K. ATSUYAMA
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THE CONNECTION BETWEEN THE SYMMETRIC SPACE $E_7/SO(12)\cdot SO(3)$ AND PROJECTIVE PLANES

BY KENJI ATSUYAMA

Introduction.

Our aim is to grasp the geometrical and intuitive image of the exceptional Lie groups. For this purpose we will solve a problem which is given by H. Freudenthal ([4], p. 175) to justify the B. A. Rozenfeld's assertions for these groups [6]. The problem asks us how to study, by making use of composition algebras, the connection between projective planes and the symmetric spaces of type EIII, EVI and EVIII in the sense of E. Cartan. As for type EIII, we in [2] dealt with the compact and simply connected symmetric space $E_6/SO(10)\cdot SO(2)$. In this paper we study, in series, the symmetric space $E_7/SO(12)\cdot SO(3)$ of type EVI. The conclusion is that the space can be considered a projective plane in the wider sense. Namely, it has the structure such that two general points are contained in three and only three lines (Theorem 5.17). The number of such lines studied in [2] is just one. In the last of this paper we mention the types of symmetric spaces which are made of the lines passing through two points in the singular position. The technique of calculations and the idea to obtain the above results are all contained in [2].

1. Preliminaries.

We explain a model, according to [1], of the compact simple Lie algebra of type E_7 to construct the symmetric space $E_7/SO(12)\cdot SO(3)$ explicitly.

Let $\mathfrak{A}$ be a composition algebra over the real field $\boldsymbol{R}$. Define in $\mathfrak{A}$ a symmetric inner product, a commutator and an associator by $(a, b)=(ab+\overline{ab})/2$, $[a, b]=ab-ba$ and $(a, b, c)=(ab)c-a(bc)$ respectively, where $a, b, c\in\mathfrak{A}$ and $-: a\to\bar{a}$ is the canonical conjugation of $\mathfrak{A}$. Then any inner derivation of $\mathfrak{A}$ can be generated by $D_{a,b}$, where $D_{a,b}(c)=[[a, b], c]-3(a, b, c)$.

Let $\mathfrak{A}^{(1)}\otimes M^3\otimes\mathfrak{A}^{(2)}$ denote a tensor product over $\boldsymbol{R}$ composed of two composition algebras $\mathfrak{A}^{(i)}$ and one 3×3 matrix algebra M^3 with coefficients in $\boldsymbol{R}$. If the confusion does not occur, we write aXu instead of $a\otimes X\otimes u$, where $a\in\mathfrak{A}^{(1)}$, $u\in\mathfrak{A}^{(2)}$ and $X\in M^3$. A product is introduced into this vector space by $xy=abXYuv$ for $x=aXu$ and $y=bYv$. Furthermore, an involution and a trace

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Tr are defined by $aXu \to \bar{a}X^T\bar{u}$ and $Tr(aXu)=a\,\mathrm{tr}(X)Iu$ respectively, where $T: X \to X^T$ is the transposed operator of matrix, $\mathrm{tr}(X)=(x_{11}+x_{22}+x_{33})/3$ for $X=(x_{ij})\in M^3$, and I is the 3×3 unit matrix.

Let $\mathfrak{M}$ denote a real vector space which is generated by all elements in $\mathfrak{A}^{(1)}\otimes M^3\otimes\mathfrak{A}^{(2)}$ with the trace Tr 0 and the skew-symmetric form with respect to the involution $aXu\to\bar{a}X^T\bar{u}$. Let $L(\mathfrak{A}^{(1)}, M^3, \mathfrak{A}^{(2)})$ be the real vector space $Der\,\mathfrak{A}^{(1)}\oplus\mathfrak{M}\oplus Der\,\mathfrak{A}^{(2)}$ (direct sum), where $Der\,\mathfrak{A}^{(i)}$ is the Lie algebra of inner derivations of $\mathfrak{A}^{(i)}$. In this space we define an anti-commutative product [,] in the following way:

$$(1)\qquad [D^{(i)}, D^{(j)}]=\begin{cases}\text{the Lie product of } Der\,\mathfrak{A}^{(i)} & (i=j)\\ 0 & (i\neq j),\end{cases}$$

$$(2)\qquad [D^{(1)}+D^{(2)}, aXu]=(D^{(1)}a)Xu+aX(D^{(2)}u),$$

(3) For $x=aXu$ and $y=bYv$ in $\mathfrak{M}$,

$$[x, y]=(X, Y)(u, v)D_{a,b}+(xy-yx-Tr(xy-yx))+(X, Y)(a, b)D_{u,v},$$

where $D^{(i)}\in Der\,\mathfrak{A}^{(i)}$ and $(X, Y)=\mathrm{tr}(X, Y)$. Then $L(\mathfrak{A}^{(1)}, M^3, \mathfrak{A}^{(2)})$ becomes a real Lie algebra by this product. If $\mathfrak{A}^{(1)}$ is the Cayley algebra $\mathfrak{C}$ (over $\boldsymbol{R}$) with the non-split type, it is a compact simple Lie algebra of type F_4, E_6, E_7 or E_8 according as $\mathfrak{A}^{(2)}$ is $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$ or $\mathfrak{C}$, where $\boldsymbol{C}$ and $\boldsymbol{Q}$ are the fields of complex and quaternion numbers with the non-split types respectively. The Killing form B of $L(\mathfrak{C}, M^3, \boldsymbol{Q})$ can be given by $B(D_1^{(1)}+aXu+D_1^{(2)}, D_2^{(1)}+bYv+D_2^{(2)})=9/2B^{(1)}(D_1^{(1)}, D_2^{(1)})+216(a, b)(X, Y)(u, v)+27B^{(2)}(D_1^{(2)}, D_2^{(2)})$, where $B^{(1)}$ and $B^{(2)}$ are the Killing forms of $Der\,\mathfrak{C}$ and $Der\,\boldsymbol{Q}$ respectively.

For the remaining sections, we give a basis of $\mathfrak{C}$ explicitly:

a basis: $e_0, e_1, \cdots, e_7$;

rules of product:

$e_1e_2=e_3,\quad e_1e_4=e_5,\quad e_6e_7=e_1,\quad e_2e_5=e_7,\quad e_3e_4=e_7,$

$e_3e_5=e_6,\quad e_6e_4=e_2,$

$e_ie_j=-e_je_i$ ($i, j\geqq 1$ and $i\neq j$), $e_ie_i=-e_0$ ($i\geqq 1$),

e_0 is the unit element,

the canonical conjugation $-$: $e_0\to e_0,\quad e_i\to -e_i$ ($1\leqq i\leqq 7$).

Then $\boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$ can be realized as subalgebras in $\mathfrak{C}$ which are generated by $\{e_0\}$, $\{e_0, e_1\}$ and $\{e_0, e_1, e_2, e_3\}$ respectively, and $Der\,\boldsymbol{Q}$ is also generated by D_{e_1,e_2}, D_{e_2,e_3} and D_{e_3,e_1}.

2. Construction of a symmetric space Π.

Let $\mathfrak{G}$ be the compact real simple Lie algebra of type E_7, i.e. $\mathfrak{G}=L(\mathfrak{C}, M^3, \boldsymbol{Q})$. We will construct a compact simply connected symmetric space Π by the same method as Section 2 in [2]. It can be realized as a subset of projections in the set $End\,\mathfrak{G}$ of endomorphisms of $\mathfrak{G}$, and its type is $E_7/SO(12)\cdot SO(3)$ as a symmetric space.

Let $\mathfrak{X}$ be the subset in $\mathfrak{G}$ consisting of all elements x which satisfy an identity $(\mathrm{ad}\,x)((\mathrm{ad}\,x)^2+1)((\mathrm{ad}\,x)^2+4)=0$, where $\mathrm{ad}\,x$ is the adjoint representation of x and 1 is the identity transformation of $\mathfrak{G}$. The eigenspaces of $\mathrm{ad}\,x$, for each $x\in\mathfrak{X}$, can be given by $\mathfrak{G}_0(x)=\{z\in\mathfrak{G}\mid(\mathrm{ad}\,x)z=0\}$ and $\mathfrak{G}_i(x)=\{z\in\mathfrak{G}\mid(\mathrm{ad}\,x)^2z=-i^2z\}$, $i=1, 2$. Three projections $\{P_i(x)\}$ of $\mathfrak{G}$, moreover, can be defined by $P_0(x)=1+5/4(\mathrm{ad}\,x)^2+1/4(\mathrm{ad}\,x)^4$, $P_1(x)=-4/3(\mathrm{ad}\,x)^2-1/3(\mathrm{ad}\,x)^4$ and $P_2(x)=1/12(\mathrm{ad}\,x)^2+1/12(\mathrm{ad}\,x)^4$. These satisfy $P_i(x)P_j(x)=0$ $(i\neq j)$ and $P_0(x)+P_1(x)+P_2(x)=1$. Each $P_i(x)$ is a projection of $\mathfrak{G}$ onto $\mathfrak{G}_i(x)$. Hence $\mathfrak{G}$ has a direct sum decomposition $\mathfrak{G}=\mathfrak{G}_0(x)\oplus\mathfrak{G}_1(x)\oplus\mathfrak{G}_2(x)$, and $(\mathfrak{G}_0(x)\oplus\mathfrak{G}_2(x))\oplus\mathfrak{G}_1(x)$ becomes a Cartan decomposition of $\mathfrak{G}$ with respect to an involutive automorphism $1-2P_1(x)$ $(=\exp\pi(\mathrm{ad}\,x))$.

EXAMPLE. If we take $K_1=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$ in $\mathfrak{M}\cap\mathfrak{X}$, then the eigenspaces $\{\mathfrak{G}_i(K_1)\}$ can be given by

$$\begin{array}{lll} & & \text{dimension}\\ \mathfrak{G}_0(K_1): & Der\,\mathfrak{C}\oplus\begin{pmatrix}2a&0&0\\0&-a&b\\0&-b&-a\end{pmatrix}\oplus Der\,\boldsymbol{Q} & 14+32+3=49,\\ \mathfrak{G}_1(K_1): & \begin{pmatrix}0&b_1&b_2\\-b_1&0&0\\-b_2&0&0\end{pmatrix}\oplus\begin{pmatrix}0&a_1&a_2\\a_1&0&0\\a_2&0&0\end{pmatrix} & 44+20=64,\\ \mathfrak{G}_2(K_1): & \begin{pmatrix}0&0&0\\0&0&a\\0&a&0\end{pmatrix}\oplus\begin{pmatrix}0&0&0\\0&a&0\\0&0&-a\end{pmatrix} & 10+10=20,\end{array}$$

where a, a_1, a_2 (resp. b, b_1, b_2) are linear combinations of $e_0\otimes e_j$ and $e_i\otimes e_0$ (resp. $e_0\otimes e_0$ and $e_i\otimes e_j$), $i=1, 2, \cdots, 7$ and $j=1, 2, 3$.

