

PROBABILISTIC CHARACTERIZATION OF CERTAIN BANACH SPACES

Dedicated to Professor Hisaharu Umegaki on his 60th birthday

BY JUN KAWABE

1. Introduction.

Let (Ω, \mathcal{A}, P) be a probability measure space, X a real separable Banach space and X^* its topological dual space. In this paper we look into the relation between the convergence of a sequence $(\xi_n)_{n \geq 1}$ of random variables with values in X (for short, random elements) and the convergence of the associated sequence $(\langle \xi_n, f \rangle)_{n \geq 1}$, $f \in X^*$, of real random variables. In [8], when X is a Hilbert space, the author has treated this problem for mean square convergence with the aid of covariance operators or more specifically, with the aid of the relative compactness of covariance operators with respect to the *trace norm*. In Chevet-Chobanjan-Linde-Tarieladze [1, 10] they showed that covariance operators on Banach spaces introduced in Vakhania [14] are nuclear operators which were defined by Grothendieck [6]. This leads us to consider the following

PROPOSITION (A) ξ_n converges in $L_2(\Omega; X)$ if and only if the following two conditions are satisfied:

(a) For each $f \in X^*$, $\langle \xi_n, f \rangle$ converges in $L_2(\Omega; \mathbf{R})$, where \mathbf{R} is the real line.

(b) The set $\{R_{\xi_n}\}$ of covariance operators is relatively compact with respect to the nuclear norm.

This Proposition (A), however, is not valid in general. The purpose of this paper is to find necessary and sufficient conditions on the structure of Banach spaces in order that the Proposition (A) holds.

In section 2 we shall give some definitions and preliminary results. In Section 3 we shall show that the Proposition (A) is not valid unless X is isomorphic to a Hilbert space. This result will be proved by an isomorphic characterization of Hilbert spaces which was obtained by Kwapien [9]. In Section 4 we shall also prove that the Proposition (A) holds only for Gaussian random elements if and only if X is of type 2.

The author thanks Professor Hisaharu Umegaki for his valuable advice and encouragement. He also thanks Professor Fumio Hiai who carefully read the

Received March 19, 1985

text of this paper and gave him some useful comments.

2. Preliminaries.

Let X be a real separable Banach space, X^* its topological dual space and $\mathcal{B}(X)$ the Borel field of X . By a *random element* in X defined on a basic probability space (Ω, \mathcal{A}, P) we mean a measurable mapping $\xi : (\Omega, \mathcal{A}) \rightarrow (X, \mathcal{B}(X))$. Every random element ξ induces on $(X, \mathcal{B}(X))$ the probability measure $\mu_\xi = P \circ \xi^{-1}$ which is called its *distribution*. Let μ be a probability measure on X . The *characteristic functional* of μ is defined by

$$\hat{\mu}(f) = \int_X \exp\{i\langle x, f \rangle\} \mu(dx), \quad f \in X^*.$$

We say that a sequence $(\mu_n)_{n \geq 1}$ of probability measures on X *converges weakly* if there exists a probability measure μ on X such that

$$\int_X \varphi(x) \mu_n(dx) \rightarrow \int_X \varphi(x) \mu(dx) \quad \text{as } n \rightarrow \infty$$

for every bounded continuous real function φ on X . It is well known ([12]) that the topology on probability measures determined by the above weak convergence is metrizable by Prokhorov metric. A set $\{\xi_\alpha\}$ of random elements in X is said to be *uniformly tight* if for each $\varepsilon > 0$, there exists a compact subset C_ε of X such that

$$\mu_{\xi_\alpha}(C_\varepsilon) > 1 - \varepsilon \quad \text{for all } \alpha.$$

Let us recall that a subset A of a metric space is said to be *relatively compact* if every sequence in A contains a convergent subsequence. According to [12] the set $\{\xi_\alpha\}$ is uniformly tight if and only if the set $\{\mu_{\xi_\alpha}\}$ is relatively compact with respect to Prokhorov metric.

