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NULL 2-TYPE SURFACES IN E^3 ARE CIRCULAR CYLINDERS

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Abstract

In this article we prove that open portions of circular cylinders are the only surfaces in E^3 which are constructed from eigenfunctions of Δ with eigenvalue 0 and an eigenvalue λ ($\neq 0$).

1. Introduction.

Let M be a connected (not necessary compact) surface in a Euclidean 3-space E^3. Denote by Δ the Laplacian of M associated with the induced metric. Then the position vector x and the mean curvature vector H of M in E^3 satisfy

$$\Delta x = -2H. \tag{1.1}$$

This formula yields the following well-known result: A surface M in E^3 is minimal if and only if all coordinate functions of E^3, restricted to M, are harmonic functions, that is,

$$\Delta x = 0. \tag{1.2}$$

In other words, minimal surfaces are constructed from eigenfunctions of Δ with eigenvalue zero.

According to the famous Douglas and Rado's solutions to the Plateau problem there exist ample examples of minimal surfaces in E^3. The study of minimal surfaces in E^3 has attracted many mathematicians for many years (cf. [3]).

On the other hand, it is easy to see that circular cylinders in E^3 are constructed from harmonic functions and eigenfunctions of Δ with a nonzero eigenvalue, say λ. The position vector of such a surface admits the following simple spectral decomposition:

$$x = x_0 + x_q, \quad \text{with } \Delta x_0 = 0 \text{ and } \Delta x_q = \lambda x_q, \tag{1.3}$$

for some non-constant maps x_0 and x_q, where λ is a non-zero constant. In the following, we simply call a surface M in a Euclidean space *a surface of null 2-type* if the position vector x of M has the spectral decomposition (1.3).

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We ask the following simple geometric question:

"Determine all surfaces in E^3 which are constructed from eigenfunctions of Δ with two eigenvalues 0 and λ $(\neq 0)$."

The purpose of this article is to give a complete solution to this question. More precisely, we shall prove the following

THEOREM. *A surface M in E^3 is of null 2-type if and only if M is an open portion of a circular cylinder.*

2. Proof of Theorem.

Let M be a surface in a Euclidean 3-space E^3. We denote by h, A, H, ∇ and D the second fundamental form, the Weingarten map, the mean curvature vector, the Riemannian connection and the normal connection of the surface M in E^3.

Let X, Y be two vector fields tangent to M. Then, for any constant vector c in E^3, we have

$$YX\langle H, c\rangle=\langle D_Y D_X H, c\rangle-\langle\nabla_Y(A_H X), c\rangle -\langle A_{D_X H}Y, c\rangle-\langle h(Y, A_H X), c\rangle, \tag{2.1}$$

where $\langle\ ,\ \rangle$ denotes the inner product in E^3. Let e_1, e_2 be an orthonormal local frame fields tangent to M. Then (2.1) implies (cf. [2, p. 271])

$$\Delta H=\Delta^D H+\|h\|^2 H+\mathrm{tr}\,(\bar{\nabla} A_H)\,, \tag{2.2}$$

where $\Delta^D H$ is the Laplacian of H with respect to the normal connection D and

$$\bar{\nabla} A_H=\nabla A_H+A_{DH}\,. \tag{2.3}$$

We need the following lemma.

LEMMA. *Let M be a surface in E^3. Then $\mathrm{tr}\,(\bar{\nabla} A_H)=0$ if and only if $\nabla\alpha^2$, the gradient of α^2 $(=\langle H, H\rangle)$, is an eigenvector of the Weingarten map A with eigenvalue $-\alpha$ whenever $\nabla\alpha^2\neq 0$, that is, we have*

$$A(\nabla\alpha^2)=-\alpha\nabla\alpha^2 \qquad on\ U=\{p\ in\ M: \nabla\alpha^2(p)\neq 0\}. \tag{2.4}$$

Proof of Lemma. Let e_1, e_2 be an orthonormal local frame field tangent to M and $\xi=e_3$ a unit local field normal to M. Denote by $\omega_A{}^B$ $(A, B=1, 2, 3)$ the connection forms associated with e_1, e_2, e_3 and by ω^1, ω^2 the dual frame of e_1, e_2. If we may choose e_1, e_2 to be eigenvectors of the Weingarten map A $(A=A_\xi)$ with eigenvalues denoted by κ_1, κ_2, respectively, then we have

$$A_H(e_i)=\alpha\kappa_i e_i\,, \qquad i, j, k=1, 2\,, \tag{2.5}$$

where $H=\alpha\xi$. For simplicity we denote ∇_{e_i} by ∇_i. From (2.5) we have

$$(\nabla_i A_H)e_j=\alpha(e_i\kappa_j)e_j+(e_i\alpha)\kappa_j e_j+\alpha\sum(\kappa_j-\kappa_k)\omega_j^k(e_i)e_k\,. \tag{2.6}$$

Thus, by Codazzi equation, we find

$$\alpha(e_i\kappa_j)e_j-\alpha(e_j\kappa_i)e_i=\sum\alpha\{(\kappa_i-\kappa_k)\omega_i^k(e_j)-(\kappa_j-\kappa_k)\omega_j^k(e_i)\}e_k \tag{2.7}$$

from which we obtain

$$\alpha(e_i\kappa_j)=\alpha(\kappa_j-\kappa_i)\omega_j^i(e_j)\qquad\text{for } i\neq j\,. \tag{2.8}$$

Combining (2.6) and (2.8) we may obtain

$$\operatorname{tr}(\nabla A_H)=\sum(\nabla_i A_H)e_i=\nabla\alpha^2+A(\nabla\alpha)\,. \tag{2.9}$$

Consequently, from (2.3) and (2.9) we get

$$\operatorname{tr}(\bar{\nabla}A_H)=\nabla\alpha^2+2A(\nabla\alpha)\,, \tag{2.10}$$

from which we obtain the lemma.