The action of the adjoint group G of $\mathfrak{G}$ on $End\,\mathfrak{G}$ is defined by $g\cdot h=ghg^{-1}$, where $g\in G$ and $h\in End\,\mathfrak{G}$. Let Π be the orbit of the projection $P_1(K_1)$ by G under this action, i.e. $\Pi=\{g\cdot P_1(K_1)\mid g\in G\}$. We note $g\cdot P_1(K_1)=P_1(gK_1)$. Then the eigenspace $\mathfrak{G}_1(gK_1)$ can be regarded as the tangent space of Π at $P_1(gK_1)$, and the eigenspace $\mathfrak{G}_0(gK_1)\oplus\mathfrak{G}_2(gK_1)$ can also be regarded as the Lie algebra of the isotropy group at $P_1(gK_1)$ for G. When we introduce a G-invariant Riemannian structure into Π by restricting the Killing form B of $\mathfrak{G}$ to each tangent space $\mathfrak{G}_1(gK_1)$, G equals to the identity component of the isometry group of Π. Since the compact connected symmetric spaces of type EVI have one locally isometry class (cf. [3], p. 411), the following assertion can be obtained finally.

PROPOSITION 2.1. *Π is a simply connected compact symmetric space of type* EVI, *that is,* $E_7/SO(12)\cdot SO(3)$ *with the dimension* 64. *Each point* $P_1(gK_1)$ *of* Π *has the geodesic symmetry* $1-2P_1(gK_1)$.

3. Maximal flat tori of Π.

From now on we will write $P(x)$ simply instead of $P_1(x)$ as points of Π. Three elements $\{K_i\}$ in $\mathfrak{X}$ are defined by

$$K_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad K_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad K_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},$$

where the unit elements e_0 and the tensor product $\otimes$ are omitted.

The matrix representation of a projection $P((\exp t(\operatorname{ad} K_2))K_1)$, $t\in\boldsymbol{R}$, is first given. We note that $\mathfrak{G}$ $(=L(\mathfrak{C}, M^3, \boldsymbol{Q}))=Der\,\mathfrak{C}\oplus\mathfrak{M}\oplus Der\,\boldsymbol{Q}$ and the set of elements of $\mathfrak{G}$ written in (2), (3) and (4) makes a basis of $\mathfrak{M}$. The following matrices are the same as ones in [2], Section 3, and the direct product of these matrices becomes the representation which we want to obtain.

(1) On $Der\,\mathfrak{C}\oplus Der\,\boldsymbol{Q}$, the form is the 0 matrix,

(2) On the each subspace consisting of $e_iK_1e_j$, $e_iK_2e_j$ and $e_iK_3e_j$ $(i, j=0$ or $i, j\geqq 1)$, the form is

$$\begin{pmatrix}\sin^2 t&0&1/2\sin 2t\\0&1&0\\1/2\sin 2t&0&\cos^2 t\end{pmatrix},$$

(3) On the each subspace consisting of $e_iI_1e_0$, $e_iI_2e_0$, $e_iF_1e_0$, $e_iF_2e_0$ and $e_iF_3e_0$ $(i\geqq 1)$, the form is

$$\begin{pmatrix}1/2\sin^2 2t&1/2\sin^2 2t&0&1/2\sin 4t&0\\1/2\sin^2 2t&1/2\sin^2 2t&0&1/2\sin 4t&0\\0&0&\sin^2 2t&0&-1/2\sin 2t\\1/4\sin 4t&1/4\sin 4t&0&\cos^2 2t&0\\0&0&-1/2\sin 2t&0&\cos^2 t\end{pmatrix},$$

where $I_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$, $I_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$, $F_1=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $F_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$ and $F_3=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$,

(4) On the each subspace consisting of $e_0I_1e_i$, $e_0I_2e_i$, $e_0F_1e_i$, $e_0F_2e_i$ and $e_0F_3e_i$ $(i\geqq 1)$, the form is the same as that in (3).

LEMMA 3.1. *The curve* $\exp t(\operatorname{ad} K_2)\cdot P(K_1)$ *in* Π *is a simply closed geodesic* *with the initial point $P(K_1)$ and the tangent vector K_2. The period is π and the length is 12π.*

Proof. We can derive the above assertions except the length l from the matrix representation of the geodesic $r(t)=\exp t(\operatorname{ad} K_2)\cdot P(K_1)$. As B is the Killing form of $\mathfrak{G}$, $-B$ gives an inner product, being positive definite, by the definition of the Riemannian structure of Π. Since $r(t)$ has the tangent vector K_2 at each point, its length is

$$l=\int_0^\pi(-B(\dot r(t),\ \dot r(t)))^{1/2}dt=\int_0^\pi(-B(K_2,\ K_2))^{1/2}dt=(-216\operatorname{tr} K_2K_2)^{1/2}\pi=12\pi\,.$$

Remark. When the tangent vector of $r(t)$ is $e_iK_2e_i$, $i=1$, 2 or 3, instead of K_2, the above lemma also holds by direct calculations (or by the same method as [2], Lemma 3.2).

Let $P(K_1)$ be the base point in Π. Since Π has the rank 4 as a symmetric space and has the tangent space $\mathfrak{G}_1(K_1)$ at $P(K_1)$, the subspace $\mathfrak{T}_0$ in $\mathfrak{G}_1(K_1)$, spanned by tangent vectors K_2, $e_1K_2e_1$, $e_2K_2e_2$ and $e_3K_2e_3$, is a maximal abelian subspace. Then the associated set $T_0=\{\exp(\operatorname{ad} x)\cdot P(K_1)\mid x\in\mathfrak{T}_0\}$ is a maximal torus in Π passing through the base point $P(K_1)$. Next we define a mapping ϕ of the 4-dimensional Euclidean space $\boldsymbol{R}^4$ onto the torus T_0 by $\phi:(t_i)\to\exp(\operatorname{ad} x)\cdot P(K_1)$, where $(t_i)=(t_1,\ t_2,\ t_3,\ t_4)$, $t_i\in\boldsymbol{R}$, and $x=\Sigma t_ie_iK_2e_i$. This mapping, however, is not injective, and so we must establish the following criterion, where $\boldsymbol{Z}$ is the ring of integers.

LEMMA 3.2. *It holds that $(t_i)\in\phi^{-1}(P(K_1))$ if and only if (1) $t_i\in\pi/2\boldsymbol{Z}$, for each i, and (2) $\Sigma t_i\in\pi\boldsymbol{Z}$.*

Proof. The necessity is first showed. Put $\alpha=\exp(\operatorname{ad}(\Sigma t_ie_iK_2e_i))$. If $\alpha\cdot P(K_1)=P(K_1)$ holds, then we have $P(K_1)\alpha^{-1}K_3=\alpha^{-1}\alpha P(K_1)\alpha^{-1}K_3=\alpha^{-1}P(K_1)K_3=\alpha^{-1}K_3$ because $P(K_1)$ leaves K_3 fixed as a projection of $\mathfrak{G}$. Hence $\alpha^{-1}K_3\in\mathfrak{G}_1(K_1)$. The same method also gives $\alpha^{-1}(e_4F_3e_0)\in\mathfrak{G}_1(K_1)$. The two relations imply the eight identities

$$\cos t_i\sin t_j\sin t_k\sin t_l=0\,,$$

$$\sin t_i\cos t_j\cos t_k\cos t_l=0\,,$$

where $\{i,\ j,\ k,\ l\}=\{1,\ 2,\ 3,\ 4\}$. These contain the three possible cases $(t_i)\in\boldsymbol{R}^4$ such that, under the condition $n_i\in\boldsymbol{Z}$ for all i,

(i) $((1/2+n_0)\pi,\ (1/2+n_1)\pi,\ (1/2+n_2)\pi,\ (1/2+n_3)\pi)$,
(ii) $((1/2+n_0)\pi,\ (1/2+n_1)\pi,\ n_2\pi,\ n_3\pi)$ and its permutations,
(iii) $(n_0\pi,\ n_1\pi,\ n_2\pi,\ n_3\pi)$.

In the each case, the above (t_i) satisfies the conditions (1) and (2) in the lemma.

Next the sufficiency is showed. If $(t_i)\in\boldsymbol{R}^4$ satisfies (1) and (2), the possible cases for (t_i) are only (i), (ii) and (iii) above. For (t_i) in the each case, that

$\phi(t_i)=P(K_1)$ can be derived from the fact that $\exp(\mathrm{ad}\,\Sigma\, n_i\pi e_iK_2e_i)\cdot P(K_1)=P(K_1)$, $n_i\in \boldsymbol{Z}$, and $\exp\pi/2\,\mathrm{ad}(e_iK_2e_i+e_jK_2e_j)\cdot P(K_1)=P(K_1)$.

COROLLARY 3.3. *It holds that* $\phi(t_i)=\phi(s_i)$ *if and only if* (1) $t_i-s_i\in\pi/2\boldsymbol{Z}$, *for each* i, *and* (2) $\Sigma(t_i-s_i)\in\pi\boldsymbol{Z}$.

In the torus T_0 we next find out the points which are commutative with the base point $P(K_1)$ in Π as endomorphisms of $\mathfrak{G}$.

LEMMA 3.4. *A point* $\exp(\mathrm{ad}\,x)\cdot P(K_1)$, $x\in\mathfrak{G}_1(K_1)$, *is commutative with* $P(K_1)$ *if and only if* $\exp(\mathrm{ad}\,2x)\cdot P(K_1)=P(K_1)$.

