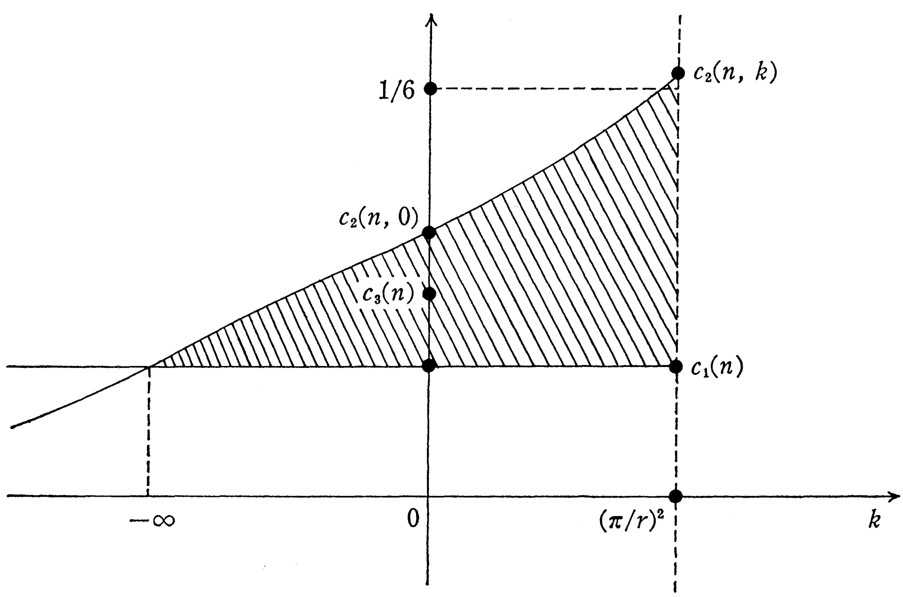

Fig. 2.

$$c_1(n)=\left[\frac{\pi\alpha(n-1)}{n\alpha(n)}\right]^n, \qquad c_2(n,\ k)=\frac{(n+3)}{6(n+1)}f(k),$$

$$c_3(n)=\frac{(n+3)}{6(n+1)}\left(\frac{2}{3}\right)^n.$$

Then we have $c_2(n, 0)>c_3(n)$. Now we give the explicit values of $c_1(n)$ and $c_3(n)$ in Table 1. From this table, we can observe that if the sectional curvature k of the metric ball is positive then, for $n\geqq 6$, our constant $c_2(n, k)$ is better than Croke's constant $c_1(n)$ (See also Fig. 2.).

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DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO
TOKYO 152, JAPAN