Now, assume M is a null 2-type surface in E^3. Then the position vector x of M takes the following form:

$$x=x_0+x_q\,,\quad \Delta x_0=0\quad\text{and}\quad \Delta x_q=\lambda x_q\,, \tag{2.11}$$

for some non-constant maps x_0 and x_q. By using (1.1), (2.11) implies that

$$\Delta H=\lambda H\,. \tag{2.12}$$

Combining (2.2) and (2.12) we find

$$\Delta^D H=(\lambda-\|h\|^2)H \tag{2.13}$$

and

$$\operatorname{tr}(\bar{\nabla}A_H)=0\,. \tag{2.14}$$

Let $U=\{p\in M:(\nabla\alpha^2)(p)\neq 0\}$. Then U is an open subset of M. Assume that U is non-empty. Then, by Lemma, the Weingarten map A has eigenvalues $-\alpha$ and 3α on U. Moreover, by Lemma, we may assume that e_1, e_2 are orthonormal local frame fields in U such that e_1 is parallel to $\nabla\alpha^2$. Then we have

$$\omega_3^1=\alpha\omega^1\,,\qquad \omega_3^2=-3\alpha\omega^2\,, \tag{2.15}$$

$$d\alpha=(e_1\alpha)\omega^1\,. \tag{2.16}$$

Taking the exterior differentiation of the first equation of (2.15) and applying (2.16) and the structure equations, we obtain

$$d\omega^1=0\,. \tag{2.17}$$

Hence we have locally

$$\omega^1 = du\,, \tag{2.18}$$

where u is a local function on U. Similarly, by taking the exterior differentiation of the second equation of (2.15), we may find

$$\omega_2{}^1(e_2) = (3e_1\alpha)/4\alpha\,. \tag{2.19}$$

From (2.18) we obtain

$$\omega_2{}^1(e_1) = 0\,. \tag{2.20}$$

From (2.16) and (2.18) we have

$$d\alpha \wedge du = 0\,. \tag{2.21}$$

This shows that α is function of u, that is $\alpha = \alpha(u)$. In particular, we have

$$d\alpha = \alpha'(u)du\,, \tag{2.22}$$

$$4\alpha\omega_2{}^1 = 3\alpha'(u)\omega^2\,. \tag{2.23}$$

Taking the exterior differentiation of (2.23), we may obtain the following second order ordinary differential equation:

$$4\alpha\alpha'' - 7(\alpha')^2 + 16\alpha^4 = 0\,. \tag{2.24}$$

Let $y = (\alpha')^2$. Then it is easy to see that equation (2.24) can be reduced to the following first order differential equation:

$$2\alpha y' - 7y = -16\alpha^4\,, \tag{2.25}$$

where y' denotes the derivative of y with respect to α. From this equation we obtain the following solution:

$$y = (\alpha')^2 = C\alpha^{7/2} - 16\alpha^4\,, \tag{2.26}$$

where C is a constant.

On the other hand, since M is of codimension one, equation (2.13) and (2.15) imply

$$-\alpha\Delta\alpha = (10\alpha^2 - \lambda)\alpha^2\,. \tag{2.27}$$

By using (2.19) we find

$$4\alpha(\nabla_2 e_2)\alpha = 3(\alpha')^2\,. \tag{2.28}$$

Therefore, by applying (2.16), (2.28) and the definition of Δ, we may obtain

$$4\alpha\Delta\alpha = 3(\alpha')^2 - 4\alpha\alpha''\,. \tag{2.29}$$

Combining (2.27) and (2.29) we find

$$4\alpha\alpha''-3(\alpha')^2+4(\lambda-10\alpha^2)\alpha^2=0\,. \tag{2.30}$$

Therefore, from (2.24) and (2.30), we obtain

$$(\alpha')^2=14\alpha^4-\lambda\alpha^2\,. \tag{2.31}$$

Comparing equations (2.26) and (2.31), we conclude that α is constant on U which contradicts to our assumption. Therefore, U is empty and consequently the null 2-type surface M has constant mean curvature α. Thus, by applying (2.13) we see that the second fundamental form h has constant length. Hence, by the constancy of the mean curvature and the equation of Gauss, we have known that the Gaussian curvature of M is also constant. Since M is assumed to be of null 2-type, these conditions imply that M is an open portion of a circular cylinder (cf. [1, p. 118]). The converse of this is trivial. (Q. E. D.)

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