Proof. Put $P=\exp(\mathrm{ad}\,x)\cdot P(K_1)$. Since the base point $P(K_1)$ has the geodesic symmetry $1-2P(K_1)$ $(=\alpha$ simply), we have $\alpha\cdot P=\alpha(\exp(\mathrm{ad}\,x))\alpha^{-1}\cdot\alpha\cdot P(K_1)=\exp(\mathrm{ad}\,\alpha x)\cdot P(K_1)=\exp(\mathrm{ad}-x)\cdot P(K_1)$. If P and $P(K_1)$ are commutative, it holds that $\alpha\cdot P=\alpha P\alpha^{-1}=P$ and, hence, $\exp(\mathrm{ad}\,2x)\cdot P(K_1)=P(K_1)$ from the above identity. Conversely, if this equation holds, $\alpha\cdot P=P$, i.e. $\alpha P=P\alpha$ can be obtained. This implies $P(K_1)P=PP(K_1)$.

LEMMA 3.5. *There are exactly fifteen points except* $P(K_1)$ *itself in the maximal torus* T_0 *which are commutative with* $P(K_1)$.

Proof. By Corollary 3.3, Lemma 3.4 and (i), (ii), (iii) in Lemma 3.2, the points in T_0 commuting with $P(K_1)$ have the coordinates (t_i) with respect to ϕ: (iv) $(\pi/4, \pi/4, \pi/4, \pi/4)$ and $(3\pi/4, \pi/4, \pi/4, \pi/4)$, (v) $(\pi/4, \pi/4, 0, 0)$, $(3\pi/4, \pi/4, 0, 0)$ and these permutations, (vi) $(\pi/2, 0, 0, 0)$ and $(0, 0, 0, 0)$. Its number is fifteen except $(0, 0, 0, 0)$.

The points in Π commuting with $P(K_1)$ can be characterized by the following assertion.

PROPOSITION 3.6. *The orbits of the points in* Π, *commuting with* $P(K_1)$, *under the isotropy group at* $P(K_1)$ *become two compact connected submanifolds which are also totally geodesic. One is a symmetric space of type* $S^2\cdot(SO(12)/U(6))$ *consisting of the midpoints (the distance* $3\sqrt{2}\,\pi$) *of the shortest closed geodesics with the initial point* $P(K_1)$. *The other is a symmetric space of type* $SO(12)/SO(8)\cdot SO(4)$ *consisting of the antipodal points (the distance* 6π) *of* $P(K_1)$.

Proof. Let U be the isotropy group at $P(K_1)$ and U_0 be its identity component. First we show that the points of (v) in Lemma 3.5 are transitive one another by U_0. Put $\alpha=\exp\pi/2(\mathrm{ad}\,D^{(1)}_{e_3,e_5})$, α is then an involutive automorphism of $\mathfrak{G}$ and $\alpha\in U_0$. The eigenvalues of α are, with respect to the Cayley numbers, 1 on the linear space $\{e_0, e_3, e_5, e_6\}$ and -1 on the linear space $\{e_1, e_2, e_4, e_7\}$. Hence we can see $\alpha\cdot\phi(\pi/4, \pi/4, 0, 0)=\alpha\cdot(\exp\pi/4\,\mathrm{ad}(K_2+e_1K_2e_1))\cdot P(K_1)=\exp\pi/4\,\mathrm{ad}(\alpha K_2+\alpha e_1K_2e_1)\cdot\alpha\cdot P(K_1)=\exp\pi/4\,\mathrm{ad}(K_2-e_1K_2e_1)\cdot P(K_1)=\phi(\pi/4, -\pi/4,$

$0, 0)=\phi(\pi/4, 3\pi/4, 0, 0)$ (by Corollary 3.3). Next, put $\alpha_1=\exp 3\pi/2(\mathrm{ad}\, e_3K_1e_3)$ and $\alpha_2=\exp \pi/2(\mathrm{ad}\, K_1)$, it then holds that $\alpha_1\alpha_2\in U_0$ and $\alpha_1\alpha_2\cdot\phi(\pi/4, \pi/4, 0, 0)=\phi(0, 0, \pi/4, \pi/4)$. The same method shows the transitivity for the others in (v). That each point in (v) is the midpoint of the shortest closed geodesic can be derived from (i), (ii), (iii) in Lemma 3.2.

Secondly, we show that the points of (iv) and (vi) in Lemma 3.5, except (0, 0, 0, 0), are transitive one another by U_0. Put $\beta=\exp \pi/4\, \mathrm{ad}(K_1+e_1K_1e_1+e_2K_1e_2+e_3K_1e_3)$. It then holds that $\beta\in U_0$ and $\beta\cdot\phi(\pi/2, 0, 0, 0)=\phi(\pi/4, -\pi/4, -\pi/4, -\pi/4)$ (by direct calculations)$=\phi(3\pi/4, \pi/4, \pi/4, \pi/4)$ (by Corollary 3.3). If $\beta_1=\exp-\pi/2(\mathrm{ad}\, e_4(I_1+I_2)e_0)$, the inclusion $e_4(I_1+I_2)e_0\in\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1)$ shows $\beta_1\in U_0$. Then, from $\beta_1K_2=-K_2$ and $\beta_1(e_iK_2e_i)=e_iK_2e_i$ $(i\geqq 1)$, we can see $\beta_1\cdot\phi(\pi/4, \pi/4, \pi/4, \pi/4)=\phi(-\pi/4, \pi/4, \pi/4, \pi/4)=\phi(3\pi/4, \pi/4, \pi/4, \pi/4)$. Lemma 3.2 implies that these points are antipodal points of $P(K_1)$.

From the above arguments and the transitivity of maximal flat tori passing through the base point $P(K_1)$, we can obtain that the points, being commutative with $P(K_1)$, make two compact connected submanifolds. That these are totally geodesic can be seen from the fact that the tangent spaces of these spaces at $P(K_1)$ are Lie triple systems (cf. [3], Lemma 2.1).

4. The roots of the symmetric space Π.

The Lie algebra $\mathfrak{G}$ has a direct sum decomposition $\mathfrak{G}=(\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1))\oplus\mathfrak{G}_1(K_1)$. The subspace $\mathfrak{G}_1(K_1)$ is the tangent space of Π at $P(K_1)$, and the subspace $\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1)$ is the Lie algebra $\mathfrak{U}$ of the isotropy group U at $P(K_1)$. The maximal flat torus T_0 has the tangent space $\mathfrak{T}_0$ at $P(K_1)$. This space is spanned by $\{e_iK_2e_i \mid i=0, 1, 2, 3\}$ and it is a maximal abelian subspace of $\mathfrak{G}_1(K_1)$. Now put $\mathfrak{H}_0=\{D^{(1)}_{e_2,e_3}+2D^{(1)}_{e_4,e_5}, D^{(1)}_{e_2,e_3}-2e_1(I_1-I_2)e_0, D^{(2)}_{e_2,e_3}-2e_0(I_1-I_2)e_1\}$, then this is a subalgebra of $\mathfrak{U}$ and gives a Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{G}$ such that $\mathfrak{H}=\mathfrak{H}_0\cup\mathfrak{T}_0$. Let Δ denote the set of roots which are obtained by the root space decomposition of $\mathfrak{G}$ with respect to $\mathfrak{H}$. We, furthermore, restrict the roots to $\mathfrak{T}_0$ and get a set $\Delta_{\mathfrak{T}_0}=\{\lambda\}$ of positive restricted roots of the symmetric space Π under an adequate ordering. Define four sets by

(1) $\mathfrak{U}(Q)=\{x\in\mathfrak{U} \mid \exp(\mathrm{ad}\, x)\cdot Q=Q\}$, for $Q\in\Pi$,

(2) $\mathfrak{U}(\mathfrak{T}_0)=\{x\in\mathfrak{U} \mid [x, \mathfrak{T}_0]=\{0\}\}$,

(3) $\mathfrak{U}_\lambda=\{x\in\mathfrak{U} \mid [y, [y, x]]=\lambda(y)^2x$ for any $y\in\mathfrak{T}_0\}$,

(4) $S_\lambda=\{Q\in T_0 \mid Q=\exp(\mathrm{ad}\, y)\cdot P(K_1)$ and $\lambda(y)\in\pi\boldsymbol{i}\boldsymbol{Z}$ for some $y\in\mathfrak{T}_0\}$, where $\boldsymbol{i}=\sqrt{-1}$.

Then we can have a useful identity $\mathfrak{U}(Q)=\mathfrak{U}(\mathfrak{T}_0)\oplus\Sigma\,\mathfrak{U}_\lambda$, where $Q\in T_0$ and the index λ runs over the positive roots λ such that $Q\in S_\lambda$ (cf. [5], p. 64). Note that the dimension of $\mathfrak{U}(\mathfrak{T}_0)$ is 9 and that of $\mathfrak{U}_\lambda$ is equal to the multiplicity of λ. If $\mathfrak{U}(Q)=\mathfrak{U}(\mathfrak{T}_0)$ holds, Q is called a regular point (with respect to the base point $P(K_1)$). If not so, Q is called a singular point. By the transitivity of maximal flat tori passing through $P(K_1)$, the definition can be applied for any point Q in Π and it is independent of the choice of maximal flat tori passing through $P(K_1)$ and Q.

Finally we list the positive roots λ with respect to the operation $\mathrm{ad}(\Sigma\, a_i e_i K_2 e_i)$, $a_i\in\boldsymbol{R}$, and also list the eigenvectors corresponding to λ, i.e. elements in $\mathfrak{U}_\lambda$. The multiplicity of λ is 1 for the roots with the type $-2(a_i\pm a_j)\boldsymbol{i}$ and is 4 for the others.

Positive roots and eigenvectors.