Denote by $L_2(\Omega; X)$ the Banach space of all random elements in X which satisfy

$$E\|\xi\|^2 \equiv \int_\Omega \|\xi(\omega)\|^2 P(d\omega) < \infty,$$

where E denotes the expectation with respect to P . We shall also denote by $M(\Omega; X)$ the Fréchet space of all random elements in X with the metric ρ given by

$$\rho(\xi, \eta) = E\left\{ \frac{\|\xi - \eta\|}{1 + \|\xi - \eta\|} \right\}, \quad \xi, \eta \in M(\Omega; X).$$

We say that a sequence $(\xi_n)_{n \geq 1}$ *converges in probability* if there exists a $\xi \in M(\Omega; X)$ such that $\rho(\xi_n, \xi) \rightarrow 0$ as $n \rightarrow \infty$. It is well known that ξ_n converges in probability to ξ if and only if, for each $\varepsilon > 0$, $P(\|\xi_n - \xi\| \geq \varepsilon) \rightarrow 0$ as $n \rightarrow \infty$. Most of the relations which exist among the various modes of convergence for real random variables are also valid for random elements. For instance, almost sure

convergence or convergence in $L_2(\Omega; X)$ imply convergence in probability, and convergence in probability implies weak convergence, and hence uniform tightness. When X is the real line we abbreviate $L_2(\Omega; X)$ and $M(\Omega; X)$ to $L_2(\Omega)$ and $M(\Omega)$ respectively. A random element ξ in X is said to be *Gaussian* if for each $f \in X^*$, $\langle \xi, f \rangle$ is a (possibly degenerate) real Gaussian random variable on (Ω, \mathcal{A}, P) . Every Gaussian random element belongs to $L_2(\Omega; X)$ (see Fernique [5]).

Let X, Y be Banach spaces. A linear operator $T: X \rightarrow Y$ is said to be *nuclear* if it admits the representation

$$Tx = \sum_{k=1}^{\infty} \langle x, f_k \rangle y_k, \quad x \in X,$$

where $f_k \in X^*$, $y_k \in Y$ and $\sum_{k=1}^{\infty} \|f_k\| \|y_k\| < \infty$. The greatest lower bound of the sum $\sum_{k=1}^{\infty} \|f_k\| \|y_k\| < \infty$ taken over all possible representations of the operator T is called the *nuclear norm* of T and denoted by $\nu(T)$ (see [6]). We shall denote by $\mathcal{N}(X, Y)$ the Banach space of all nuclear operators from X into Y with the nuclear norm $\nu(\cdot)$. A linear operator $R: X^* \rightarrow X$ is said to be *positive* if $\langle Rf, f \rangle \geq 0$ for all $f \in X^*$ and *symmetric* if $\langle Rf, g \rangle = \langle Rg, f \rangle$ for all $f, g \in X^*$. According to [1, 10], for every $\xi \in L_2(\Omega; X)$ there exist a *mean vector* $m_\xi \in X$ and a *covariance operator* $R_\xi \in \mathcal{N}(X^*, X)$ such that

$$\langle m_\xi, f \rangle = E \langle \xi, f \rangle, \quad f \in X^*$$

and

$$\langle R_\xi f, g \rangle = E \langle \xi - m_\xi, f \rangle \langle \xi - m_\xi, g \rangle, \quad f, g \in X^*.$$

Then R_ξ is positive and symmetric, and has the estimate

$$\nu(R_\xi) \leq E \|\xi - m_\xi\|^2.$$

Let $(\gamma_n)_{n \geq 1}$ be a sequence of independent standard Gaussian random variables. A Banach space X is said to be of *type 2* if for each sequence $(x_n)_{n \geq 1}$ in X with the property $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$, the series $\sum_{n=1}^{\infty} \gamma_n x_n$ is convergent a. s. (=almost surely) and is said to be of *cotype 2* if for each sequence $(x_n)_{n \geq 1}$ satisfying that the series $\sum_{n=1}^{\infty} \gamma_n x_n$ is convergent a. s., we have always $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. For instance, the spaces l_p, L_p ($2 \leq p < \infty$) are of type 2 and they are of cotype 2 when $1 \leq p \leq 2$. We know that if X is of type 2 and of cotype 2 then it is isomorphic to a Hilbert space (see [9]).

3. Characterization of Hilbert spaces.

We now characterize Hilbert spaces by mean square convergence of random elements.

THEOREM 1. *For a real separable Banach space X , the following assertions (1)–(3) are equivalent:*

- (1) X is isomorphic to a Hilbert space.