$-2a_0\boldsymbol{i}$: $e_k\otimes(I_1+I_2)\otimes e_0$

$-2a_j\boldsymbol{i}$: $D^{(1)}_{e_j,e_k}+e_je_k\otimes(I_1-I_2)\otimes e_0$

$-2(a_0\pm a_j)\boldsymbol{i}$: $e_j\otimes(I_1+I_2)\otimes e_0\mp e_0\otimes(I_1+I_2)\otimes e_j$ ($j=1, 2, 3$ and $k=4, 5, 6, 7$)

$-2(a_1\pm a_2)\boldsymbol{i}$: $D^{(1)}_{e_1,e_2}+e_3\otimes(I_1-I_2)\otimes e_0\mp e_0\otimes(I_1-I_2)\otimes e_3\mp D^{(2)}_{e_1,e_2}$

$-2(a_1\pm a_3)\boldsymbol{i}$: $D^{(1)}_{e_3,e_1}+e_2\otimes(I_1-I_2)\otimes e_0\mp e_0\otimes(I_1-I_2)\otimes e_2\mp D^{(2)}_{e_3,e_1}$

$-2(a_2\pm a_3)\boldsymbol{i}$: $D^{(1)}_{e_2,e_3}+e_1\otimes(I_1-I_2)\otimes e_0\mp e_0\otimes(I_1-I_2)\otimes e_1\mp D^{(2)}_{e_2,e_3}$

$-(a_0+\varepsilon_1a_1+\varepsilon_2a_2+\varepsilon_3a_3)\boldsymbol{i}$:

$$\begin{aligned}
&e_0\otimes K_1\otimes e_0+\varepsilon_1e_1\otimes K_1\otimes e_1+\varepsilon_2e_2\otimes K_1\otimes e_2+\varepsilon_3e_3\otimes K_1\otimes e_3\\
&-e_1\otimes F_1\otimes e_0+\varepsilon_1e_0\otimes F_1\otimes e_1-\varepsilon_2e_3\otimes K_1\otimes e_2+\varepsilon_3e_2\otimes K_1\otimes e_3\\
&-e_2\otimes F_1\otimes e_0+\varepsilon_1e_3\otimes K_1\otimes e_1+\varepsilon_2e_0\otimes F_1\otimes e_2-\varepsilon_3e_1\otimes K_1\otimes e_3\\
&-e_3\otimes F_1\otimes e_0-\varepsilon_1e_2\otimes K_1\otimes e_1+\varepsilon_2e_1\otimes K_1\otimes e_2+\varepsilon_3e_0\otimes F_1\otimes e_3
\end{aligned}$$

$-(a_0-\varepsilon_1a_1-\varepsilon_2a_2-\varepsilon_3a_3)\boldsymbol{i}$:

$$\begin{aligned}
&-e_4\otimes F_1\otimes e_0-\varepsilon_1e_5\otimes K_1\otimes e_1+\varepsilon_2e_6\otimes K_1\otimes e_2-\varepsilon_3e_7\otimes K_1\otimes e_3\\
&-e_5\otimes F_1\otimes e_0+\varepsilon_1e_4\otimes K_1\otimes e_1-\varepsilon_2e_7\otimes K_1\otimes e_2-\varepsilon_3e_6\otimes K_1\otimes e_3\\
&-e_6\otimes F_1\otimes e_0-\varepsilon_1e_7\otimes K_1\otimes e_1-\varepsilon_2e_4\otimes K_1\otimes e_2+\varepsilon_3e_5\otimes K_1\otimes e_3\\
&-e_7\otimes F_1\otimes e_0+\varepsilon_1e_6\otimes K_1\otimes e_1+\varepsilon_2e_5\otimes K_1\otimes e_2+\varepsilon_3e_4\otimes K_1\otimes e_3
\end{aligned}$$

(ε_1, ε_2, $\varepsilon_3=1$ or -1 and $\varepsilon_1\varepsilon_2\varepsilon_3=1$)

$$\begin{aligned}
\mathfrak{U}(\mathfrak{T}_0)=\mathfrak{H}_0\cup\{&D^{(1)}_{e_1,e_2}+2D^{(1)}_{e_4,e_7},\ D^{(1)}_{e_3,e_1}+2D^{(1)}_{e_6,e_4},\ D^{(1)}_{e_1,e_2}-2e_3\otimes(I_1-I_2)\otimes e_0,\\
&D^{(1)}_{e_3,e_1}-2e_2\otimes(I_1-I_2)\otimes e_0,\ D^{(2)}_{e_1,e_2}-2e_0\otimes(I_1-I_2)\otimes e_3,\\
&D^{(2)}_{e_3,e_1}-2e_0\otimes(I_1-I_2)\otimes e_2\}.
\end{aligned}$$

5. The connection between Π and projective planes.

We first introduce two geometrical objects, points and lines, into the symmetric space Π by the same method as Section 5 in [2] and study the connection between Π and projective planes. The aim is to solve a problem by H.

Freudenthal ([4], p. 175), but the result is different slightly from his conjecture, namely, we assert that there are exactly three lines passing through two general points.

Let $L(P)$ denote the set of antipodal points of P. It coincides the set of points which are commutative with P and have the distance 6π from P. We call $L(P)$ *a line* (associated with P) and call P *a point* again in the sense of projective geometry. The incidence structure is defined by the inclusion relation of sets. Let Π^L be the set of all lines in Π, then the structure of a manifold can be introduced into Π^L from Π because the correspondence $L: P \to L(P)$ gives a bijection between Π and Π^L (see Lemma 5.1). Since all lines are transitive one another by the isometry group of Π, they are diffeomorphic to the line $L(P(K_1))$ as manifolds. Therefore, each line is a compact connected symmetric space with the type $SO(12)/SO(8)\cdot SO(4)$ (from Prop. 3.6) and has the dimension 32.

From now on we will study the number of lines passing through two points in Π. Our result can be summed up as Theorem 5.17. For this purpose we begin to prepare some facts. Let $U(Q)$ be the subgroup of U which leaves Q fixed, where U is the isotropy group at the base point $P(K_1)$. Then $\mathfrak{U}(Q)$ in Section 4 is the Lie algebra of $U(Q)$. Put $Q_i = P(1/2(K_i - e_1K_ie_1 - e_2K_ie_2 - e_3K_ie_3))$, $i=1, 2, 3$, it then holds by direct calculations that $Q_1 = \phi(\pi/4, \pi/4, \pi/4, \pi/4)$ and $Q_3 = \phi(3\pi/4, \pi/4, \pi/4, \pi/4)$. Hence $Q_1, Q_3 \in T_0$ (but $Q_2 \notin T_0$). We can see later that the set $\{\alpha\cdot Q_1 \mid \alpha \in U(P(K_3))\}$, denoted as Ω, is a totally geodesic submanifold in Π and becomes a compact connected symmetric space with the type $SO(8)/SO(4)\cdot SO(4)$. Moreover put $R_i = P(1/2(K_i + e_1K_ie_1 + e_2K_ie_2 + e_3K_ie_3))$, $i=1$, 2, 3, it can be also shown in Lemma 5.3 and Corollary 5.6 that $R_1 = R_2 = R_3$, $R_1, Q_i \in \Omega$ and the fact that four points R_1, Q_i are different from one another. Note that two groups $U(P(K_2))$ and $U(P(K_3))$ are the same. This fact can be derived from the identity $(1-2P(K_1))(1-2P(K_2))(1-2P(K_3))=1$ and the commutativity of these geodesic symmetries. The Lie algebra of $U(P(K_3))$ has a direct sum decomposition $\mathfrak{U}(P(K_3)) = \mathfrak{L}_0 \oplus \mathfrak{L}_1$ $(\cong so(8)\oplus so(4))$. The basis of $\mathfrak{L}_0$ consists of $Der\,\mathfrak{C}$, $e_i(2I_1+I_2)e_0$ $(i\geqq 1)$, $e_iI_2e_0$ $(i\geqq 1)$, and its dimension is 28 $(=14+7+7)$. $\mathfrak{L}_1$ has a basis consisting of $e_0(2I_1+I_2)e_i - D^{(2)}_{e_j, e_k}$, $e_0I_2e_i$, where (i, j, k) runs over the even permutations of (1, 2, 3), and its dimension is 6 $(=3+3)$. Since $\exp(\mathrm{ad}\,\mathfrak{L}_1)$ leaves Q_1 fixed, this becomes only an identity transformation as isometries of Ω. Finally we make three involutive automorphisms of $\mathfrak{G}$ as follows. Put $A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and define a transformation δ_i of matrices X for each i by $\delta_i : X \to A_iXA_i$. Since δ_i becomes an automorphism of the matrix algebra M^3, it can be extended as an automorphism of $\mathfrak{G}$ by $\delta_i : D^{(1)} + aXu + D^{(2)} \to D^{(1)} + a(\delta_iX)u + D^{(2)}$. This extended map is also denoted by δ_i.

LEMMA 5.1. *The correspondence $L: \Pi \to \Pi^L$ is a bijective map and also gives the duality for the incidence structure.*

Proof. From the transitivity of points in Π, it is sufficient to show that $L(P(K_1))=L(Q)$ implies $P(K_1)=Q$. Then there exists $\alpha\in U(P(K_1))$ such that $\alpha\cdot Q\in T_0$ by the transitivity of maximal flat tori passing through $P(K_1)$. Since $\alpha\cdot L(P(K_1))=\alpha\cdot L(Q)$ means $L(P(K_1))=L(\alpha\cdot Q)$, the point $\alpha\cdot Q$ is commutative with any point in $L(P(K_1))$, especially with Q_1, Q_3 and $P(K_3)$ in T_0. Moreover $\alpha\cdot Q$ has the distance 6π from these points. Hence, from the transitivity of points in T_0 and the proof of Lemma 3.5, it holds that $\alpha\cdot Q=P(K_1)$, i.e. $Q=\alpha^{-1}\cdot P(K_1)=P(K_1)$. The equivalence of $P\in L(Q)$ and $L(P)\ni Q$ is an easy consequence of the definition for lines.

LEMMA 5.2. *For any point Q in Π, let V_1 and V_2 are maximal flat tori passing through $P(K_1)$ and Q. Then there exists $z\in\mathfrak{U}(Q)$ such that* $\exp(\mathrm{ad}\,z)\cdot V_1=V_2$.

Proof. This can be shown by the same method as Lemma 5.9 in [2] essentially.

LEMMA 5.3. *The followings hold:* (1) $Q_1=P(x_2)=P(x_3)$, $Q_2=P(x_1)=P(y_3)$ *and* $Q_3=P(y_1)=P(y_2)$, *where* $x_i=1/2(e_4F_ie_0+e_5K_ie_1-e_6K_ie_2+e_7K_ie_3)$ *and* $y_i=1/2(e_4F_ie_0-e_5K_ie_1+e_6K_ie_2-e_7K_ie_3)$. (2) $R_1=R_2=R_3$. (3) R_1, $Q_i\in\Omega$.

Proof. We first show Q_2, $Q_3\in\Omega$. Put $\alpha=\exp\pi/2(\mathrm{ad}\,e_4(I_1+I_2)e_0)$, then $\alpha\in U(P(K_3))$ holds because $e_4(I_1+I_2)e_0\in(\mathfrak{G}_0(K_1)\oplus\mathfrak{G}_2(K_1))\cap(\mathfrak{G}_0(K_3)\oplus\mathfrak{G}_2(K_3))$. Furthermore, we have $\alpha\cdot Q_1=\alpha\cdot\phi(\pi/4,\pi/4,\pi/4,\pi/4)=\phi(-\pi/4,\pi/4,\pi/4,\pi/4)=Q_3$. This implies $Q_3\in\Omega$. Next, put $\beta=\exp\pi/2(\mathrm{ad}\,e_4I_1e_0)$, then $\beta\in U(P(K_3))$ and $\beta\cdot Q_1=Q_2$ hold similarly. We obtain $\alpha^{-1}\cdot Q_3=P(x_3)$ by direct calculations. This gives $Q_1=P(x_3)$. When the automorphism δ_1 acts on the each side of $Q_1=P(x_3)$, we obtain $Q_1=P(x_2)$ because δ_1 maps $e_iK_1e_i$, $e_4F_3e_0$, $e_iK_3e_j$ to $-e_iK_1e_i$, $e_4F_2e_0$, $e_iK_2e_j$ respectively. To make use of δ_2 and δ_3 shows similarly the remaining equations in (1). By operating $\exp\pi/2(\mathrm{ad}\,e_4I_2e_0)$ on the both sides of $Q_1=P(x_3)$, we can see $R_1\in\Omega$ and $R_1=R_3$ from $e_4I_2e_0\in\mathfrak{U}(P(K_3))$. Finally $R_1=R_2$ follows from $R_1=\delta_1\cdot R_1$ and $R_2=\delta_1\cdot R_3$.

LEMMA 5.4. $L(P(K_i))\cap L(P(K_j))=\{P(K_k)\}\cup\Omega$ *holds, where* $\{i,j,k\}=\{1,2,3\}$.

Proof. We show the lemma in the case of $i=1$, $j=2$ and $k=3$. The result $P(K_3)\in L(P(K_1))$ is an easy consequence from $P(K_3)=\phi(\pi/2,0,0,0)$ and (vi) in Lemma 3.5. Operating δ_3 on the each side of this relation, we obtain $P(K_3)\in L(P(K_2))$ because $\delta_3K_3=-K_3$ and $\delta_3K_1=K_2$. Furthermore we can derive $P(K_1)\in L(P(K_2))$ from $\delta_2K_3=-K_1$ and $\delta_2K_2=-K_2$. By applying $\exp\pi/4\,\mathrm{ad}(K_2+e_1K_2e_1+e_2K_2e_2+e_3K_2e_3)$ to the both sides of $P(K_1)\in L(P(K_2))$, we have $Q_1\in L(P(K_2))$ because this transformation leaves $P(K_2)$ fixed. $L(P(K_1))$ contains Q_1 from (iv) in Lemma 3.5. By the above arguments, we get $L(P(K_1))\cap L(P(K_2))\ni P(K_3)$, Q_1. From $U(P(K_3))=U(P(K_2))$ and the definition of Ω, the inclusion $L(P(K_1))\cap L(P(K_2))\supset\{P(K_3)\}\cup\Omega$ follows.

Next the converse is shown. If Q is any point in $L(P(K_1))\cap L(P(K_2))$, there exists a 4-dimensional maximal flat torus $T\subset L(P(K_2))$ such that $P(K_1)$, $Q\in T$ because the line $L(P(K_2))$ has the rank 4 as a symmetric space. On the other hand, since $P(K_1)\in T_0\subset L(P(K_2))$, there exists an element α in the identity component of the isometry group of $L(P(K_2))$ (i.e. in a subgroup of $U(P(K_2))$) such that $\alpha\cdot T=T_0$ by the transitivity of maximal flat tori in $L(P(K_2))$ passing through $P(K_1)$. This implies $\alpha\cdot Q\in T_0$. Hence $\alpha\cdot Q$ is commutative with $P(K_1)$ and has the distance 6π from $P(K_1)$. Such points in T_0 are only Q_1, Q_3 and $P(K_3)$ by Lemma 3.5 and Prop. 3.6. If $\alpha\cdot Q=P(K_3)$, $Q=\alpha^{-1}\cdot P(K_3)=P(K_3)$ holds because $U(P(K_2))=U(P(K_3))$. If $\alpha\cdot Q=Q_1$, we obtain $Q=\alpha^{-1}\cdot Q_1\in\Omega$. If $\alpha\cdot Q=Q_3$, $Q=\alpha^{-1}\beta\cdot Q_1\in\Omega$ holds, where $\beta=\exp\pi/2(\mathrm{ad}\, e_4(I_1+I_2)e_0$ and hence, $\beta\in U(P(K_3))$. By the above arguments, we can see $L(P(K_1))\cup L(P(K_2))\subset\{P(K_3)\}\cup\Omega$. Note that $P(K_1)$, $P(K_2)$, $P(K_3)\notin\Omega$.

In other cases for i, j, k, we can show the lemma by applying the automorphisms δ_1 and δ_3 to the identical equation showed already. Then note that $\delta_m\cdot\Omega=\Omega$. This fact can be given by the following method. First we have easily $\delta_m\cdot\Omega=\{\delta_m\alpha\cdot Q_1 \mid \alpha\in U(P(K_3))\}=\{\beta\cdot\delta_m\cdot Q_1 \mid \beta\in U(\delta_m\cdot P(K_3))\}$. If $\delta_m=\delta_1$, this set becomes Ω because $\delta_1\cdot Q_1=Q_1$ and $U(\delta_1\cdot P(K_3))=U(P(K_2))=U(P(K_3))$. If $\delta_m=\delta_3$, this set equals $\{\beta\cdot Q_2 \mid \beta\in U(P(K_3))\}=\{\beta\beta_1\cdot Q_1 \mid \beta\in U(P(K_3))\}=\Omega$, where $\beta_1=\exp\pi/2(\mathrm{ad}\, e_4I_1e_0)$ and, hence, $\beta_1\in U(P(K_3))$. The proof is completed.

We will study further the submanifold Ω in Π. Ω is defined as the orbit of Q_1 under the group $U(P(K_3))$. Let Q_1 be the base point of Ω. The Lie algebra of $U(P(K_3))$ is $\mathfrak{L}_0\oplus\mathfrak{L}_1$ as before and $\exp(\mathrm{ad}\,\mathfrak{L}_1)$ acts on Ω only as an identity transformation. Hence $\mathfrak{L}_0$ is the Lie algebra of the isometry group of Ω, and the Lie algebra of the isotropy group at Q_1 with respect to the group $\exp(\mathrm{ad}\,\mathfrak{L}_0)$ becomes $\mathfrak{L}_{0,0}\oplus\mathfrak{L}_{0,1}$ $(\cong so(4)\oplus so(4))$: $\mathfrak{L}_{0,0}$ has a basis consisting of $e_i(I_1+I_2)e_0$, $D^{(1)}_{e_i,e_j}+e_ie_j(I_1-I_2)e_0$ and $\mathfrak{L}_{0,1}$ has a basis consisting of $D^{(1)}_{e_i,e_j}-2D^{(1)}_{e_5e_i,e_5e_j}$, $D^{(1)}_{e_i,e_j}-2e_ie_j(I_1-I_2)e_0$, where $(i,j)=(1,2)$, $(2,3)$ and $(3,1)$. Then the tangent space of Ω at Q_1 is spanned by sixteen vectors $e_j(I_1+I_2)e_0$, $D^{(1)}_{e_i,e_j}+e_ie_j(I_1-I_2)e_0$, where $\imath=1,2,3$ and $j=4,5,6,7$. This space becomes a Lie triple system in the tangent space of Π at Q_1. Hence Ω is also a compact connected symmetric space with the type $SO(8)/SO(4)\cdot SO(4)$ which has the rank 4. Let $\mathfrak{T}_\Omega$ be the maximal abelian subspace spanned by four tangent vectors $e_4(I_1+I_2)e_0$, $D^{(1)}_{e_i,e_4e_i}+e_4(I_1-I_2)e_0$ at Q_1, and denote the maximal flat torus in Ω associated with $\mathfrak{T}_\Omega$ as T_Ω. We make here a correspondence γ between T_Ω and T_0. Put $\gamma=\exp\pi/4\,\mathrm{ad}(e_4F_2e_0+e_5K_2e_1-e_6K_2e_2+e_7K_2e_3)$, then this is an isometry of Π.

LEMMA 5.5. (1) *$\gamma\cdot T_0=T_\Omega$ holds· especially $\gamma\cdot P(K_1)=R_1$, $\gamma\cdot P(K_3)=Q_2$, $\gamma\cdot Q_1=Q_1$ and $\gamma\cdot Q_3=Q_3$.* (2) *$\gamma^2=-1$ on T_0.*

Proof. We can see $\gamma\cdot T_0=T_\Omega$ from $\gamma K_2=-e_4(I_1+I_2)e_0$ and $\gamma(e_iK_2e_\imath)=1/3(D^{(1)}_{e_i,e_4e_i}+e_4(I_1-I_2)e_0)$. That $\gamma^2(e_iK_2e_i)=-e_iK_2e_\imath$ implies $\gamma^2=-1$ on T_0. Since $\gamma\cdot Q_1=Q_1$ and $\gamma\cdot P(K_1)=R_2$ can be obtained easily by direct calculations, we have $\gamma\cdot Q_3=Q_3$: $\gamma\cdot Q_3=\exp\pi/2(\mathrm{ad}\, e_4F_2e_0)\exp\pi/4(\mathrm{ad}(-e_4F_2e_0+e_5K_2e_1-e_6K_2e_2+e_7K_2e_3))\cdot Q_3=\exp\pi/2(\mathrm{ad}\, e_4F_2e_0)\cdot\delta_2\gamma\delta_2^{-1}\cdot(\delta_2\cdot Q_1)$ (by $\delta_2\cdot Q_1=Q_3$)$=\exp\pi/2(\mathrm{ad}\, e_4F_2e_0)\cdot Q_3$ (by $\gamma\cdot Q_1=Q_1$)$=P(y_1)$ (by direct calculations)$=Q_3$ (by Lemma 5.3). Next we give

$\gamma\cdot P(K_3)=Q_2$ by the similar method: $\gamma\cdot P(K_3)=\exp\pi/2(\mathrm{ad}\ e_4F_2e_0)\cdot\exp\pi/2(\mathrm{ad}\ D_{e_1,e_2})\cdot\delta_2\cdot(\gamma\cdot P(K_1))=\exp\pi/2(\mathrm{ad}\ e_4F_2e_0)\cdot R_3=P(x_1)$ (by direct calculations)$=Q_2$ (by Lemma 5.3), where the second equality is derived from $\gamma\cdot P(K_1)=R_1$, $\delta_2\cdot R_1=R_3$ and $\exp\pi/2(\mathrm{ad}\ D_{e_1,e_2})\cdot R_3=R_3$.