- (2) For any subset K of mean-zero random elements in $L_2(\Omega; X)$, K is relatively compact if (and only if) the following two conditions are satisfied
- (a) For each $f \in X^*$, the set $\{\langle \xi, f \rangle; \xi \in K\}$ is relatively compact in $L_2(\Omega)$.
 - (b) The set $\{R_\xi; \xi \in K\}$ is relatively compact in $\mathcal{N}(X^*, X)$.
- (3) For any sequence $(\xi_n)_{n \geq 1}$ of mean-zero random elements in $L_2(\Omega; X)$, ξ_n converges if (and only if) the following two conditions are satisfied
- (c) For each $f \in X^*$, $\langle \xi_n, f \rangle$ converges in $L_2(\Omega)$.
 - (d) The set $\{R_{\xi_n}\}$ is relatively compact in $\mathcal{N}(X^*, X)$.

We need the following two results to prove Theorem 1.

THEOREM \mathcal{K} ([8; Proposition 1]). *Let H be a real separable Hilbert space with inner product (\cdot, \cdot) . A set $K \subset L_2(\Omega; H)$ is relatively compact if and only if the following two conditions are satisfied:*

- (a) For each $x \in H$, the set $\{(\xi, x); \xi \in K\}$ is relatively compact in $L_2(\Omega)$.
- (b) The set $\{m_\xi; \xi \in K\}$ is relatively compact in H and the set $\{R_\xi; \xi \in K\}$ is relatively compact in $\mathcal{N}(H, H)$ (i. e. the space of all trace class operators on H).

The following lemma follows immediately from the definition of the nuclear norm and we state it without proof.

LEMMA 1. *Let X be a real separable Banach space and $\xi \in L_2(\Omega; X)$. Then we have for every $f \in X^*$,*

$$E|\langle \xi - m_\xi, f \rangle|^2 \leq \|f\|^2 \cdot \nu(R_\xi).$$

Proof of Theorem 1. The implication (1) \Rightarrow (2) follows from Theorem \mathcal{K} . (2) \Rightarrow (3): It suffices to show that ξ_n converges in $L_2(\Omega; X)$ if we assume conditions (c) and (d). Since $\{\xi_n\}$ satisfies the conditions (a) and (b) of the assumption (2) we see that the set $\{\xi_n\}$ is relatively compact in $L_2(\Omega; X)$. Now we shall assume that ξ_n is not convergent. From this and the relative compactness of the set $\{\xi_n\}$ we can find two subsequences $(\xi_{n'})$ and $(\xi_{n''})$ converging in $L_2(\Omega; X)$ to distinct elements ξ_1 and ξ_2 respectively. Then from (c) it follows that for each $f \in X^*$ we have $\langle \xi_{n'}(\omega), f \rangle = \langle \xi_1(\omega), f \rangle$ a. s. On the other hand, since X is separable, by the Hahn-Banach theorem we can find a countable subset $\{g_k\}$ of X^* which separates points in X . Then the standard argument shows that there exists a P -null set N such that, if $\omega \notin N$, $\langle \xi_1(\omega), g_k \rangle = \langle \xi_2(\omega), g_k \rangle$ for all $k \geq 1$. Thus $\xi_1 = \xi_2$ a. s. and this is a contradiction.

(3) \Rightarrow (1): From the result of [9] we shall prove that X is of type 2 and of cotype 2. First we prove that X is of type 2. Let $(x_i)_{i \geq 1}$ be a sequence in X such that

$$\sum_{i=1}^{\infty} \|x_i\|^2 < \infty \tag{3.1}$$

and we define mean-zero Gaussian random elements as follows:

$$\xi_n(\cdot) = \sum_{i=1}^n \gamma_i(\cdot) x_i \quad n=1, 2, \dots, \quad (3.2)$$

where $(\gamma_i)_{i \geq 1}$ is a sequence of independent standard Gaussian random variables. Then by a routine calculation using the fact that

$$E[\gamma_i \gamma_j] = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \quad (3.3)$$

we have

$$R_{\xi_n} f = \sum_{i=1}^n \langle x_i, f \rangle x_i, \quad f \in X^*. \quad (3.4)$$

Then $(\xi_n)_{n \geq 1}$ satisfies the conditions (c) and (d) of the assumption (3). To see this it is sufficient to show that $(\langle \xi_n, f \rangle)_{n \geq 1}$ is a Cauchy sequence in $L_2(\mathcal{Q})$ for every $f \in X^*$ and $(R_{\xi_n})_{n \geq 1}$ is also a Cauchy sequence in $\mathcal{N}(X^*, X)$. By (3.3) we have for each $f \in X^*$ and all integers $n > m \geq 1$,

$$\begin{aligned} E|\langle \xi_n, f \rangle - \langle \xi_m, f \rangle|^2 &= E \left| \sum_{i=m+1}^n \langle x_i, f \rangle \gamma_i \right|^2 \\ &= \sum_{i,j=m+1}^n \langle x_i, f \rangle \langle x_j, f \rangle E[\gamma_i \gamma_j] \\ &= \sum_{i=m+1}^n \langle x_i, f \rangle^2 \leq \|f\|^2 \sum_{i=m+1}^n \|x_i\|^2. \end{aligned} \quad (3.5)$$

On the other hand, by (3.4) we have for all $n > m \geq 1$

$$(R_{\xi_n} - R_{\xi_m})f = \sum_{i=m+1}^n \langle x_i, f \rangle x_i.$$

Therefore from the definition of the nuclear norm we get

$$\nu(R_{\xi_n} - R_{\xi_m}) \leq \sum_{i=m+1}^n \|x_i\|^2. \quad (3.6)$$

Thus by (3.1), (3.5) and (3.6) we see that $(\langle \xi_n, f \rangle)_{n \geq 1}$ and $(R_{\xi_n})_{n \geq 1}$ are Cauchy sequences in $L_2(\mathcal{Q})$ and in $\mathcal{N}(X^*, X)$ respectively. Therefore $(\xi_n)_{n \geq 1}$ satisfies conditions (c) and (d), so that $\xi_n = \sum_{i=1}^n \gamma_i x_i$ converges in $L_2(\mathcal{Q}; X)$ by the assumption (3). Consequently, using Itô-Nisio's theorem [7], $\sum_{i=1}^{\infty} \gamma_i x_i$ is convergent a. s., that is, X is of type 2.

Next we prove that X is of cotype 2. Assume that there exists a sequence $(x_i)_{i \geq 1}$ such that $\sum_{i=1}^{\infty} \gamma_i x_i$ is convergent a. s., but $\sum_{i=1}^{\infty} \|x_i\|^2 = \infty$. If we set $a_k = \sum_{i=1}^k \|x_i\|^2$ then $a_k \rightarrow \infty$ and also $\sum_{k=1}^{\infty} \|x_k\|^2 / a_k = \infty$. Define a sequence $(\zeta_k)_{k \geq 1}$ of independent mean-zero random elements in $L_2(\mathcal{Q}; X)$ with distributions such that

$$P\left(\zeta_k = -\frac{a_k^{1/2} x_k}{\|x_k\|}\right) = P\left(\zeta_k = \frac{a_k^{1/2} x_k}{\|x_k\|}\right) = \frac{1}{2} \cdot \frac{\|x_k\|^2}{a_k},$$

$$P(\zeta_k=0)=1-\frac{\|x_k\|^2}{a_k}.$$

Then, by the Borel-Cantelli lemma, ζ_k does not converge to 0 a. s. so that

$$\eta_n \equiv \sum_{k=1}^n \zeta_k \quad \text{diverges a. s.} \tag{3.7}$$

Let us consider the sequence $(\xi_n)_{n \geq 1}$ of mean-zero Gaussian random elements defined by (3.2). Then, since $\xi_n = \sum_{i=1}^n \gamma_i x_i$ is convergent a. s., by [3; Theorem 1] R_{ξ_n} converges in $\mathcal{N}(X^*, X)$, and hence the set $\{R_{\xi_n}\}$ is relatively compact in $\mathcal{N}(X^*, X)$. On the other hand, from (3.4) and Lemma 1 we have for all $n \geq 1$

$$\begin{aligned} \sum_{i=1}^n \langle x_i, f \rangle^2 &= E|\langle \xi_n, f \rangle|^2 \\ &\leq \|f\|^2 \cdot \nu(R_{\xi_n}) \leq \|f\|^2 \cdot \sup_n \nu(R_{\xi_n}) < \infty, \end{aligned}$$

so that

$$\sum_{i=1}^{\infty} \langle x_i, f \rangle^2 \quad \text{is convergent.} \tag{3.8}$$