COROLLARY 5.6. (1) *Four points R_1, Q_1, Q_2 and Q_3 are different from one another.* (2) *$\gamma\cdot T_\Omega=T_0$ holds· especially $\gamma\cdot R_1=P(K_1)$, $\gamma\cdot Q_2=P(K_3)$, $\gamma\cdot Q_1=Q_1$ and $\gamma\cdot Q_3=Q_3$.*

LEMMA 5.7. *If λ is a root of type $-2(a_i\pm a_j)\boldsymbol{i}$, the set* $\exp(\mathrm{ad}\,U_\lambda)$ *is contained in $U(Q_1)\cap U(P(K_3))$.*

Proof. If λ is such a root, both Q_1 and $P(K_3)$ are contained in S_λ because $Q_1=\phi(\pi/4, \pi/4, \pi/4, \pi/4)$ and $P(K_3)=\phi(\pi/2, 0, 0, 0)$. Therefore the inclusion $\mathfrak{U}_\lambda\subset\mathfrak{U}(Q_1)\cap\mathfrak{U}(P(K_3))$ holds by the identity $\mathfrak{U}(Q)=\mathfrak{U}(\mathfrak{T}_0)\oplus\Sigma\,\mathfrak{U}_\lambda$. This gives the lemma.

LEMMA 5.8. *Three points Q_2, Q_3 and R_1 are fixed by the identity component of the isotropy group at Q_1 with respect to the isometry group of Ω.*

Proof. Let $I(P)$ denote the isotropy group at P with respect to the isometry group of Π. Note that $U(P(K_2))=U(P(K_3))$ is equivalent to $I(P(K_1))\cap I(P(K_2))=I(P(K_1))\cap I(P(K_3))$. By operating an isometry $\exp\pi/4\ \mathrm{ad}(K_1+e_1K_1e_1+e_2K_1e_2+e_3K_1e_3)$ on this relation, we have $I(P(K_1))\cap I(Q_2)=I(P(K_1))\cap I(Q_3)$. By making use of δ_2 further, $I(P(K_3))\cap I(Q_2)=I(P(K_3))\cap I(Q_1)$ can be found. It shows $I(Q_1)\cap U(P(K_3))\subset I(Q_2)$ which asserts the lemma for Q_2. For Q_3, by the action of δ_1 on this inclusion relation and by $U(P(K_2))=U(P(K_3))$, we can see $I(Q_1)\cap U(P(K_3))\subset I(Q_3)$. For the case of R_1, by operating $\gamma\delta_2\gamma$ on $I(P(K_1))\cap I(Q_2)=I(P(K_1))\cap I(Q_3)$, we also obtain $I(P(K_1))\cap I(R_1)=I(P(K_1))\cap I(Q_1)$ from Lemma 5.5 and Corollary 5.6, where γ is the same as the one in Lemma 5.5. This implies $I(Q_1)\cap U(P(K_3))\subset I(R_1)$.

LEMMA 5.9. *If a point $P\in\Omega$ is commutative with Q_1 and Q_3 and P has the distance 6π from these points, then $P=Q_2$ or R_1 hold.*

Proof. Let $P\in\Omega$ satisfy the assumption in the lemma. Then there exists α in the identity component of the isotropy group at Q_1 (with respect to the isometry group of Ω) such that $\alpha\cdot P\in T_\Omega$ by the transitivity of maximal flat tori of Ω passing through Q_1. Since $\alpha\cdot Q_3=Q_3$ by Lemma 5.8, $\alpha\cdot P$ satisfies the same assumption as P. Hence we obtain $\alpha\cdot P=Q_2$ or R_1 from Lemma 3.5 and Corollary 5.6. This means $P=Q_2$ or R_1 by Lemma 5.8. Conversely we can see easily from Corollary 5.6 that Q_2 and R_1 satisfy the assumption in the lemma. The proof is completed.

For $Q\in T_0$, three sets $\{\Xi_i\}$ are defined by $\Xi_1=\{\alpha\cdot P(K_2)\mid\alpha\in U(Q)_0\}$, $\Xi_2=\{\alpha\cdot Q_2\mid\alpha\in U(Q)_0\}$ and $\Xi_3=\{\alpha\cdot R_1\mid\alpha\in U(Q)_0\}$, where 0 means the identity component of $U(Q)$. Then we have the following.

PROPOSITION 5.10. *Let $Q\in T_0$. Then a line $L(P)$ passes through two distinct points $P(K_1)$ and Q if and only if $P\in\Xi_1\cup\Xi_2\cup\Xi_3$ holds.*

Proof. First the necessity is showed. If $L(P)$ is such a line, there exists in $L(P)$ a maximal flat torus T with the dimension 4 such that $P(K_1)$, $Q\in T$ because the rank of $L(P)$ is 4 as a symmetric space. Moreover, there exists $z\in\mathfrak{U}(Q)$ by Lemma 5.2 such that $\alpha\cdot T_0=T$, where $\alpha=\exp(\mathrm{ad}\, z)$ and so $\alpha\in U(Q)_0$. This means $T_0\subset L(\alpha^{-1}\cdot P)$. Hence $\alpha^{-1}\cdot P$ is commutative with $P(K_1)$, $P(K_3)$, Q_1 and Q_3, and $\alpha^{-1}\cdot P$ has the distance 6π from the points. From these facts it holds $\alpha^{-1}\cdot P\in L(P(K_1))\cap L(P(K_3))$ and, therefore, we have $\alpha^{-1}\cdot P=P(K_2)$ or $\alpha^{-1}\cdot P\in\Omega$ by Lemma 5.4. In the first case, $P=\alpha\cdot P(K_2)\in\Xi_1$. In the latter case, $\alpha^{-1}\cdot P=Q_2$ or R_1 by Lemma 5.9. This implies $P\in\Xi_2\cup\Xi_3$.

Next the sufficiency is showed. Let P be contained, for instance, in Ξ_3. Then there exists $\alpha\in U(Q)_0$ such that $P=\alpha\cdot R_1$. On the other hand, since $R_1\in\Omega\subset L(P(K_1))$ from Lemma 5.4, we have $P(K_1)\in L(R_1)$ by the duality of L (see Lemma 5.1). Since T_0 is spanned by $\{\exp t(\mathrm{ad}\, e_iK_2e_i)\}$ as an orbit of $P(K_1)$ and these transformations leave R_2 $(=R_1)$ fixed, we obtain $T_0\subset L(R_1)$. Hence $\alpha\cdot T_0\subset L(P)$. This shows that the line $L(P)$ passes through $P(K_1)$ and Q because α leaves $P(K_1)$ and Q fixed. In the case of $P\in\Xi_1$ or Ξ_2, the assertion can be showed similarly. The proof is completed.

Let Q be a regular point in T_0, i.e. satisfying $\mathfrak{U}(Q)=\mathfrak{U}(\mathfrak{T}_0)$. Since $U(Q)_0=\exp(\mathrm{ad}\,\mathfrak{U}(Q))$ and $\mathfrak{U}(Q)\subset\mathfrak{U}(Q_1)\cap\mathfrak{U}(P(K_3))$ hold, we obtain $U(Q)_0\subset U(Q_1)\cap U(P(K_3))$. This implies by Lemma 5.8 that $U(Q)_0$ leaves Q_2 and R_1 fixed. $U(Q)_0$ also does $P(K_2)$ fixed because $U(P(K_2))=U(P(K_3))$. Therefore, for the above lemma, we can assert the following.

COROLLARY 5.11. *If $Q\in T_0$ is a regular point for $P(K_1)$, there exist exactly three lines $L(P(K_2))$, $L(Q_2)$ and $L(R_1)$ which pass through $P(K_1)$ and Q.*

For any positive root $\lambda\in\Delta_{T_0}$ the set S_λ becomes a 3-dimensional flat torus in T_0 because there exists $x\in\mathfrak{T}_0$ such that $\lambda(x)=\pi\boldsymbol{i}$ and $\exp(\mathrm{ad}\, x)\cdot P(K_1)=P(K_1)$. Hence S_λ is said to be the torus associated with λ. If $Q\in S_\lambda$, we say that λ passes through Q. From the list of the positive roots in Section 4, each S_λ has three shortest closed geodesics of Π as generating elements. If $\lambda=-2a_0\boldsymbol{i}$, for example, such the geodesics $\{r_i(t)\}$ can be defined by $r_1(t)=\exp t(\mathrm{ad}(e_1K_2e_1+e_2K_2e_2))\cdot P(K_1)$, $r_2(t)=\exp t(\mathrm{ad}(e_1K_2e_1-e_2K_2e_2))\cdot P(K_1)$ and $r_3(t)=\exp t(\mathrm{ad}(e_2K_2e_2+e_3K_2e_3))\cdot P(K_1)$. The volume of each torus S_λ is $432\pi^3$, $432\sqrt{2}\pi^3$ or $432\pi^3$ according as the root λ has the type $-2a_i\boldsymbol{i}$, $-2(a_i\pm a_j)\boldsymbol{i}$ or $-(a_0\pm a_1\pm a_2\pm a_3)\boldsymbol{i}$.