Now we show that $(\eta_n)_{n \geq 1}$ defined by (3.7) satisfies the conditions (c) and (d) of the assumption (3). To see this it suffices to show that $(\langle \eta_n, f \rangle)_{n \geq 1}$ is a Cauchy sequence in $L_2(\mathcal{Q})$ for every $f \in X^*$ and the set $\{R_{\eta_n}\}$ is relatively compact in $\mathcal{N}(X^*, X)$. A routine calculation using the fact that

$$E\langle \zeta_i, f \rangle \langle \zeta_j, g \rangle = \begin{cases} \langle x_i, f \rangle \langle x_j, g \rangle & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \tag{3.9}$$

shows that for each $f \in X^*$ and all integers $n > m \geq 1$,

$$\begin{aligned} E|\langle \eta_n, f \rangle - \langle \eta_m, f \rangle|^2 &= E \left| \sum_{k=m+1}^n \langle \zeta_k, f \rangle \right|^2 \\ &= \sum_{k=m+1}^n \langle x_k, f \rangle^2. \end{aligned}$$

We see from (3.8) that $(\langle \eta_n, f \rangle)_{n \geq 1}$ is a Cauchy sequence in $L_2(\mathcal{Q})$. On the other hand, by (3.4) and (3.9) it is easy to show that $R_{\xi_n} = R_{\eta_n}$ for all $n \geq 1$. From this and the relative compactness of the set $\{R_{\xi_n}\}$, the set $\{R_{\eta_n}\}$ must be relatively compact in $\mathcal{N}(X^*, X)$. Thus from the assumption (3) it follows that $\eta_n = \sum_{k=1}^n \zeta_k$ is convergent in $L_2(\mathcal{Q}; X)$. Consequently, by Itô-Nisio's theorem [7], $\sum_{k=1}^{\infty} \zeta_k$ is convergent a. s. and this contradicts (3.7). The proof is now complete.

4. Characterization of Banach spaces of type 2.

We now turn to characterization of Banach spaces of type 2 by the convergence in probability of (mean-zero) Gaussian random elements. Let X be

a real separable Banach space. A subset G of X^* is said to be *total* if G separates points in X , that is, $\langle x, g \rangle = 0$ for all $g \in G$ implies $x = 0$. Before formulating Theorem 2 below we shall prove the following results of interest in themselves.

LEMMA 2. *Let $(\xi_n)_{n \geq 1}$ be a sequence in $M(\Omega; X)$. Then ξ_n converges in probability if and only if the following two conditions are satisfied*

(a) *For each $g \in G$, $\langle \xi_n, g \rangle$ converges in probability, where G is a total subset of X^* .*

(b) *The set $\{\xi_n\}$ is uniformly tight.*

Proof. The proof of the necessity is obvious. We proceed by a variation of an argument in [7; p. 38] in order to prove the sufficiency. Let us denote the distributions of $\xi_n - \xi_m$ by $\mu_{m,n}$, $m, n \geq 1$. Then from (b) it is easy to show that the set $\{\mu_{m,n}; m, n \geq 1\}$ is uniformly tight. We shall now prove that ξ_n converges in probability. Suppose to the contrary that there exist $\varepsilon > 0$ and a subsequence $(\mu_{m',n'})$ such that

$$\mu_{m',n'}(B_\varepsilon) \leq 1 - \varepsilon \quad \text{for all } m', n',$$

where B_ε denotes the open ε -neighborhood of the origin in X . Since $\{\mu_{m,n}\}$ is uniformly tight, we can assume that $\mu_{m',n'}$ converges weakly to a probability measure μ on X , so that

$$\mu(B_\varepsilon) \leq \liminf_{\substack{m' \rightarrow \infty \\ n' \rightarrow \infty}} \mu_{m',n'}(B_\varepsilon) \leq 1 - \varepsilon. \quad (4.1)$$

On the other hand we have

$$\widehat{\mu_{m',n'}}(f) = E[\exp(i \langle \xi_{n'} - \xi_{m'}, f \rangle)], \quad f \in X^*. \quad (4.2)$$

If we denote by $S(G)$ the linear subspace spanned by G in X^* then it is easy from (a) to show that $\langle \xi_{n'} - \xi_{m'}, f \rangle$ converges to 0 in probability for every $f \in S(G)$. Consequently, letting $m', n' \rightarrow \infty$ in (4.2), we have $\widehat{\mu}(f) = 1$ for every $f \in S(G)$, so that μ concentrates at the origin in X by Perlman [11; Theorem 9]. This contradicts (4.1) and the proof is now complete.