From now on we will study the converse of the above facts. The result is given in Prop. 5.13. Put $x=a_i\pi/2e_iK_2e_i+a_j\pi/2\, e_jK_2e_j$, where $i\neq j$ and a_i, $a_j\in\boldsymbol{Z}-\{0\}$. Assume that the geodesic $r(t)=\exp t(\mathrm{ad}\, x)\cdot P(K_1)$ satisfies $r(1)=P(K_1)$.

LEMMA 5.12. *If $r(t)$ first returns to $P(K_1)$ at $t=1$, one has $\langle a_i, a_j\rangle=1$ or 2, where $\langle\,,\rangle$ means the greatest common divior.*

Proof. Put $\langle a_i, a_j\rangle=2^l n$, where n is a positive integer such that $\langle n, 2\rangle=1$. Since $r(1)=P(K_1)$, we have $a_i+a_j=2m$, $m\in\boldsymbol{Z}$, from Lemma 3.2. There exists $m_0\in\boldsymbol{Z}$ such that $m=nm_0$ because $\langle n, 2\rangle=1$. Then a_i/n, $a_j/n\in\boldsymbol{Z}$ and $a_i/n+a_j/n=2m_0$ hold. This gives $r(1/n)=P(K_1)$ by Lemma 3.2. If $n\neq 1$, it contradicts the our assumption because $0<1/n<1$. So we may consider only the case $\langle a_i, a_j\rangle=2^l$. If $l\geqq 2$, $r(1/2)=P(K_1)$ again by the same reason as above. This also contradicts ours. Therefore we obtain $l=0$ or 1, i.e. $\langle a_i, a_j\rangle=1$ or 2.

PROPOSITION 5.13. *Let T^3 be any 3-dimensional torus in T_0. Assume T^3 contains $P(K_1)$ and has the minimal value of volume. Then T^3 is oue of the twelve tori associated with the roots of type $-2a_i\boldsymbol{i}$ and $-(a_0\pm a_1\pm a_2\pm a_3)\boldsymbol{i}$. The minimal value is $432\pi^3$.*

Proof. Let T^3 be such a torus in T_0. T^3 has three geodesics $\exp t(\mathrm{ad}\, z_i)\cdot P(K_1)$ as generating elements, where $z_1=\Sigma\, a_i\pi/2\, e_iK_2e_i$, $z_2=\Sigma\, b_i\pi/2\, e_iK_2e_i$ and $z_3=\Sigma\, c_i\pi/2\, e_iK_2e_i$. Assume these geodesics first return to $P(K_1)$ at $t=1$. Then we obtain from Lemma 3.2 that a_i, b_i, $c_i\in\boldsymbol{Z}$ and $\Sigma\, a_i$, $\Sigma\, b_i$, $\Sigma\, c_i\in 2\boldsymbol{Z}$. Define a mapping ψ of the 3-dimensional Euclidean space $\boldsymbol{R}^3$ onto T^3 by $\psi(t_1, t_2, t_3)=\exp(\mathrm{ad}(t_1z_1+t_2z_2+t_3z_3))\cdot P(K_1)$.

First we consider the case of $a_3=b_3=c_3=0$. Moreover, if $a_2=b_2=c_2=0$, this leads to a contradiction because $\{z_i\}$ are linearly independent. So we may assume $a_2\neq 0$ without the loss of generality. If $b_2\neq 0$, put $w_2=a_2z_2-b_2z_1$. Then $w_2\in\mathfrak{T}_0$ and $\exp t(\mathrm{ad}\, w_2)\cdot P(K_1)\in T^3$. Note that z_1, w_2, z_3 are also linearly independent. Since $\exp(\mathrm{ad}\, w_2)\cdot P(K_1)=P(K_1)$, there exists the minimal value $t_0\in(0, 1]$ such that $\exp t_0(\mathrm{ad}\, w_2)\cdot P(K_1)=P(K_1)$. Write again z_1, t_0w_2, z_3 as z_1, z_2, z_3 respectively, then b_2 can be considered to be 0. By the same reason, $c_2=0$. Since $b_1\neq 0$ and $c_0\neq 0$ can be assumed, we may say $a_1=c_1=0$ and $a_0=b_0=0$. After all, the above argument asserts that T^3 can have three tangent vectors $z_1=a_2\pi/2e_2K_2e_2$, $z_2=b_1\pi/2e_1K_2e_1$ and $z_3=c_0\pi/2K_2$ (a_2, b_1, $c_0>0$) such that each geodesic $\exp t(\mathrm{ad}\, z_i)\cdot P(K_1)$ first returns to $P(K_1)$ at $t=1$. Then Lemma 3.2 gives $a_2=b_1=c_0=2$. T^3 turns out the torus associated with the root $-2a_3\boldsymbol{i}$. The volume $\mathrm{vol}(T^3)$ can be calculated by making use of the fact that ψ is a bijective map for $0\leqq t_1<1$ and $0\leqq t_2$, $t_3<1/2$:

$$\mathrm{vol}(T^3)=\int_0^{1/2}\int_0^{1/2}\int_0^1\sqrt{g}\; dt_1dt_2dt_3=432\pi^3,$$

where $g=\det(g_{ij})$ with $g_{ij}=-B(z_i, z_j)$.

Secondly suppose that one of a_3, b_3, c_3 is not 0 at least. Then there remain in essential three cases to study. We consider these by the same method as the first case.

(i) In this case $\{z_i\}$ satisfy that $z_1=a_2\pi/2e_2K_2e_2+a_3\pi/2e_3K_2e_3$, $z_2=\pi e_1K_2e_1$ and $z_3=\pi K_2$, where a_2, $a_3\in\boldsymbol{Z}-\{0\}$ and $a_2+a_3\in 2\boldsymbol{Z}$. If a_2, a_3 are even numbers and $a_2+a_3+2\in 4\boldsymbol{Z}$, ψ is an injective map on the set $\{(t_1, t_2, t_3)\in\boldsymbol{R}^3 \mid 0\leqq t_1<1, 0\leqq t_2<1/2, 0\leqq t_3<1/2\}$. Then we have $\mathrm{vol}(T^3)=216(a_2^2+a_3^2)^{1/2}\pi^3\geqq 432\sqrt{5}\pi^3$. The equality holds, for instance, when $a_2=4$ and $a_3=2$. If a_2, a_3 are not so, since ψ is injective for $0\leqq t_1<1$, $0\leqq t_2<1/2$, $0\leqq t_3<1$, we get $\mathrm{vol}(T^3)\geqq 432\sqrt{2}\pi^3$. The equality can be given by $a_2=1$ and $a_3=-1$.

(ii) This is the case that $\{z_i\}$ have the forms that $z_1=a_0\pi/2K_2+a_3\pi/2e_3K_2e_3$, $z_2=b_0\pi/2K_2+b_2\pi/2e_2K_2e_2$ and $z_3=\pi e_1K_2e_1$, where $a_0, a_3, b_0, b_2\in\boldsymbol{Z}-\{0\}$ and a_0+a_3, $b_0+b_2\in 2\boldsymbol{Z}$. Since both z_1 and z_2 satisfy the assumption in Lemma 5.12, we obtain $\langle a_0, a_3\rangle, \langle b_0, b_2\rangle=1$ or 2. If $|b_2|=1$, ψ is an injective map on the set $\{(t_1, t_2, t_3)\in\boldsymbol{R}^3 \mid 0\leqq t_1<1/|a_3|, 0\leqq t_2<1, 0\leqq t_3<1\}$. Hence $\mathrm{vol}(T^3)\geqq 432((a_0b_2/a_3)^2+b_0^2+b_2^2)^{1/2}\pi^3>432\sqrt{2}\pi^3$. If $|b_2|>1$, since ψ is injective for $0\leqq t_1<1/|a_3|$, $0\leqq t_2<1$, $0\leqq t_3<1/2$, we have $\mathrm{vol}(T^3)\geqq 216\sqrt{5}\pi^3>432\pi^3$.

(iii) In this case $\{z_i\}$ have the forms that $z_1=a_0\pi/2K_2+a_3\pi/2e_3K_2e_3$, $z_2=b_0\pi/2K_2+b_2\pi/2e_2K_2e_2$ and $z_3=c_1\pi/2e_1K_2e_1+c_2\pi/2e_2K_2e_2$, where $a_0, a_3, \cdots, c_2\in\boldsymbol{Z}-\{0\}$ and a_0+a_3, b_0+b_2, $c_1+c_2\in 2\boldsymbol{Z}$. Lemma 5.12 gives $\langle a_0, a_3\rangle, \langle b_0, b_2\rangle, \langle c_1, c_2\rangle=1$ or 2. If $|b_0|=|b_2|=1$ does not hold, ψ is an injective map on the set $\{(t_1, t_2, t_3)\in\boldsymbol{R}^3 \mid 0\leqq t_1<1/|a_3|, 0\leqq t_2<1, 0\leqq t_3<1/|c_1|\}$. Hence $\mathrm{vol}(T^3)\geqq 216((a_0b_2/a_3)^2+(b_0c_2/c_1)^2+b_0^2+b_2^2)^{1/2}\pi^3>432\pi^3$. If $|b_0|=|b_2|=1$ holds and $|a_0|=|a_3|=1$ does not hold, ψ is injective for $0\leqq t_1<1$, $0\leqq t_2<1$, $0\leqq t_3<1/|c_1|$. Then we obtain $\mathrm{vol}(T^3)\geqq 216((a_3c_2/c_1)^2+a_0^2+2a_3^2)^{1/2}\pi^3>432\pi^3$. Finally, if $|b_0|=|b_2|=|a_0|=|a_3|=1$, since ψ is injective for $0\leqq t_1, t_2, t_3<1$, we have $\mathrm{vol}(T^3)\geqq 216(3c_1^2+c_2^2)^{1/2}\pi^3\geqq 432\pi^3$. The equality can be established when $|c_1|=|c_2|=1$. Then T^3 is associated with a root of type $-(a_0\pm a_1\pm a_2\pm a_3)\boldsymbol{i}$.

The above argument shows that the minimal volume of 3-dimensional flat tori in T_0 is $432\pi^3$ and its value is attained by the tori associated with the roots of type $-2a_i\boldsymbol{i}$ or $-(a_0\pm a_1\pm a_2\pm a_3)\boldsymbol{i}$. The proof is completed.