COROLLARY 1. *A set $K \subset M(\Omega; X)$ is relatively compact if and only if the following two conditions are satisfied:*

(a) *For each $g \in G$, the set $\{\langle \xi, g \rangle; \xi \in K\}$ is relatively compact in $M(\Omega)$, where G is a total subset of X^* .*

(b) *K is uniformly tight.*

Proof. The proof of the necessity is obvious. To prove the sufficiency, let $(\xi_n)_{n \geq 1}$ be an arbitrary sequence in K , and show that $(\xi_n)_{n \geq 1}$ contains a subsequence converging in probability. Since X is separable, by the Hahn-Banach theorem we can get a countable total subset G of X^* . By (a), using a diagonal selection argument, we can find a subsequence $(\xi_{n'})$ such that for each $g \in G$,

$\langle \xi_{n'}, g \rangle$ converges in probability. On the other hand, it is obvious from (b) that the set $\{\xi_{n'}\}$ is uniformly tight. Thus $\xi_{n'}$ converges in probability since $(\xi_{n'})$ satisfies the conditions (a) and (b) of Lemma 2. The proof is now complete.

We say that a Banach space X has the *approximation property* if, for every compact set C in X and every $\varepsilon > 0$, there is a continuous linear operator $T : X \rightarrow X$ of finite rank such that $\|Tx - x\| \leq \varepsilon$ for every $x \in C$. Though, by Enflo's example, not all Banach spaces enjoy this property, many spaces, for instance, l_p , L_p ($1 \leq p \leq \infty$) and $C[0, 1]$, have enough structure to allow this approximation by operators of finite rank. (See [6] and Diestel-Uhl [4].)

Let ξ and ξ_n , $n=1, 2, \dots$, be mean-zero Gaussian random elements. It is known in [3] that $\lim_{n \rightarrow \infty} \nu(R_{\xi_n} - R_{\xi}) = 0$ if μ_{ξ_n} converges weakly to μ_{ξ} , but the converse implication is not valid in general. However, we have the following theorem which was given by Chevet.

THEOREM C ([2]). *For a real separable Banach space X which has the approximation property, the following assertions (1)-(3) are equivalent:*

- (1) X is of type 2.
- (2) For any set K of mean-zero Gaussian random elements in X , K is uniformly tight if (and only if) the set $\{R_{\xi}; \xi \in K\}$ is relatively compact in $\mathfrak{N}(X^*, X)$.
- (3) For any sequence $(\xi_n)_{n \geq 0}$ of mean-zero Gaussian random elements in X , μ_{ξ_n} converges weakly to μ_{ξ_0} if (and only if) $\lim_{n \rightarrow \infty} \nu(R_{\xi_n} - R_{\xi_0}) = 0$.

For the case of convergence in probability we have the following:

THEOREM 2. *For a real separable Banach space X which has the approximation property, the following assertions (1)-(3) are equivalent:*

- (1) X is of type 2.
- (2) For any set K of mean-zero Gaussian random elements in X , K is relatively compact in $M(\Omega; X)$ if (and only if) the following two conditions are satisfied:
 - (a) For each $f \in X^*$, the set $\{\langle \xi, f \rangle; \xi \in K\}$ is relatively compact in $M(\Omega)$.
 - (b) The set $\{R_{\xi}; \xi \in K\}$ is relatively compact in $\mathfrak{N}(X^*, X)$.
- (3) For any sequence $(\xi_n)_{n \geq 1}$ of mean-zero Gaussian random elements in X , ξ_n converges in probability if (and only if) the following two conditions are satisfied:
 - (c) For each $f \in X^*$, $\langle \xi_n, f \rangle$ converges in probability.
 - (d) The set $\{R_{\xi_n}\}$ is relatively compact in $\mathfrak{N}(X^*, X)$.

Proof. The implication (1) \Rightarrow (2) follows from Theorem C and Corollary 1. (2) \Rightarrow (3): We proceed in a way similar to the proof of the implication (2) \Rightarrow (3) of Theorem 1. (3) \Rightarrow (1): We have already shown this in the proof of the implication (3) \Rightarrow (1) of Theorem 1.

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