COROLLARY 5.14. *Let T^3 be a 3-dimensional torus in Π. If T^3 has the minimal volume, it has three shortest closed geodesics as generating elements.*

Definition. (1) Two distinct points in Π are said to be in the general position if any 3-dimensional flat torus with the minimal volume does not contain both of them. If not so, they are said to be in the singular position. (2) Two distinct lines $L(P)$ and $L(Q)$ in Π are said to be in the general (resp. singular) position if P and Q are in the general (resp. singular) position.

PROPOSITION 5.15. *A point Q in Π is a singular point with respect to $P(K_1)$ if and only if there exists a 3-dimensional flat torus passing through $P(K_1)$ and Q such that it has three shortest closed geodesics with the initial point $P(K_1)$ as generating elements.*

Proof. We first show the necessity. Let Q be such a singular point. There exists $\alpha\in U$ such that $\alpha\cdot Q\in T_0$, where U is the isotropy group at $P(K_1)$. Since $\alpha\cdot Q$ is also a singular point, we can find a root $\lambda\in\Delta_{T_0}$ such that $\alpha\cdot Q\in S_\lambda$. S_λ

is generated by three shortest closed geodesics. Hence $\alpha^{-1}\cdot S_\lambda$ contains Q and satisfies the condition in the proposition. Next the sufficiency is showed. Let T^3 be the torus satisfying the condition. From the transitivity of maximal flat tori, we may assume $Q\in T^3\subset T_0$. (v) in Lemma 3.5 gives all the shortest closed geodesics in T_0 with the initial point $P(K_1)$: $\phi(t\pi/4, t\pi/4, 0, 0), \cdots, \phi(0, 0, t\pi/4, -t\pi/4)$, where $0\leqq t<2$. The number of the geodesics is 12. Moreover, since any 3-dimensional flat torus determined by three geodesics in them is certainly associated with some positive root λ, therefore $Q\in T^3=S_\lambda$ holds. This means that Q is a singular point.

COROLLARY 5.16. *If $P(K_1)$ and Q are in the singular position, Q is a singular point with respect to $P(K_1)$. The converse is not always true.*

THEOREM 5.17. *Π is a projective plane in the wider sense, that is, Π satisfies the following properties:*

(1) *For two distinct points there exist exactly three lines passing through them if the points are in the general position. If in the singular position, the set of lines passing through the points forms a symmetric space as a manifold.*

(2) *The correspondence L asserts the duality of* (1) *for two distinct lines.*

Proof. Since (2) can be derived from (1) and Lemma 5.1, we show only (1). Let P and Q be two distinct points in Π. We may assume $P=P(K_1)$ and $Q\in T_0$ by the transitivity of points and of maximal flat tori. Let a line $L(R)$ pass through $P(K_1)$ and Q. Then $R\in\Xi_1\cup\Xi_2\cup\Xi_3$ by Prop. 5.10. If $P(K_1)$ and Q are in the general position, Prop. 5.13 gives (i) Q is a regular point with respect to $P(K_1)$ or (ii) Q is the point which only the roots of type $-2(\boldsymbol{a}_i\pm \boldsymbol{a}_j)\boldsymbol{i}$ pass through. If (i) holds, Corollary 5.11 shows $R=P(K_2)$, Q_2 or R_1. If (ii) holds, Lemma 5.7 and 5.8 give the fact again since $\mathfrak{U}(Q)=\mathfrak{U}(\mathfrak{T}_0)\oplus\Sigma\,\mathfrak{U}_\lambda$ for some λ of type $-2(\boldsymbol{a}_i\pm \boldsymbol{a}_j)\boldsymbol{i}$. On the other hand, if $P(K_1)$ and Q are in the singular position, the following lemma finishes the proof.

LEMMA 5.18. *If $P(K_1)$ and Q are in the singular position, the set of lines passing through them makes six kinds of symmetric spaces as submanifolds in Π^L, that is,* (1) *$SO(n+4)/SO(n)\cdot SO(4)\cup\{$one isolated point$\}$ $(n=1, 2, 3, 4)$,* (2) *$Sp(3)/Sp(2)\cdot Sp(1)$ and $SU(6)/S(U(4)\cdot U(2))$.*

Proof. Let Π^L have the differential structure introduced by L from Π. Let $\Gamma\subset\Pi^L$ be the set of lines passing through $P(K_1)$ and Q. We may assume $Q\in T_0$. Denote by $n(\lambda)$ the number of positive roots λ such that $Q\in S_\lambda$. First we consider the case of $n(\lambda)=1$. If $\lambda=-2\boldsymbol{a}_0\boldsymbol{i}$, $U(Q)_0$ leaves $P(K_2)$ fixed because $\mathfrak{U}(Q)=\mathfrak{U}(\mathfrak{T}_0)\oplus\mathfrak{U}_{-2\boldsymbol{a}_0\boldsymbol{i}}$ and hence $\mathfrak{U}(Q)\subset\mathfrak{G}_0(K_2)\oplus\mathfrak{G}_2(K_2)$ by the list in Section 4. Moreover, $\exp\pi/2(\mathrm{ad}\,x)\cdot Q_2=R_2$ holds, where $x=e_4(I_1+I_2)e_0$ and so $x\in\mathfrak{U}_{-2\boldsymbol{a}_0\boldsymbol{i}}$. This shows by Prop. 5.10 that (i) Ξ_1 is an isolated point $P(K_2)$ and (ii) $\Xi_2\cup\Xi_3$ is a connected symmetric space with the type $\mathfrak{U}(Q)/\mathfrak{U}(\mathfrak{T}_0)$ $(\cong so(5)/so(4))$. In fact $\Xi_2\cup\Xi_3$ turns out to be the 4-dimensional sphere S^4. Therefore $L^{-1}(\Gamma)=$

$S^4\cup\{P(K_2)\}$. When λ has the type $-2a_i\boldsymbol{i}$ ($i\geqq 1$) or $-(a_0\pm a_1\pm a_2\pm a_3)\boldsymbol{i}$, we also get the same result. For instance, if $\lambda=-(a_0-a_1-a_2-a_3)\boldsymbol{i}$, put $\alpha=\exp -\pi/4\,\mathrm{ad}(K_1+e_1K_1e_1+e_2K_1e_2+e_3K_1e_3)$. Then $\alpha\cdot T_0=T_0$, $\alpha\cdot P(K_1)=P(K_1)$ and $\alpha\mathfrak{U}_\lambda=\mathfrak{U}_{-2a_0i}$ hold. Hence, by this α, the argument for λ comes back to that for $-2a_0\boldsymbol{i}$. Therefore, since $\alpha\cdot P(K_2)=Q_2$, $\alpha\cdot Q_2=P(K_2)$ and $\alpha\cdot R_1=R_1$, we can obtain $L^{-1}(\Gamma)=S^4\cup\{Q_2\}$. Next, if $\lambda=-(a_0+a_1+a_2+a_3)\boldsymbol{i}$, put $\alpha=\exp -\pi/4\,\mathrm{ad}(e_4F_1e_0+e_5K_1e_1-e_6K_1e_2+e_7K_1e_3)$. Noting that $\alpha\cdot P(K_2)=R_2$, $\alpha\cdot R_2=P(K_2)$ and $\alpha\cdot Q_2=Q_2$, we have $L^{-1}(\Gamma)=S^4\cup\{R_1\}$ by the same method.

Secondly we consider the case of $n(\lambda)\geqq 2$. Then there remain in essential six cases to study. (i) $\{\lambda\}=\{-2a_0\boldsymbol{i}, -2a_1\boldsymbol{i}, -2(a_0\pm a_1)\boldsymbol{i}\}$: Then $L^{-1}(\Gamma)=SO(6)/SO(2)\cdot SO(4)\cup\{P(K_2)\}$ holds. (ii) $\{\lambda\}=\{-2a_0\boldsymbol{i}, -2(a_1+a_2)\boldsymbol{i}\}$: This is the same case as $\{\lambda\}=\{-2a_0\boldsymbol{i}\}$. Hence $L^{-1}(\Gamma)=S^4\cup\{P(K_2)\}$. (iii) $\{\lambda\}=\{-2a_0\boldsymbol{i}, -(a_0+a_1-a_2-a_3)\boldsymbol{i}, -(a_0-a_1+a_2+a_3)\boldsymbol{i}\}$: We have $L^{-1}(\Gamma)=Sp(3)/Sp(2)\cdot Sp(1)$. This is the quaternion projective plane. (iv) $\{\lambda\}=\{-2a_i\boldsymbol{i}, -2(a_i+a_j)\boldsymbol{i}\}$, $(i, j=0, 1, 2)$: Then $L^{-1}(\Gamma)=SO(7)/SO(3)\cdot SO(4)\cup\{P(K_2)\}$. (v) $\{\lambda\}=\{-2a_0\boldsymbol{i}, -2a_1\boldsymbol{i}, -2(a_0\pm a_1)\boldsymbol{i}, -2(a_2+a_3)\boldsymbol{i}, -(a_0\pm a_1-a_2-a_3)\boldsymbol{i}, -(a_0\pm a_1+a_2+a_3)\boldsymbol{i}\}$: We have $L^{-1}(\Gamma)=SU(6)/S(U(4)\cdot U(2))$. This is a maximal submanifold in $L^{-1}(\Gamma)$ with respect to the inclusion relation. (vi) $\{\lambda\}=\{-2a_i\boldsymbol{i}, -2(a_i\pm a_j)\boldsymbol{i}\}$, $(i, j\geqq 0)$: We obtain $L^{-1}(\Gamma)=SO(8)/SO(4)\cdot SO(4)$. This is maximal too.

As a consequence, we can assert the following. If Q is a singular point, $S^4\cup\{P\}$ is minimal in $\{L^{-1}(\Gamma)\}$, where P is some isolated point. This manifold has three possible kinds of extension: (i) $SO(n+4)/SO(n)\cdot SO(4)\cup\{P\}$ $(n=2, 3, 4)$. (ii) $Sp(3)/Sp(2)\cdot Sp(1)\subset SU(6)/S(U(4)\cdot U(2))$. (iii) $SO(6)/SO(2)\cdot SO(4)\subset SU(6)/S(U(4)\cdot U(2))$.

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KUMAMOTO INSTITUTE OF TECHNOLOGY
IKEDA, KUMAMOTO 860
JAPAN